Fourth-order nonlinear evolution equations for a capillary-gravity wave packet in the presence of another wave packet in deep water

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(Received 3 February 2007; accepted 13 July 2007; published online 26 September 2007)

Starting from the Zakharov integral equation, two coupled fourth-order nonlinear equations have been derived for the evolution of the amplitudes of two capillary-gravity wave packets propagating in the same direction. The two evolution equations are used to investigate the stability of a uniform capillary-gravity wave train in the presence of another having the same group velocity. The relative changes in phase velocity of each uniform wave train due to the presence of the other one have been shown in figures for different wave numbers. The condition of instability of a wave of greater wavelength in the presence of a wave of shorter wavelength is obtained. It is observed that the instability region for a surface gravity wave train in the presence of a capillary-gravity wave train expands with the increase of wave steepness of the capillary-gravity wave train. It is found that the presence of a uniform capillary-gravity wave train causes an increase in the growth rate of instability of a uniform surface gravity wave train.
[DOI: 10.1063/1.2772252]

I. INTRODUCTION

One approach to studying the stability of finite-amplitude surface gravity and also capillary-gravity waves in deep water is through the application of the lowest-order nonlinear evolution equation, which is the nonlinear Schrödinger equation (hereafter referred to as the NLS equation). But such an analysis is valid only for small wave steepness. When wave steepness exceeds the value 0.15, the results from the NLS equation do not agree with the exact numerical results of Longuet-Higgins.[1,2] Dysthe[3] derived an equation that is one order higher than the NLS equation and is thus the fourth-order nonlinear evolution equation known also as the modified NLS equation. The stability analysis made from this equation gives results consistent with the exact results of Longuet-Higgins[1,2] and also with the experimental results of Benjamin and Feir[4] for waves of wave steepness up to 0.25. The fourth-order effect gives surprising improvement compared to only the NLS effect in many respects; some of these points have been elaborated on by Janssen.[5] The dominant new effect that comes in the fourth order is the influence of wave-induced mean flow, which produces a significant deviation in the stability character. From these observations, it follows that the fourth-order nonlinear evolution is quite appropriate for studying nonlinear effects of surface gravity and capillary-gravity wave packets in deep water. Considering the importance of stability analysis of the uniform wave train made from fourth-order nonlinear evolution equations, many authors [Stiassnie,[6] Hogan,[7] Dhar and Das,[8,9] and many others] have derived fourth-order nonlinear evolution equations for surface gravity and capillary-gravity wave packets in different contexts and have made stability analysis by the use of these evolution equations.

What has been said in the preceding paragraph is for the evolution of a single wave packet. It is of considerable importance to extend the stability analysis of a wave packet in the presence of another wave packet. For such an analysis, nonlinear evolution equations for a wave packet in the presence of another wave packet are needed. The stability analysis of a surface gravity wave in the presence of a second wave was made by Roskes[10] based on lowest-order nonlinear evolution equations, which consist of two coupled NLS equations. In this investigation, the modulational perturbation is restricted to a direction along which the group velocity projections of the two waves overlap. Dhar and Das[8] made the same analysis of Roskes[10] making use of two coupled fourth-order nonlinear evolution equations that they derived for two wave packets having the same characteristic wave number. The same analysis including the effect of capillarity was later made by Dhar and Das.[9] For the derivation of evolution equations, Dhar and Das[8,9] used the multiple scale method. They observed significant deviations from the results obtained from coupled cubic nonlinear Schrödinger equations.

Pierce and Knobloch[11] derived asymptotically exact third-order evolution equations for counterpropagating capillary-gravity wave trains having equal characteristic wave number and frequency propagating over finite depth water. They also investigated stability properties of stationary standing and quasiperiodic waves. Later on, Debsarma and Das[12] derived asymptotically exact nonlocal fourth-order evolution equations for two counterpropagating capillary-gravity wave packets on the surface of water of infinite depth.

In the present investigation, we have derived two coupled fourth-order nonlinear evolution equations for two

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capillary-gravity wave packets propagating in the same direction and having unequal wave numbers. Unlike Dhar and Das,[8,9] the evolution equations are derived here using Zakharov integral equation. Stiassnie[6] and Hogan[7] also used the Zakharov integral equation for the derivation of fourth-order nonlinear evolution equations for a surface gravity wave packet and a capillary-gravity wave packet, respectively. In deriving the two coupled evolution equations, we make an extension of the paper by Hogan *et al.*,[13] who derived the change in phase speed of one capillary-gravity wave train in the presence of another starting from the Zakharov integral equation. The expression for the change in phase speed for the case of gravity waves was first obtained by Longuet-Higgins and Phillips[14] by the perturbation method. Onorato *et al.*[15] also derived third-order evolution equations to study the problem of interaction of two wave systems in deep water with equal characteristic wave number and propagating in two different directions. They found that the introduction of a second wave results in an increase of the instability growth rates and causes enlargement of the instability region.

From the uniform wave train solution of the two evolution equations obtained here, we have derived the change in phase speed of one capillary-gravity wave train in the presence of another. Then by the use of the two coupled nonlinear evolution equations derived here, the stability analysis of a uniform surface gravity wave train in the presence of a uniform capillary-gravity wave train is investigated, when the group velocities of the two wave trains coincide. The condition of instability of a uniform gravity wave train in the presence of a capillary-gravity wave train is obtained. We have plotted stable-unstable regions for four sets of values of wave numbers. In each case, the instability regions are compared with those in the absence of the second wave. An expression for the growth rate of instability of a surface gravity wave train in the presence of a capillary-gravity wave train is obtained here. It is found that the growth rate of instability of a uniform surface gravity wave train increases due to the presence of a capillary-gravity wave train, and it increases with the increase in wave steepness of the capillary-gravity wave train. In all the figures for stable-unstable regions and growth rate of instability against the perturbation wave number, we find a comparison between results from fourth-order evolution equations and those from third-order evolution equations. It is observed that at fourth order, the instability region is shortened slightly. It is also found that the effect of fourth-order terms is to enhance the growth rate of instability. This is in contrast with the results for a single wave packet.

II. DERIVATION OF EVOLUTION EQUATIONS

The two coupled nonlinear evolution equations are derived here using the Zakharov integral equation, which is the following:

$$i\frac{\partial B(\vec{k},t)}{\partial t} = \iint\int_{-\infty}^{\infty} T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)B^*(\vec{k}_1,t)B(\vec{k}_2,t)B(\vec{k}_3,t) \times \delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)\exp\{i[\omega(\vec{k})+\omega(\vec{k}_1) - \omega(\vec{k}_2)-\omega(\vec{k}_3)]t\}d\vec{k}_1 d\vec{k}_2 d\vec{k}_3, \quad (1)$$

where $B(\vec{k},t)$ is related to the free surface elevation $\zeta(\vec{x},t)$ by

$$\zeta(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{|\vec{k}|}{2\omega(\vec{k})}\right)^{1/2}\{B(\vec{k},t)\exp\{i[\vec{k}\cdot\vec{x}-\omega(\vec{k})t]\} + \text{c.c.}\}d\vec{k}. \quad (2)$$

Here c.c. denotes complex conjugate, $\vec{k}=(k,l)$ is the wave vector, $\vec{x}=(x,y)$ is the horizontal spatial vector, and ω is the linearized wave frequency related to $\vec{k}$ through the following linear dispersion relation:

$$\omega(\vec{k}) = [g|\vec{k}| + \nu|\vec{k}|^3]^{1/2} = [g|\vec{k}|(1+c|\vec{k}|^2)]^{1/2}, \quad (3)$$

g being the acceleration due to gravity, ν being the ratio of the surface tension coefficient $T^{(S)}$ to the water density ρ, and $c=\nu/g$. $T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)$ is a scalar function given first by Zakharov.[16] Its expression is given in Appendix of Crawford *et al.*[17] for surface gravity waves. For capillary-gravity waves, the corrected form of the Zakharov kernel was first given in the Appendix of Hogan *et al.*[13] Later on, Krasitskii[18] derived the expression for the kernel that corresponds exactly to the Hamiltonian and possesses symmetry with respect to its subscripts. In the present paper, we have used the expression given by Krasitskii[18] for the kernel $T(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)$ appearing in Eq. (1).

We consider two narrow capillary-gravity wave packets centered around the wave numbers $\vec{k}_a$ and $\vec{k}_b$, which we call, respectively, the first and second wave packet. A nonzero contribution to the integral in Eq. (1) can be obtained, when the following condition for wave numbers is satisfied:

$$\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3=0, \quad (4)$$

where $\vec{k}$, $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ are wave vectors of the four waves.

A. Evolution equation for the first wave packet

With $\vec{k}=\vec{k}_a$, the condition (4) is satisfied for two waves with wave vectors $\vec{k}_a$ and $\vec{k}_b$ in the following three cases:

$$\text{(i)}\vec{k}_1=\vec{k}_b,\vec{k}_2=\vec{k}_b,\vec{k}_3=\vec{k}_a.$$
$$\text{(ii)}\vec{k}_1=\vec{k}_b,\vec{k}_2=\vec{k}_a,\vec{k}_3=\vec{k}_b. \quad (5)$$
$$\text{(iii)}\vec{k}_1=\vec{k}_a,\vec{k}_2=\vec{k}_a,\vec{k}_3=\vec{k}_a.$$

So, for the evolution equation of the first wave packet, we are to take $\vec{k}=\vec{k}_a+\vec{\chi}$ in Eq. (1) and the triple integral on the right-hand side is to be replaced by the sum of the three triple integrals with the replacements $\vec{k}_1=\vec{k}_b+\vec{\chi}_1$, $\vec{k}_2=\vec{k}_b+\vec{\chi}_2$, $\vec{k}_3=\vec{k}_a+\vec{\chi}_3$ for the first triple integral; $\vec{k}_1=\vec{k}_b+\vec{\chi}_1$, $\vec{k}_2=\vec{k}_a+\vec{\chi}_2$, $\vec{k}_3=\vec{k}_b+\vec{\chi}_3$ for the second; and $\vec{k}_1=\vec{k}_a+\vec{\chi}_1$, $\vec{k}_2=\vec{k}_a+\vec{\chi}_2$, $\vec{k}_3=\vec{k}_a+\vec{\chi}_3$ for the third. Therefore, Eq. (1) for the first wave packet becomes

$$i\frac{\partial B(\vec{k}_a+\vec{\chi},t)}{\partial t}=\iint\int_{-\infty}^{\infty}T(\vec{k}_a+\vec{\chi},\vec{k}_b+\vec{\chi}_1,\vec{k}_b+\vec{\chi}_2,\vec{k}_a+\vec{\chi}_3)B^*(\vec{k}_b+\vec{\chi}_1,t)B(\vec{k}_b+\vec{\chi}_2,t)B(\vec{k}_a+\vec{\chi}_3,t)$$
$$\times\,\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2-\vec{\chi}_3)\exp\{i[\omega(\vec{k}_a+\vec{\chi})+\omega(\vec{k}_b+\vec{\chi}_1)-\omega(\vec{k}_b+\vec{\chi}_2)-\omega(\vec{k}_a+\vec{\chi}_3)]t\}d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3$$
$$+\iint\int_{-\infty}^{\infty}T(\vec{k}_a+\vec{\chi},\vec{k}_b+\vec{\chi}_1,\vec{k}_a+\vec{\chi}_2,\vec{k}_b+\vec{\chi}_3)B^*(\vec{k}_b+\vec{\chi}_1,t)B(\vec{k}_a+\vec{\chi}_2,t)B(\vec{k}_b+\vec{\chi}_3,t)$$
$$\times\,\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2-\vec{\chi}_3)\exp\{i[\omega(\vec{k}_a+\vec{\chi})+\omega(\vec{k}_b+\vec{\chi}_1)-\omega(\vec{k}_a+\vec{\chi}_2)-\omega(\vec{k}_b+\vec{\chi}_3)]t\}d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3$$
$$+\iint\int_{-\infty}^{\infty}T(\vec{k}_a+\vec{\chi},\vec{k}_a+\vec{\chi}_1,\vec{k}_a+\vec{\chi}_2,\vec{k}_a+\vec{\chi}_3)B^*(\vec{k}_a+\vec{\chi}_1,t)B(\vec{k}_a+\vec{\chi}_2,t)B(\vec{k}_a+\vec{\chi}_3,t)$$
$$\times\,\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2-\vec{\chi}_3)\exp\{i[\omega(\vec{k}_a+\vec{\chi})+\omega(\vec{k}_a+\vec{\chi}_1)-\omega(\vec{k}_a+\vec{\chi}_2)-\omega(\vec{k}_a+\vec{\chi}_3)]t\}d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3. \quad (6)$$

Introducing new variables $A_1(\vec{\chi},t)$ and $A_2(\vec{\chi},t)$ defined by

$$A_1(\vec{\chi},t)=B(\vec{k}_a+\vec{\chi},t)\exp\{-i[\omega(\vec{k}_a+\vec{\chi})-\omega(\vec{k}_a)]t\},$$
$$A_2(\vec{\chi},t)=B(\vec{k}_b+\vec{\chi},t)\exp\{-i[\omega(\vec{k}_b+\vec{\chi})-\omega(\vec{k}_b)]t\}, \quad (7)$$

Eq. (6) can be written as

$$i\frac{\partial A_1(\vec{\chi},t)}{\partial t}-A_1(\vec{\chi},t)[\omega(\vec{k}_a+\vec{\chi})-\omega(\vec{k}_a)]$$
$$=\iint\int_{-\infty}^{\infty}T(\vec{k}_a+\vec{\chi},\vec{k}_b+\vec{\chi}_1,\vec{k}_b+\vec{\chi}_2,\vec{k}_a+\vec{\chi}_3)A_2^*(\vec{\chi}_1,t)A_2(\vec{\chi}_2,t)A_1(\vec{\chi}_3,t)\times\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2-\vec{\chi}_3)d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3$$
$$+\iint\int_{-\infty}^{\infty}T(\vec{k}_a+\vec{\chi},\vec{k}_b+\vec{\chi}_1,\vec{k}_a+\vec{\chi}_2,\vec{k}_b+\vec{\chi}_3)A_2^*(\vec{\chi}_1,t)A_1(\vec{\chi}_2,t)A_2(\vec{\chi}_3,t)\times\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2-\vec{\chi}_3)d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3$$
$$+\iint\int_{-\infty}^{\infty}T(\vec{k}_a+\vec{\chi},\vec{k}_a+\vec{\chi}_1,\vec{k}_a+\vec{\chi}_2,\vec{k}_a+\vec{\chi}_3)A_1^*(\vec{\chi}_1,t)A_1(\vec{\chi}_2,t)A_1(\vec{\chi}_3,t)\times\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2-\vec{\chi}_3)d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3. \quad (8)$$

Replacing $\vec{k}$ by $\vec{k}_a+\vec{\chi}$ in Eq. (2), we get the surface elevation $\zeta_1(\vec{x},t)$ as given below for the wave packet whose wave numbers are centered around $\vec{k}_a$,

$$\zeta_1(\vec{x},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{|\vec{k}_a+\vec{\chi}|}{2\omega(\vec{k}_a+\vec{\chi})}\right]^{1/2}\{B(\vec{k}_a+\vec{\chi},t)$$
$$\times\exp i[(\vec{k}_a+\vec{\chi})\cdot\vec{x}-\omega(\vec{k}_a+\vec{\chi})t]+\text{c.c.}\}d\vec{\chi}. \quad (9)$$

Similarly, the surface elevation $\zeta_2(\vec{x},t)$ for the wave packet whose wave numbers are centered around $\vec{k}_b$ is given by

$$\zeta_2(\vec{x},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{|\vec{k}_b+\vec{\chi}|}{2\omega(\vec{k}_b+\vec{\chi})}\right]^{1/2}\{B(\vec{k}_b+\vec{\chi},t)$$
$$\times\exp i[(\vec{k}_b+\vec{\chi})\cdot\vec{x}-\omega(\vec{k}_b+\vec{\chi})t]+\text{c.c.}\}d\vec{\chi}. \quad (10)$$

In terms of the new variables defined by Eq. (7), $\zeta_1(\vec{x},t)$ and $\zeta_2(\vec{x},t)$ given, respectively, by Eqs. (9) and (10) can be expressed as

$$\zeta_1(\vec{x},t)=\exp i[\vec{k}_a\cdot\vec{x}-\omega(\vec{k}_a)t\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\vec{k}_a+\vec{\chi}|^{1/4}}{[4g(1+c|\vec{k}_a+\vec{\chi}|^2)]^{1/4}}A_1(\vec{\chi},t)\exp i(\vec{\chi}\cdot\vec{x})d\vec{\chi}\right\}+\text{c.c.}, \quad (11)$$

$$\zeta_2(\vec{x},t)=\exp i[\vec{k}_b\cdot\vec{x}-\omega(\vec{k}_b)t\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\vec{k}_b+\vec{\chi}|^{1/4}}{[4g(1+c|\vec{k}_b+\vec{\chi}|^2)]^{1/4}}A_2(\vec{\chi},t)expi(\vec{\chi}\cdot\vec{x})d\vec{\chi}\right\}+\text{c.c.} \quad (12)$$

In the subsequent development, we shall consider two wave packets both propagating along the x axis and having wave numbers k_1 and k_2, where $k_1 > k_2$. Thus we take the two wave vectors $\vec{k}_a$, $\vec{k}_b$ as follows:

$$\vec{k}_a \equiv \vec{k}_1 = k_1\hat{x}, \quad \vec{k}_b \equiv \vec{k}_2 = k_2\hat{x}. \tag{13}$$

Here $\hat{x}$ denotes a unit vector along the x axis. Also we shall consider modulational perturbation only along the x axis, i.e., in the direction of propagation of the two wave packets. In view of this, we can take $\vec{\chi}=\chi\hat{x}$ in Eq.(11) and (12).

Writing $\vec{k}_a=k_1\hat{x}$, $\vec{\chi}=\chi\hat{x}$ in the expression $\frac{|\vec{k}_a+\vec{\chi}|^{1/4}}{[4g(1+c|\vec{k}_a+\vec{\chi}|^2)]^{1/4}}$ appearing in Eq. (11) and then Taylor-expanding this in powers of χ keeping only linear terms in χ, which is the requirement for the derivation of the fourth-order nonlinear evolution equation (Stiassnie[6]), we get the following expression for $\zeta_1(\vec{x},t)$ given by Eq. (11):

$$\zeta_1(\vec{x},t) = \tfrac{1}{2}[a_1(\vec{x},t)\exp\{i[\vec{k}_1\cdot\vec{x}-\omega(\vec{k}_1)t]\} + a_1^*(\vec{x},t)\exp\{-i[\vec{k}_1\cdot\vec{x}-\omega(\vec{k}_1)t]\}], \tag{14}$$

where

$$a_1(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha_1(\vec{\chi},t)e^{i\vec{\chi}\cdot\vec{x}}d\vec{\chi} \tag{15}$$

and

$$\alpha_1(\vec{\chi},t) = \left[\frac{4k_1}{g}\right]^{1/4}\left[1+\frac{(1-ck_1^2)\chi}{4k_1(1+ck_1^2)}\right] \times (1+ck_1^2)^{-1/4}A_1(\vec{\chi},t). \tag{16}$$

Similarly, we get the following expression for $\zeta_2(\vec{x},t)$ given by Eq. (12):

$$\zeta_2(\vec{x},t) = \tfrac{1}{2}[a_2(\vec{x},t)\exp\{i[\vec{k}_2\cdot\vec{x}-\omega(\vec{k}_2)t]\} + a_2^*(\vec{x},t)\exp\{-i[\vec{k}_2\cdot\vec{x}-\omega(\vec{k}_2)t]\}], \tag{17}$$

where

$$a_2(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha_2(\vec{\chi},t)e^{i\vec{\chi}\cdot\vec{x}}d\vec{\chi}, \tag{18}$$

$$\alpha_2(\vec{\chi},t) = \left[\frac{4k_2}{g}\right]^{1/4}\left[1+\frac{(1-ck_2^2)\chi}{4k_2(1+ck_2^2)}\right] \times (1+ck_2^2)^{-1/4}A_2(\vec{\chi},t). \tag{19}$$

Now, to derive the nonlinear evolution equation for the wave packet whose wave numbers are centered around $\vec{k}_a \equiv \vec{k}_1 = k_1\hat{x}$, we multiply both sides of Eq. (8) by

$$\frac{1}{2\pi}\cdot\left[\frac{4k_1}{g}\right]^{1/4}\left[1+\frac{(1-ck_1^2)\chi}{4k_1(1+ck_1^2)}\right](1+ck_1^2)^{-1/4}\exp(i\vec{\chi}\cdot\vec{x})$$

and then integrate with respect to $\vec{\chi}$ after replacing $A_1(\vec{\chi},t)$ and $A_2(\vec{\chi},t)$ by $\alpha_1(\vec{\chi},t)$ and $\alpha_2(\vec{\chi},t)$, respectively, given by the relations (16) and (19). Then the evolution equation (8) assumes the following form, where we perform integration with respect to $\vec{\chi}$ on the right-hand side:

$$\begin{aligned}
&i\frac{\partial}{\partial t}\frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha_1(\vec{\chi},t)e^{i\vec{\chi}\cdot\vec{x}}d\vec{\chi} - \frac{1}{2\pi}\int_{-\infty}^{\infty}[\omega(\vec{k}_1+\vec{\chi})-\omega(\vec{k}_1)]\alpha_1(\vec{\chi},t)e^{i\vec{\chi}\cdot\vec{x}}d\vec{\chi} \\
&= \frac{1}{2\pi}\iint\int_{-\infty}^{\infty} T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_2+\vec{\chi}_1,\vec{k}_2+\vec{\chi}_2,\vec{k}_1+\vec{\chi}_3)\alpha_2^*(\vec{\chi}_1,t)\alpha_2(\vec{\chi}_2,t)\alpha_1(\vec{\chi}_3,t) \\
&\quad\times\frac{\omega_2}{2k_2}\left[1+\frac{(1-ck_1^2)}{4k_1(1+ck_1^2)}(\chi_2-\chi_1)-\frac{(1-ck_2^2)}{4k_2(1+ck_2^2)}(\chi_1+\chi_2)\right]e^{i(\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1)\cdot\vec{x}}d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3 \\
&\quad+\frac{1}{2\pi}\iint\int_{-\infty}^{\infty} T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_2+\vec{\chi}_1,\vec{k}_1+\vec{\chi}_2,\vec{k}_2+\vec{\chi}_3)\alpha_2^*(\vec{\chi}_1,t)\alpha_1(\vec{\chi}_2,t)\alpha_2(\vec{\chi}_3,t) \\
&\quad\times\frac{\omega_2}{2k_2}\left[1+\frac{(1-ck_1^2)}{4k_1(1+ck_1^2)}(\chi_3-\chi_1)-\frac{(1-ck_2^2)}{4k_2(1+ck_2^2)}(\chi_3+\chi_1)\right]e^{i(\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1)\cdot\vec{x}}d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3 \\
&\quad+\frac{1}{2\pi}\iint\int_{-\infty}^{\infty} T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_1+\vec{\chi}_1,\vec{k}_1+\vec{\chi}_2,\vec{k}_1+\vec{\chi}_3)\alpha_1^*(\vec{\chi}_1,t)\alpha_1(\vec{\chi}_2,t)\alpha_1(\vec{\chi}_3,t) \\
&\quad\times\frac{\omega_1}{2k_1}\left[1-\frac{(1-ck_1^2)}{2k_1(1+ck_1^2)}\chi_1\right]e^{i(\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1)\cdot\vec{x}}d\vec{\chi}_1d\vec{\chi}_2d\vec{\chi}_3.
\end{aligned} \tag{20}$$

We first evaluate the left-hand side of Eq. (20). Following the same argument as already stated, we set $\vec{\chi}=\chi\hat{x}$ in $\omega(\vec{k}_1+\vec{\chi})$ and expand this in powers of χ keeping terms up to third degree in χ. Then the left-hand side of Eq. (20) becomes the following after evaluation of the inversion integrals:

$$\text{L.H.S.} = i\frac{\partial a_1}{\partial t} + i\frac{\omega_1}{k_1}\left[\frac{1+3ck_1^2}{2(1+ck_1^2)}\right]\frac{\partial a_1}{\partial x} - \frac{\omega_1}{k_1^2}\left[\frac{1-6ck_1^2-3(ck_1^2)^2}{8(1+ck_1^2)^2}\right]\frac{\partial^2 a_1}{\partial x^2} - i\frac{\omega_1}{k_1^3}\left[\frac{1+5ck_1^2-5(ck_1^2)^2-(ck_1^2)^3}{16(1+ck_1^2)^3}\right]\frac{\partial^3 a_1}{\partial x^3}. \quad (21)$$

Since the modulational perturbations are assumed to take place only along the x axis, we set $\vec{\chi}_1=\chi_1\hat{x}$, $\vec{\chi}_2=\chi_2\hat{x}$, $\vec{\chi}_3=\chi_3\hat{x}$ in the arguments of T appearing on the right-hand side of Eq. (20) and make Taylor expansions of them in powers of χ_1, χ_2, χ_3 keeping terms up to linear in these variables. These expansions are given in Appendix A, in which we have used the following notations:

$$m(k) = \frac{T^{(S)}k^2}{\rho g} = \frac{ck^2}{g}, \quad s(k) = \frac{m(k)}{1+m(k)},$$

$$\omega(k) = [gk+\nu k^3]^{1/2}, \quad c_g = \frac{d\omega(k)}{dk},$$

$$m_i = m(k_i), \quad s_i = s(k_i), \quad \omega_i = \omega(k_i),$$

$$(c_g)_i = \left(\frac{d\omega(k)}{dk}\right)_{k=k_i},$$

$$k_{i\pm j} = k_i \pm k_j, \quad m_{i\pm j} = m(k_{i\pm j}), \quad \omega_{i\pm j} = \omega(k_{i\pm j}),$$

$$\kappa = \frac{k_2}{k_1}, \quad r = \frac{\omega_2}{\omega_1},$$

$$r_1 = \frac{r\omega_{1-2}^2}{\omega_{1-2}^2-(\omega_1-\omega_2)^2},$$

$$r_2 = \frac{r\{\omega_{1-2}^2+(\omega_1-\omega_2)^2\}}{\omega_{1-2}^2-(\omega_1-\omega_2)^2},$$

$$r_3 = \frac{r\omega_1^2}{\omega_{1+2}^2-(\omega_1+\omega_2)^2},$$

$$d_1^{(\pm)} = \frac{2(k_1\pm k_2)}{\omega_{1\pm 2}^2-(\omega_1\pm\omega_2)^2}\{\omega_{1\pm 2}(c_g)_{1\pm 2}-(\omega_1\pm\omega_2)(c_g)_1\},$$

$$d_2^{(\pm)} = \frac{2(k_1\pm k_2)}{\omega_{1\pm 2}^2-(\omega_1\pm\omega_2)^2}\{\omega_{1\pm 2}(c_g)_{1\pm 2}-(\omega_1\pm\omega_2)(c_g)_2\},$$

$$d_3^{(\pm)} = \frac{(k_1\pm k_2)(\omega_1\pm\omega_2)}{\omega_{1\pm 2}^2-(\omega_1\pm\omega_2)^2}\{(c_g)_1-(c_g)_2\}. \quad (22)$$

We then find the expressions

$$T^{(1)} \equiv T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1, \vec{k}_2+\vec{\chi}_1, \vec{k}_2+\vec{\chi}_2, \vec{k}_1+\vec{\chi}_3) \times \left[1+\frac{(1-m_1)}{4(1+m_1)}\cdot\frac{(\chi_2-\chi_1)}{k_1}-\frac{(1-m_2)}{4(1+m_2)}\cdot\frac{(\chi_2+\chi_1)}{k_2}\right],$$

$$T^{(2)} \equiv T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1, \vec{k}_2+\vec{\chi}_1, \vec{k}_1+\vec{\chi}_2, \vec{k}_2+\vec{\chi}_3) \times \left[1+\frac{(1-m_1)}{4(1+m_1)}\cdot\frac{(\chi_3-\chi_1)}{k_1}-\frac{(1-m_2)}{4(1+m_2)}\cdot\frac{(\chi_3+\chi_1)}{k_2}\right],$$

$$T^{(3)} \equiv T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1, \vec{k}_1+\vec{\chi}_1, \vec{k}_1+\vec{\chi}_2, \vec{k}_1+\vec{\chi}_3) \times \left[1-\frac{(1-m_1)}{2(1+m_1)}\cdot\frac{\chi_1}{k_1}\right]$$

appearing in the integrals on the right-hand side of Eq. (20) in powers of χ_1, χ_2, and χ_3 keeping terms up to linear in these variables. The right-hand side of Eq. (20) now becomes the following:

$$\text{R.H.S.} = \frac{1}{2\pi}\iint\int_{-\infty}^{\infty}\left(\frac{\omega_2}{2k_2}\right)T^{(1)}\alpha_2^*(\vec{\chi}_1,t)\alpha_2(\vec{\chi}_2,t)\alpha_1(\vec{\chi}_3,t)e^{i(\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1)\cdot\vec{x}}d\vec{\chi}_1 d\vec{\chi}_2 d\vec{\chi}_3 + \frac{1}{2\pi}\iint\int_{-\infty}^{\infty}\left(\frac{\omega_2}{2k_2}\right)T^{(2)}\alpha_2^*(\vec{\chi}_1,t)\alpha_1(\vec{\chi}_2,t)\alpha_2(\vec{\chi}_3,t)e^{i(\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1)\cdot\vec{x}}d\vec{\chi}_1 d\vec{\chi}_2 d\vec{\chi}_3 + \frac{1}{2\pi}\iint\int_{-\infty}^{\infty}\left(\frac{\omega_1}{2k_1}\right)T^{(3)}\alpha_1^*(\vec{\chi}_1,t)\alpha_1(\vec{\chi}_2,t)\alpha_1(\vec{\chi}_3,t)e^{i(\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1)\cdot\vec{x}}d\vec{\chi}_1 d\vec{\chi}_2 d\vec{\chi}_3. \quad (23)$$

Performing the Fourier inversion integrals in the above three integrals, we get the following expression for the right-hand side of Eq. (20):

$$
\begin{aligned}
\text{R.H.S.} = \frac{\omega_2 k_1}{2\kappa}\Bigg[& k_1\left\{\kappa^2 + \frac{\kappa}{2}(1-\kappa)(r-2r_1) - \kappa(1+\kappa)r_3 - \frac{3rm_2}{4(1+m_2)}\right\}a_1 a_2 a_2^* \\
& + i\left\{\kappa(1+\kappa)r_3 - \kappa(1-\kappa)(r-2r_1)\cdot\frac{1-m_1}{4(1+m_1)} - \kappa^2\cdot\frac{3+m_1}{4(1+m_1)} - (1-\kappa)(r-2r_1)\cdot\frac{1+3m_2}{4(1+m_2)}\right. \\
& + (1+\kappa)r_3\cdot\frac{1+3m_2}{2(1+m_2)} - \frac{\kappa}{4}\cdot\frac{3+5m_2}{1+m_2} + \frac{3rm_2}{4\kappa(1+m_2)} + \frac{3rm_2(3+m_1)}{8(1+m_2)(1+m_1)} - \kappa r_3 d_2^{(+)} + \kappa r_1 d_3^{(-)} \\
& \left. + \kappa r_3 d_3^{(+)}\right\}a_1 a_2^*\frac{\partial a_2}{\partial x} + i\left\{-\frac{\kappa r}{2} + \kappa(1+\kappa)r_3 - \kappa(1-\kappa)\cdot\frac{(r-2r_1)(1-m_1)}{4(1+m_1)} - \kappa^2\cdot\frac{3+m_1}{4(1+m_1)}\right. \\
& + (1-\kappa)(r-2r_1)\cdot\frac{1+3m_2}{4(1+m_2)} - (1+\kappa)r_3\cdot\frac{1+3m_2}{2(1+m_2)} + \kappa\cdot\frac{3+5m_2}{4(1+m_2)} - \frac{3rm_2}{4\kappa(1+m_2)} + \frac{3rm_2(3+m_1)}{8(1+m_2)(1+m_1)} \\
& \left. + 2\kappa r_1\cdot\frac{1+2m_{1-2}}{1+m_{1-2}} - \kappa r_1 d_2^{(-)} + \kappa r_1 d_3^{(-)} + \kappa r_3 d_3^{(+)}\right\}a_1 a_2\frac{\partial a_2^*}{\partial x} + i\left\{-\frac{\kappa r}{2} - \kappa(1-\kappa)(r-2r_1)\cdot\frac{1-m_1}{4(1+m_1)}\right. \\
& \left. + 2\kappa r_1\cdot\frac{1+2m_{1-2}}{1+m_{1-2}} + \kappa(1+\kappa)r_3\cdot\frac{3+5m_1}{2(1+m_1)} + \frac{3rm_2(5+3m_1)}{8(1+m_2)(1+m_1)} - \kappa^2 - \kappa r_1 d_1^{(-)} - \kappa r_3 d_1^{(+)}\right\}a_2 a_2^*\frac{\partial a_1}{\partial x} \\
& + \kappa a_1 H\frac{\partial}{\partial x}(a_2 a_2^*)\Bigg] + \frac{\omega_1 k_1}{4}\Bigg[k_1\left\{1 - \frac{3m_1}{4(1+m_1)} + \frac{1+m_1}{1-2m_1}\right\}a_1 a_1^* - i\left\{3 - \frac{3m_1(5+3m_1)}{4(1+m_1)^2}\right. \\
& \left. + \frac{3+15m_1+6m_1^2-24m_1^3}{(1-2m_1)^2(1+4m_1)}\right\}a_1 a_1^*\frac{\partial a_1}{\partial x} + i\left\{-\frac{1-m_1}{2(1+m_1)} - \frac{1-m_1}{2(1-2m_1)} + \frac{3m_1(1-m_1)}{8(1+m_1)^2}\right\}a_1^2\frac{\partial a_1^*}{\partial x} \\
& + 2a_1 H\frac{\partial}{\partial x}(a_1 a_1^*)\Bigg]. \qquad (24)
\end{aligned}
$$

We now introduce the following dimensionless variables:

$$a_1' = k_1 a_1, \quad a_2' = k_2 a_2, \quad x' = k_2 x, \quad t' = \omega_2 t \qquad (25)$$

on the L.H.S. and R.H.S. of Eq. (20) as given by Eqs. (21) and (24), respectively. Finally dropping the primes, we get the following nonlinear evolution equation for the first wave packet in the presence of the second wave packet:

$$
\begin{aligned}
i\frac{\partial a_1}{\partial t} + i\beta_0^{(1)}\frac{\partial a_1}{\partial x} + \beta_1^{(1)}\frac{\partial^2 a_1}{\partial x^2} + i\beta_2^{(1)}\frac{\partial^3 a_1}{\partial x^3} &= \Lambda_1^{(1)} a_1^2 a_1^* \\
& + i\Lambda_2^{(1)} a_1 a_1^*\frac{\partial a_1}{\partial x} + i\Lambda_3^{(1)} a_1^2\frac{\partial a_1^*}{\partial x} + \Lambda_4^{(1)} a_1 H\left\{\frac{\partial}{\partial x}(a_1 a_1^*)\right\} \\
& + \mu_1^{(1)} a_1 a_2 a_2^* + i\mu_2^{(1)} a_2 a_2^*\frac{\partial a_1}{\partial x} + i\mu_3^{(1)} a_1 a_2^*\frac{\partial a_2}{\partial x} \\
& + i\mu_4^{(1)} a_1 a_2\frac{\partial a_2^*}{\partial x} + \mu_5^{(1)} a_1 H\left\{\frac{\partial}{\partial x}(a_2 a_2^*)\right\}, \qquad (26)
\end{aligned}
$$

where H denotes the Hilbert transform operator in one dimension defined by

$$H\psi = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\psi(\xi)}{\xi - x}d\xi. \qquad (27)$$

The coefficients $\beta_i^{(1)}$, $\Lambda_i^{(1)}$, $\mu_i^{(1)}$ appearing in Eq. (26) are given in Appendix B. For a check of our calculations, we have found the coefficient $\mu_1^{(1)}$ by the multiple scale method which is given in Appendix C. Setting $a_2=0$ and introducing appropriate dimensionless variables in Eq. (26), one can recover the evolution equation (2.20) of Hogan[7] for a single capillary-gravity wave train.

B. Evolution equation for the second wave packet

Making an interchange between the suffixes a and b in the evolution equation (6) for the first wave packet, we get the following evolution equation for the second wave packet:

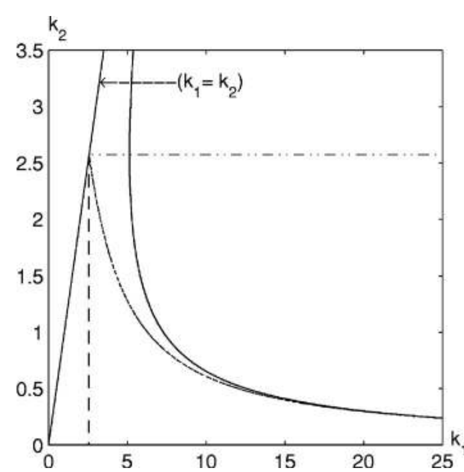

FIG. 1. Curves of invalid wave numbers: (- - - -) $m_1=0.5$, (-.-.-.-) $m_2=0.5$, (—–) $\omega_{1-2}^2=(\omega_1-\omega_2)^2$, (……) $\omega_{1+2}^2=(\omega_1+\omega_2)^2$.

$$i\frac{\partial B(\vec{k}_b+\vec{\chi},t)}{\partial t} = \iint\int_{-\infty}^{\infty} T(\vec{k}_b+\vec{\chi},\vec{k}_a+\vec{\chi}_1,\vec{k}_a+\vec{\chi}_2,\vec{k}_b+\vec{\chi}_3)B^*(\vec{k}_a+\vec{\chi}_1,t)B(\vec{k}_a+\vec{\chi}_2,t)B(\vec{k}_b+\vec{\chi}_3,t)\times\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2 - \vec{\chi}_3)\exp\{i[\omega(\vec{k}_b+\vec{\chi})+\omega(\vec{k}_a+\vec{\chi}_1)-\omega(\vec{k}_a+\vec{\chi}_2)-\omega(\vec{k}_b+\vec{\chi}_3)]t\}d\vec{\chi}_1 d\vec{\chi}_2 d\vec{\chi}_3$$
$$+\iint\int_{-\infty}^{\infty} T(\vec{k}_b+\vec{\chi},\vec{k}_a+\vec{\chi}_1,\vec{k}_b+\vec{\chi}_2,\vec{k}_a+\vec{\chi}_3)B^*(\vec{k}_a+\vec{\chi}_1,t)B(\vec{k}_b+\vec{\chi}_2,t)B(\vec{k}_a+\vec{\chi}_3,t)\times\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2 - \vec{\chi}_3)\exp\{i[\omega(\vec{k}_b+\vec{\chi})+\omega(\vec{k}_a+\vec{\chi}_1)-\omega(\vec{k}_b+\vec{\chi}_2)-\omega(\vec{k}_a+\vec{\chi}_3)]t\}d\vec{\chi}_1 d\vec{\chi}_2 d\vec{\chi}_3$$
$$+\iint\int_{-\infty}^{\infty} T(\vec{k}_b+\vec{\chi},\vec{k}_b+\vec{\chi}_1,\vec{k}_b+\vec{\chi}_2,\vec{k}_b+\vec{\chi}_3)B^*(\vec{k}_b+\vec{\chi}_1,t)B(\vec{k}_b+\vec{\chi}_2,t)B(\vec{k}_b+\vec{\chi}_3,t)\times\delta(\vec{\chi}+\vec{\chi}_1-\vec{\chi}_2 - \vec{\chi}_3)\exp\{i[\omega(\vec{k}_b+\vec{\chi})+\omega(\vec{k}_b+\vec{\chi}_1)-\omega(\vec{k}_b+\vec{\chi}_2)-\omega(\vec{k}_b+\vec{\chi}_3)]t\}d\vec{\chi}_1 d\vec{\chi}_2 d\vec{\chi}_3. \quad (28)$$

Now, to derive the second evolution equation, we follow the same procedure of deriving Eq. (26) from Eq. (6). In the process of the derivation of the second evolution equation, we need the expressions for the kernels appearing in the three integrals on the right-hand side of Eq. (28). These expressions are given in Appendix A, in which we have used the notation (22).

Finally, we arrive at the following evolution equation for the second wave packet from Eq. (28),

$$i\frac{\partial a_2}{\partial t}+i\beta_0^{(2)}\frac{\partial a_2}{\partial x}+\beta_1^{(2)}\frac{\partial^2 a_2}{\partial x^2}+i\beta_2^{(2)}\frac{\partial^3 a_2}{\partial x^3} = \Lambda_1^{(2)}a_2^2a_2^*+i\Lambda_2^{(2)}a_2a_2^*\frac{\partial a_2}{\partial x}+i\Lambda_3^{(2)}a_2^2\frac{\partial a_2^*}{\partial x} + \Lambda_4^{(2)}a_2H\left\{\frac{\partial}{\partial x}(a_2a_2^*)\right\}+\mu_1^{(2)}a_2a_1a_1^*+i\mu_2^{(2)}a_1a_1^*\frac{\partial a_2}{\partial x} + i\mu_3^{(2)}a_2a_1^*\frac{\partial a_1}{\partial x}+i\mu_4^{(2)}a_2a_1\frac{\partial a_1^*}{\partial x} + \mu_5^{(2)}a_2H\left\{\frac{\partial}{\partial x}(a_1a_1^*)\right\}, \quad (29)$$

where the coefficients are given in Appendix B. Setting $a_1=0$ and introducing appropriate dimensionless variables in Eq. (29), one can recover the evolution equation (2.20) of Hogan[7] for a single capillary-gravity wave train.

The evolution equations (26) and (29) do not remain valid when any one of the factors $1-2m_1$, $1-2m_2$, $\omega_{1-2}^2-(\omega_1-\omega_2)^2$, and $\omega_{1+2}^2-(\omega_1+\omega_2)^2$ appearing in the denominators of the coefficients vanishes. Curves on which these factors vanish in the (k_1,k_2) plane are shown in Fig. 1.

III. STABILITY OF UNIFORM WAVE TRAINS

For a uniform wave train solution of the two coupled evolution equations (26) and (29), we take

$$a_1 \equiv a_1^{(0)} = a_{o1}\exp(-it\Delta\omega_1), \quad a_2 \equiv a_2^{(0)} = a_{o2}\exp(-it\Delta\omega_2), \quad (30)$$

where a_{01} and a_{02} are two real constants. Substituting Eq. (30) in Eqs. (26) and (29), we get

$$\Delta\omega_1 = \Lambda_1^{(1)}a_{01}^2+\mu_1^{(1)}a_{02}^2, \quad \Delta\omega_2 = \Lambda_1^{(2)}a_{02}^2+\mu_1^{(2)}a_{01}^2, \quad (31)$$

which are the amplitude-dependent frequency shifts of the two waves.

Since the dimensionless wave numbers of the first and second wave are, respectively, k_1/k_2 and 1, the amplitude-dependent shift in phase speeds Δc_1 and Δc_2 of the two waves is given by

$$\Delta c_1 = \frac{\Delta\omega_1}{k_1/k_2} = \kappa(\Lambda_1^{(1)}a_{01}^2+\mu_1^{(1)}a_{02}^2), \quad \Delta c_2 = \Delta\omega_2 = \Lambda_1^{(2)}a_{02}^2+\mu_1^{(2)}a_{01}^2. \quad (32)$$

In the absence of nonlinearity, the phase speeds of the first and second waves are, respectively, $k_2\omega_1/k_1\omega_2$ and 1. Hence, the relative change in phase speed Δc_a of the first wave due to the presence of the second wave is

$$\Delta c_a = \kappa^{1/2}\left[\frac{1+m_2}{1+m_1}\right]^{1/2}\mu_1^{(1)}a_{02}^2. \quad (33)$$

A similar expression for the relative change in phase speed Δc_b of the second wave due to the presence of the first wave is

$$\Delta c_b = \mu_1^{(2)} a_{01}^2. \tag{34}$$

Variations of Δc_a against a_{02} and Δc_b against a_{01} given, respectively, by Eqs. (33) and (34) are shown in Figs. 2(a) and 2(b) for four sets of values of (k_1, k_2).

To make a stability analysis of the two uniform wave trains given by Eq. (30), we set

$$a_1 = a_1^{(0)}[1 + \hat{a}_1(\xi, t)],$$
$$a_2 = a_2^{(0)}[1 + \hat{a}_2(\xi, t)], \tag{35}$$

where $\hat{a}_1$ and $\hat{a}_2$ are complex.

Substituting Eq. (35) in the two evolution equations (26) and (29) and then linearizing with respect to $\hat{a}_1$ and $\hat{a}_2$, the following two equations are obtained:

$$i\frac{\partial \hat{a}_1}{\partial t} + i\beta_0^{(1)}\frac{\partial \hat{a}_1}{\partial x} + \beta_1^{(1)}\frac{\partial^2 \hat{a}_1}{\partial x^2} + i\beta_2^{(1)}\frac{\partial^3 \hat{a}_1}{\partial x^3} = \Lambda_1^{(1)} a_{01}^2(\hat{a}_1 + \hat{a}_1^*) + i\Lambda_2^{(1)} a_{01}^2 \frac{\partial \hat{a}_1}{\partial x} + i\Lambda_3^{(1)} a_{01}^2 \frac{\partial \hat{a}_1^*}{\partial x} + \Lambda_4^{(1)} a_{01}^2 H\left\{\frac{\partial \hat{a}_1}{\partial x} + \frac{\partial \hat{a}_1^*}{\partial x}\right\}$$
$$+ \mu_1^{(1)} a_{02}^2(\hat{a}_2 + \hat{a}_2^*) + i\mu_2^{(1)} a_{02}^2 \frac{\partial \hat{a}_1}{\partial x} + i\mu_3^{(1)} a_{02}^2 \frac{\partial \hat{a}_2}{\partial x} + i\mu_4^{(1)} a_{02}^2 \frac{\partial \hat{a}_2^*}{\partial x} + \mu_5^{(1)} a_{02}^2 H\left\{\frac{\partial \hat{a}_2}{\partial x} + \frac{\partial \hat{a}_2^*}{\partial x}\right\}, \tag{36}$$

$$i\frac{\partial \hat{a}_2}{\partial t} + i\beta_0^{(2)}\frac{\partial \hat{a}_2}{\partial x} + \beta_1^{(2)}\frac{\partial^2 \hat{a}_2}{\partial x^2} + i\beta_2^{(2)}\frac{\partial^3 \hat{a}_2}{\partial x^3} = \Lambda_1^{(2)} a_{02}^2(\hat{a}_2 + \hat{a}_2^*) + i\Lambda_2^{(2)} a_{02}^2 \frac{\partial \hat{a}_2}{\partial x} + i\Lambda_3^{(2)} a_{02}^2 \frac{\partial \hat{a}_2^*}{\partial x} + \Lambda_4^{(2)} a_{02}^2 H\left\{\frac{\partial \hat{a}_2}{\partial x} + \frac{\partial \hat{a}_2^*}{\partial x}\right\}$$
$$+ \mu_1^{(2)} a_{01}^2(\hat{a}_1 + \hat{a}_1^*) + i\mu_2^{(2)} a_{01}^2 \frac{\partial \hat{a}_2}{\partial x} + i\mu_3^{(2)} a_{01}^2 \frac{\partial \hat{a}_1}{\partial x} + i\mu_4^{(2)} a_{01}^2 \frac{\partial \hat{a}_1^*}{\partial x} + \mu_5^{(2)} a_{01}^2 H\left\{\frac{\partial \hat{a}_1}{\partial x} + \frac{\partial \hat{a}_1^*}{\partial x}\right\}. \tag{37}$$

Now setting $\hat{a}_1 = a_r^{(1)} + i a_i^{(1)}$ and $\hat{a}_2 = a_r^{(2)} + i a_i^{(2)}$ in the two equations (36) and (37), where $a_r^{(1)}$, $a_i^{(1)}$, $a_r^{(2)}$, and $a_i^{(2)}$ are real, and then equating real and imaginary parts on both sides of each equation, we get four coupled linear equations for $a_r^{(1)}$, $a_i^{(1)}$, $a_r^{(2)}$, and $a_i^{(2)}$. Next assuming that the space-time dependence of $a_r^{(1)}$, $a_i^{(1)}$, $a_r^{(2)}$, and $a_i^{(2)}$ is of the form $\exp i(\lambda x - \Omega t)$, we get the following four coupled equations:

$$R_1 a_r^{(1)} + S_1 a_r^{(2)} - i(\Omega - P_1^{(+)}) a_i^{(1)} - i Q_1^{(-)} a_i^{(2)} = 0, \tag{38}$$

$$i(\Omega - P_1^{(-)}) a_r^{(1)} + i Q_1^{(+)} a_r^{(2)} + U_1 a_i^{(1)} = 0, \tag{39}$$

$$S_2 a_r^{(1)} + R_2 a_r^{(2)} - i(\Omega - P_2^{(+)}) a_i^{(2)} - i Q_2^{(-)} a_i^{(1)} = 0, \tag{40}$$

$$i Q_2^{(+)} a_r^{(1)} + i(\Omega - P_2^{(-)}) a_r^{(2)} + U_2 a_i^{(2)} = 0. \tag{41}$$

In the last four equations, $P_1^{(\pm)}$, $Q_1^{(\pm)}$, R_1, S_1, U_1, and $P_2^{(\pm)}$, $Q_2^{(\pm)}$, R_2, S_2, U_2 are given by

$$P_1^{(\pm)} = \beta_0^{(1)}\lambda - \beta_2^{(1)}\lambda^3 - \Lambda_2^{(1)}\lambda a_{01}^2 \pm \Lambda_3^{(1)}\lambda a_{01}^2 - \mu_2^{(1)}\lambda a_{02}^2,$$

$$Q_1^{(\pm)} = (\mu_3^{(1)} \pm \mu_4^{(1)})\lambda a_{02}^2,$$

$$R_1 = \beta_1^{(1)}\lambda^2 + 2\Lambda_1^{(1)} a_{01}^2 - 2\Lambda_4^{(1)}|\lambda| a_{01}^2,$$

$$S_1 = 2(\mu_1^{(1)} - \mu_5^{(1)}|\lambda|) a_{02}^2,$$

$$U_1 = \beta_1^{(1)}\lambda^2,$$

$$P_2^{(\pm)} = \beta_0^{(2)}\lambda - \beta_2^{(2)}\lambda^3 - \Lambda_2^{(2)}\lambda a_{02}^2 \pm \Lambda_3^{(2)}\lambda a_{02}^2 - \mu_2^{(2)}\lambda a_{01}^2, \tag{42}$$

$$Q_2^{(\pm)} = (\mu_3^{(2)} \pm \mu_4^{(2)})\lambda a_{01}^2,$$

$$R_2 = \beta_1^{(2)}\lambda^2 + 2\Lambda_1^{(2)} a_{02}^2 - 2\Lambda_4^{(2)}|\lambda| a_{02}^2,$$

$$S_2 = 2(\mu_1^{(2)} - \mu_5^{(2)}|\lambda|) a_{01}^2,$$

$$U_2 = \beta_1^{(2)}\lambda^2.$$

For nontrivial solution of the above system of equations (38)–(41), we get the following condition:

$$(\Omega - P_1^{(+)})(\Omega - P_1^{(-)})(\Omega - P_2^{(+)})(\Omega - P_2^{(-)}) - R_2 U_2(\Omega - P_1^{(+)})(\Omega - P_1^{(-)}) - R_1 U_1(\Omega - P_2^{(+)})(\Omega - P_2^{(-)}) - Q_1^{(+)} Q_2^{(+)}(\Omega - P_1^{(+)})$$
$$\times(\Omega - P_2^{(+)}) - Q_1^{(-)} Q_2^{(-)}(\Omega - P_1^{(-)})(\Omega - P_2^{(-)}) + S_2 U_2 Q_1^{(+)}(\Omega - P_1^{(+)}) + S_1 U_2 Q_2^{(-)}(\Omega - P_1^{(-)}) + S_1 U_1 Q_2^{(-)}(\Omega - P_2^{(+)})$$
$$+ S_2 U_1 Q_1^{(-)}(\Omega - P_2^{(-)}) + R_1 R_2 U_1 U_2 - U_1 U_2 S_1 S_2 - R_1 U_2 Q_1^{(+)} Q_2^{(-)} - R_2 U_1 Q_1^{(-)} Q_2^{(+)} + Q_1^{(+)} Q_1^{(-)} Q_2^{(+)} Q_2^{(-)} = 0. \tag{43}$$

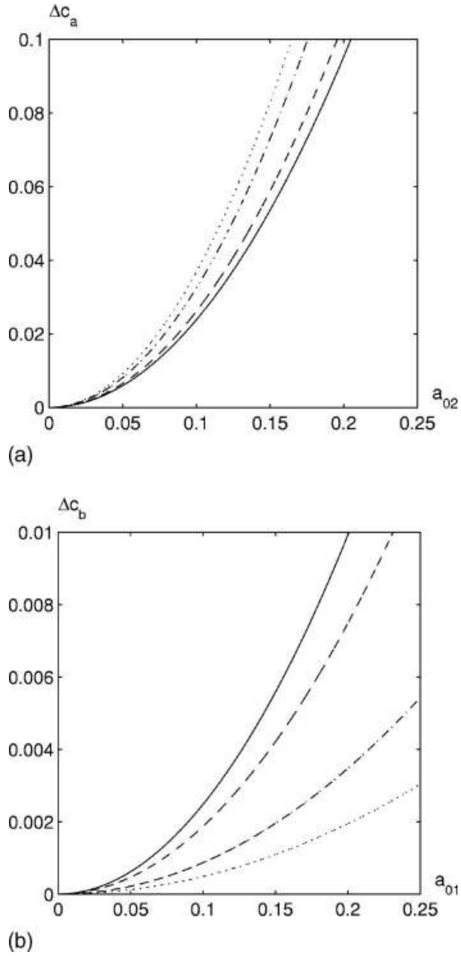

FIG. 2. Relative change in phase velocity of first wave train: (——) $(k_1,k_2)=(2.70005,0.71323)$, (- - - -) $(k_1,k_2)=(3.00078,0.62890)$, (-.-.-.-) $(k_1,k_2)=(4.00176,0.44390)$, (……) $(k_1,k_2)=(5.00010,0.33944)$. Relative change in phase velocity of the second wave train: (——) $(k_1,k_2)=(2.70005,0.71323)$, (- - - -) $(k_1,k_2)=(3.00078,0.62890)$, (-.-.-.-) $(k_1,k_2)=(4.00176,0.44390)$, (……) $(k_1,k_2)=(5.00010,0.33944)$.

Neglecting higher-order terms, the above dispersion relation can be written as follows:

$$[(\Omega-P_1)^2-R_1U_1][(\Omega-P_2)^2-R_2U_2] = L(\Omega-P_1)(\Omega-P_2)-M_1(\Omega-P_1) - M_2(\Omega-P_2)+N, \quad (44)$$

where

$$P_1=\beta_0^{(1)}\lambda-\beta_2^{(1)}\lambda^3-\Lambda_2^{(1)}\lambda a_{01}^2-\mu_2^{(1)}\lambda a_{02}^2,$$

$$P_2=\beta_0^{(2)}\lambda-\beta_2^{(2)}\lambda^3-\Lambda_2^{(2)}\lambda a_{02}^2-\mu_2^{(2)}\lambda a_{01}^2,$$

$$L=2(\mu_3^{(1)}\mu_3^{(2)}+\mu_4^{(1)}\mu_4^{(2)})\lambda^2 a_{01}^2 a_{02}^2,$$

$$M_1=2\beta_1^{(2)}[\mu_1^{(2)}(\mu_3^{(1)}+\mu_4^{(1)})+\mu_1^{(1)}(\mu_3^{(2)}-\mu_4^{(2)})]\lambda^3 a_{01}^2 a_{02}^2, \quad (45)$$

$$M_2=2\beta_1^{(1)}[\mu_1^{(1)}(\mu_3^{(2)}+\mu_4^{(2)})+\mu_1^{(2)}(\mu_3^{(1)}-\mu_4^{(1)})]\lambda^3 a_{01}^2 a_{02}^2,$$

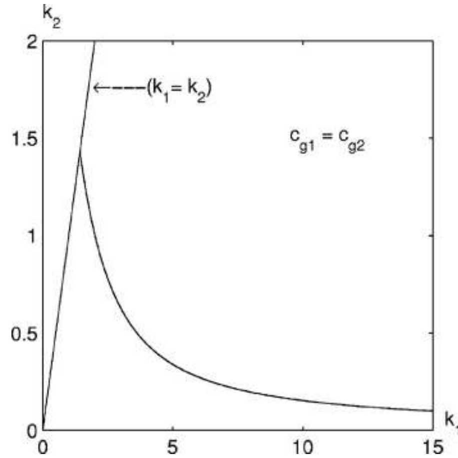

FIG. 3. Curve on which group velocities of the two wave trains are equal in the $k_1>k_2$ plane.

$$N=4\beta_1^{(1)}\beta_1^{(2)}[\mu_1^{(1)}\mu_1^{(2)}-(\mu_1^{(1)}\mu_5^{(2)}+\mu_1^{(2)}\mu_5^{(1)})\lambda]\lambda^4 a_{01}^2 a_{02}^2.$$

We now restrict our stability analysis to the case of nearly equal group velocities of the two waves and to the investigation of stability of the second wave in the presence of the first wave. The curve on which the group velocities c_{g1} and c_{g2} of the two waves become equal is shown in Fig. 3.

Let ϵ be a small parameter ordering the smallness of a_{01}, a_{02}, and λ. If we assume nearly equal group velocity of the two waves, i.e., $\beta_0^{(1)}\approx\beta_0^{(2)}$, then it can be shown easily from the two evolution equations that

$$\Omega-\beta_0^{(1)}\lambda=O(\epsilon^2) \quad \text{and} \quad \Omega-\beta_0^{(2)}\lambda=O(\epsilon^2).$$

In view of this ordering, the dispersion relation (44) can be put in the following form keeping terms only up to $O(\epsilon^9)$:

$$[(\Omega-P_1)^2-R_1U_1][(\Omega-P_2)^2-R_2U_2]+M_1(\Omega-P_1) + M_2(\Omega-P_2)-N=0. \quad (46)$$

At lowest order, the solution of the dispersion relation (46) is given by

$$\Omega^{(1)}=\beta_0^{(1)}\lambda\pm\{\beta_1^{(1)}\lambda^2(\beta_1^{(1)}\lambda^2+2\Lambda_1^{(1)}a_{01}^2)\}^{1/2},$$

$$\Omega^{(2)}=\beta_0^{(2)}\lambda\pm\{\beta_1^{(2)}\lambda^2(\beta_1^{(2)}\lambda^2+2\Lambda_1^{(2)}a_{02}^2)\}^{1/2}. \quad (47)$$

At fourth order, the dispersion relation (46) can be solved for the second wave train as follows:

$$\left[\Omega-P_2+\frac{M_2}{2\{(\Omega^{(2)}-P_1)^2-R_1U_1\}}\right]^2 = R_2U_2+\frac{N-M_1(\Omega^{(2)}-\beta_0^{(1)}\lambda)}{(\Omega^{(2)}-P_1)^2-R_1U_1}. \quad (48)$$

From Eq. (48) we conclude that the condition of instability of the second wave train in the presence of the first wave train is

$$R_2U_2+\frac{N-M_1(\Omega^{(2)}-\beta_0^{(1)}\lambda)}{(\Omega^{(2)}-P_1)^2-R_1U_1}<0. \quad (49)$$

In the absence of the first wave train, the instability condition (49) becomes

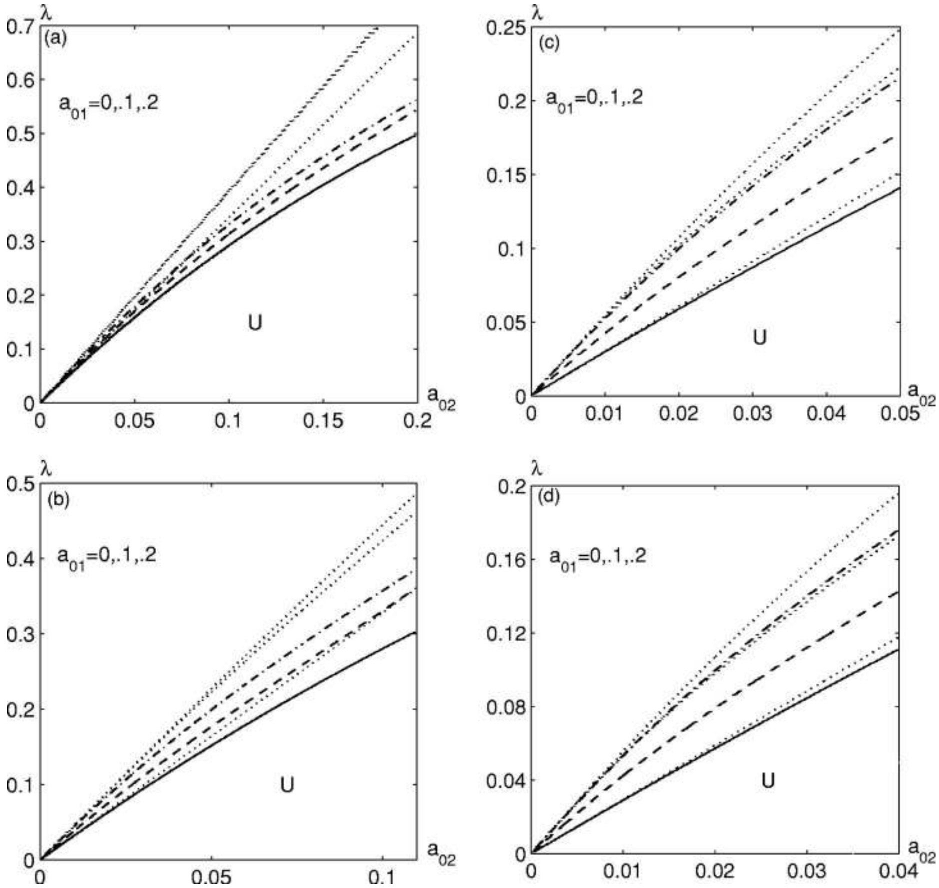

FIG. 4. Stable-unstable regions of the second wave train; (——) $a_{01}=0$, (- - - -) $a_{01}=0.1$, (-.-.-.-) $a_{01}=0.2$; (……) corresponding curves for third-order evolution equations; U denotes the unstable region; $(k_1,k_2)=(2.70005,0.71323)$ in (a), $(k_1,k_2)=(3.00078,0.62890)$ in (b), $(k_1,k_2)=(4.00176,0.44390)$ in (c), $(k_1,k_2)=(5.00010,0.33944)$ in (d).

$R_2U_2<0$,

that is,

$$\beta_1^{(2)}\lambda^2(\beta_1^{(2)}\lambda^2+2\Lambda_1^{(2)}a_{02}^2-2\Lambda_4^{(2)}|\lambda|a_{02}^2)<0, \qquad (50)$$

which looks similar to the condition of instability of a single wave packet. In fact, the instability condition (50) is identical with the instability condition obtained from Eq. (11) of Debsarma and Das[19] in the absence of thermocline and for the case in which the angle $\theta=0$. Also in the absence of capillarity, the instability condition (50) gives the instability condition (3.8) of Dysthe.[3] Stable-unstable regions of the second wave train in the presence of the first wave train are shown in Figs. 4(a)–4(d) for four different sets of values of wave numbers: $(k_1,k_2)=(2.70005,0.71323)$ in (a), $(k_1,k_2)=(3.00078,0.62890)$ in (b), $(k_1,k_2)=(4.00176,0.44390)$ in (c), and $(k_1,k_2)=(5.00010,0.33944)$ in (d). In all these figures, we have plotted marginal stability curves for the second wave train of smaller wave numbers assuming an amplitude of the first wave train of greater wave numbers to be $a_{01}=0.1$ and $a_{01}=0.2$. We have also plotted the marginal stability curve of the second wave train in the absence of the first wave train in all these figures. It is found that the instabilityregion of the second wave train expands due to the presence of the first wave train. Figures 4(a)–4(d) reveal that the instability region of the second wave train of longer wavelength expands with the increase of the amplitude of the first wave train of shorter wavelength. In each of these figures, graphs have also been included neglecting fourth-order terms to see the role of those terms on instability. We observe that the instability region is shortened slightly by inclusion of fourth-order terms.

In the case of instability, the growth rate of instability of the wave train of longer wavelength is given by

$$G=\left[-R_2U_2-\frac{N-M_1(\Omega^{(2)}-\beta_0^{(1)}\lambda)}{(\Omega^{(2)}-P_1)^2-R_1U_1}\right]^{1/2}. \qquad (51)$$

We have plotted the growth rate of instability of the second wave train against the perturbation wave number for different values of the amplitude of the first wave train for two fixed values of the amplitude of the second wave train, namely $a_{02}=0.05$ and 0.1. These curves are shown in Figs. 5(a) and 5(b) and 6(a) and 6(b) for the wave numbers $(k_1,k_2)=(3.00078,0.62890)$ and $(k_1,k_2)=(5.00010,0.33944)$, respectively. In these figures, we have also included the growth rate curves for the second wave train in the absence of the first wave train. From these figures, it is clear that the growth rate of instability of the second wave train increases due to the presence of the first wave train and it increases with the increase of the amplitude of the first wave train. In Figs. 5(a) and 5(b) and 6(a) and 6(b), we have also plotted the corresponding curves that can be obtained from third-order evolution equations. We observe that the influence of fourth-order terms is to increase the growth rate of instability.

For the wave numbers that we have chosen here to draw Figs. 4(a)–4(d), 5(a) and 5(b), and 6(a) and 6(b), the first and

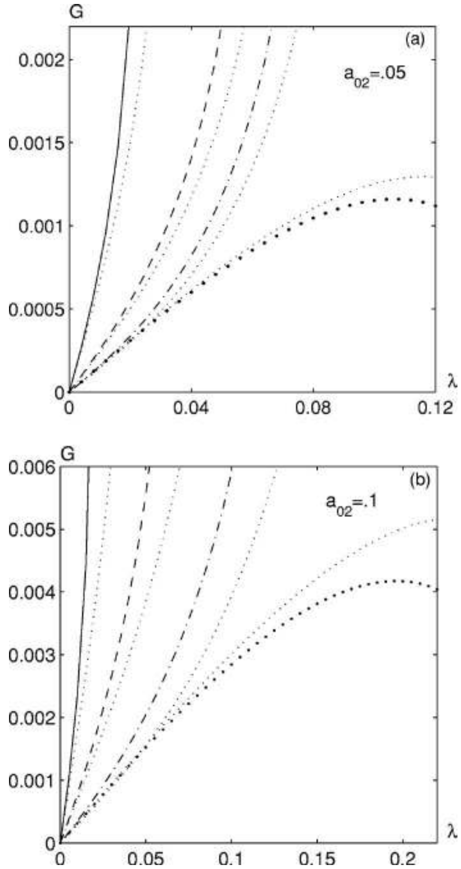

FIG. 5. Growth rate of instability of the second wave train against the perturbation wave number when $(k_1,k_2)=(3.00078,0.62890)$; in (a) $a_{02}=0.05$ and (* * * *) $a_{01}=0$, (-.-.-.-) $a_{01}=0.035$, (- - - -) $a_{01}=0.038$, (——) $a_{01}=0.042$; in (b) $a_{02}=0.10$ and (* * * *) $a_{01}=0$, (-.-.-.-) $a_{01}=0.07$, (- - - -) $a_{01}=0.08$, (——) $a_{01}=0.085$; (……) corresponding curves for third-order evolution equations.

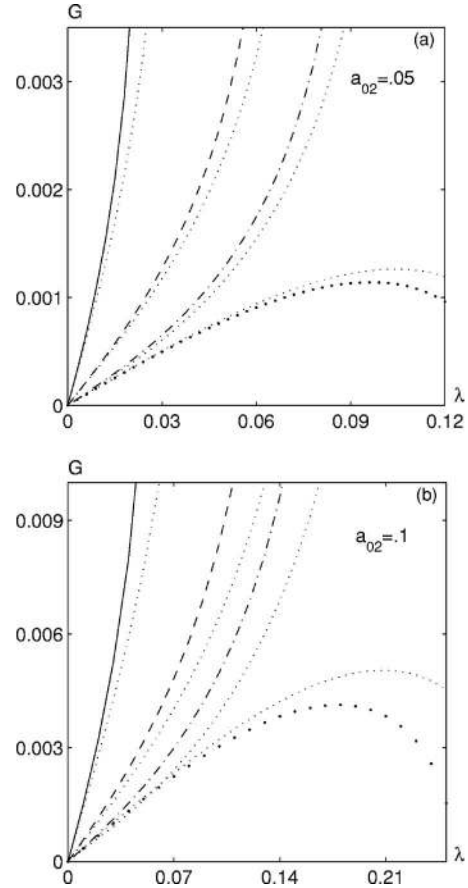

FIG. 6. Growth rate of instability of the second wave train against the perturbation wave number when $(k_1,k_2)=(5.00010,0.33944)$; in (a) $a_{02}=0.05$ and (* * * *) $a_{01}=0$, (-.-.-.-) $a_{01}=0.09$, (- - - -) $a_{01}=0.11$, (——) $a_{01}=0.13$; in (b) $a_{02}=0.10$ and (* * * *) $a_{01}=0$, (-.-.-.-)$a_{01}=0.17$, (- - - -) $a_{01}=0.20$, (——) $a_{01}=0.25$; (……) corresponding curves for third-order evolution equations.

second waves fall in the categories of capillary-gravity wave and gravity wave, respectively. Consequently, these figures show stable-unstable regions and growth rate of instabilities of a surface-gravity wave train in the presence of a capillary-gravity wave train.

IV. CONCLUSIONS

Considering the importance of stability analysis of uniform wave trains made from fourth-order nonlinear evolution equations, we have derived in this paper two coupled fourth-order nonlinear evolution equations for the evolution of the amplitudes of two capillary-gravity wave packets propagating in the same direction. For the derivation of evolution equations, we have made use of the Zakharov integral equation. The two evolution equations are used to investigate the stability properties of a uniform surface gravity wave train in the presence of another capillary-gravity wave train having the same group velocity. The condition of instability of a wave of greater wavelength in the presence of a wave of shorter wavelength is obtained. Instability regions are shown in figures for four sets of values of wave numbers. It is seen that the instability regions for a surface gravity wave train in the presence of a capillary-gravity wave train expand with the increase of wave steepness of the capillary-gravity wave train. The growth rate of instability of a uniform wave train with smaller wave number has been plotted against the perturbation wave number for different values of the amplitude of the wave train of greater wave number. It is found that the presence of a wave train of smaller wavelength increases the growth rate of instability of a uniform wave train of larger wavelength. Also with the increase of the amplitude of the wave train of smaller wavelength, the growth rate of instability of a uniform wave train of larger wavelength increases. In the figures for the stable-unstable region and for instability growth rates, we also find a comparison between third-order results and fourth-order results. We find that at fourth order, instability regions get reduced slightly and instability growth rates are increased slightly.

ACKNOWLEDGMENTS

The authors are grateful to the referees for their valuable suggestions for improvement of the earlier version of this paper.

APPENDIX A: EXPRESSIONS FOR KERNELS

$$T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_2+\vec{\chi}_1,\vec{k}_2+\vec{\chi}_2,\vec{k}_1+\vec{\chi}_3)$$
$$=\frac{k_1k_2^2}{8\pi^2}\left[1+\frac{1}{2}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_1}+\frac{\chi_1+\chi_2}{k_2}-\frac{|\chi_1-\chi_2|}{k_2}\right]-\frac{3k_1^3r}{32\pi^2}\cdot\frac{m_2}{1+m_2}\left[1+\frac{5+3m_1}{4(1+m_1)}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_1}\right.$$
$$\left.+\frac{5+3m_2}{4(1+m_2)}\cdot\frac{\chi_1+\chi_2}{k_2}\right]+\frac{k_1k_2(k_1-k_2)r}{16\pi^2}\left[1+\frac{\chi_3-\chi_1}{k_1-k_2}+\frac{1-m_1}{4(1+m_1)}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_1}+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_1+\chi_2}{k_2}\right]$$
$$-\frac{k_1k_2(k_1-k_2)r_1}{8\pi^2}\left[1+\frac{1-m_1}{4(1+m_1)}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_1}+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_1+\chi_2}{k_2}+\frac{2(1+2m_{1-2})}{1+m_{1-2}}\cdot\frac{\chi_3-\chi_1}{k_1-k_2}\right.$$
$$\left.-\frac{d_1^{(-)}\chi_3-d_2^{(-)}\chi_1-d_3^{(-)}(\chi_2-\chi_1)}{k_1-k_2}\right]-\frac{k_1k_2(k_1+k_2)r_3}{8\pi^2}\left[1+\frac{\chi_2+\chi_3}{k_1+k_2}+\frac{3+5m_1}{4(1+m_1)}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_1}\right.$$
$$\left.+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_1+\chi_2}{k_2}-\frac{d_1^{(+)}\chi_3+d_2^{(+)}\chi_2-d_3^{(+)}(\chi_2-\chi_1)}{k_1+k_2}\right],$$

$$T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_2+\vec{\chi}_1,\vec{k}_1+\vec{\chi}_2,\vec{k}_2+\vec{\chi}_3)$$
$$=\frac{k_1k_2^2}{8\pi^2}\left[1+\frac{1}{2}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_1}+\frac{\chi_1+\chi_3}{k_2}-\frac{|\chi_1-\chi_3|}{k_2}\right]-\frac{3k_1^3r}{32\pi^2}\cdot\frac{m_2}{1+m_2}\left[1+\frac{5+3m_1}{4(1+m_1)}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_1}\right.$$
$$\left.+\frac{5+3m_2}{4(1+m_2)}\cdot\frac{\chi_1+\chi_3}{k_2}\right]+\frac{k_1k_2(k_1-k_2)r}{16\pi^2}\left[1+\frac{\chi_2-\chi_1}{k_1-k_2}+\frac{1-m_1}{4(1+m_1)}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_1}+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_1+\chi_3}{k_2}\right]$$
$$-\frac{k_1k_2(k_1-k_2)r_1}{8\pi^2}\left[1+\frac{1-m_1}{4(1+m_1)}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_1}+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_1+\chi_3}{k_2}+\frac{2(1+2m_{1-2})}{1+m_{1-2}}\cdot\frac{\chi_2-\chi_1}{k_1-k_2}\right.$$
$$\left.-\frac{d_1^{(-)}\chi_2-d_2^{(-)}\chi_1-d_3^{(-)}(\chi_3-\chi_1)}{k_1-k_2}\right]-\frac{k_1k_2(k_1+k_2)r_3}{8\pi^2}\left[1+\frac{\chi_2+\chi_3}{k_1+k_2}+\frac{3+5m_1}{4(1+m_1)}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_1}\right.$$
$$\left.+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_1+\chi_3}{k_2}-\frac{d_1^{(+)}\chi_2+d_2^{(+)}\chi_3-d_3^{(+)}(\chi_3-\chi_1)}{k_1+k_2}\right],$$

$$T(\vec{k}_1+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_1+\vec{\chi}_1,\vec{k}_1+\vec{\chi}_2,\vec{k}_1+\vec{\chi}_3)$$
$$=\frac{k_1^3}{8\pi^2}\left[1+\frac{3}{2}\cdot\frac{\chi_2+\chi_3}{k_1}-\frac{|\chi_3-\chi_1|}{k_1}-\frac{|\chi_2-\chi_1|}{k_1}\right]-\frac{3k_1^3}{32\pi^2}\cdot\frac{m_1}{1+m_1}\left[1+\frac{5+3m_1}{2(1+m_1)}\cdot\frac{\chi_2+\chi_3}{k_1}\right]$$
$$+\frac{k_1^3}{8\pi^2}\cdot\frac{1+m_1}{1-2m_1}\left[1+\frac{3+15m_1+6m_1^2-24m_1^3}{2(1+m_1)(1+4m_1)(1-2m_1)}\cdot\frac{\chi_2+\chi_3}{k_1}\right],$$

$$T(\vec{k}_2+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_1+\vec{\chi}_1,\vec{k}_1+\vec{\chi}_2,\vec{k}_2+\vec{\chi}_3)$$
$$=\frac{k_1k_2^2}{8\pi^2}\left[1+\frac{\chi_2+2\chi_3-\chi_1}{k_2}+\frac{1}{2}\cdot\frac{\chi_1+\chi_2}{k_1}-\frac{|\chi_1-\chi_2|}{k_2}\right]-\frac{3k_1^3r}{32\pi^2}\cdot\frac{m_2}{1+m_2}\left[1+\frac{5+3m_2}{4(1+m_2)}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_2}\right.$$
$$\left.+\frac{5+3m_1}{4(1+m_1)}\cdot\frac{\chi_1+\chi_2}{k_1}\right]+\frac{k_1k_2(k_1-k_2)r}{16\pi^2}\left[1+\frac{\chi_1-\chi_3}{k_1-k_2}+\frac{1-m_1}{4(1+m_1)}\cdot\frac{\chi_1+\chi_2}{k_1}+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_2}\right]$$
$$-\frac{k_1k_2(k_1-k_2)r_1}{8\pi^2}\left[1+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_2}+\frac{1-m_1}{4(1+m_1)}\cdot\frac{\chi_1+\chi_2}{k_1}+\frac{2(1+2m_{1-2})}{1+m_{1-2}}\cdot\frac{\chi_1-\chi_3}{k_1-k_2}\right.$$
$$\left.-\frac{d_1^{(-)}\chi_1-d_2^{(-)}\chi_3-d_3^{(-)}(\chi_2-\chi_1)}{k_1-k_2}\right]-\frac{k_1k_2(k_1+k_2)r_3}{8\pi^2}\left[1+\frac{\chi_2+\chi_3}{k_1+k_2}+\frac{3+5m_1}{4(1+m_1)}\cdot\frac{\chi_1+\chi_2}{k_1}\right.$$
$$\left.+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_2+2\chi_3-\chi_1}{k_2}-\frac{d_1^{(+)}\chi_2+d_2^{(+)}\chi_3+d_3^{(+)}(\chi_2-\chi_1)}{k_1+k_2}\right],$$

$$T(\vec{k}_2+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_1+\vec{\chi}_1,\vec{k}_2+\vec{\chi}_2,\vec{k}_1+\vec{\chi}_3)$$
$$=\frac{k_1k_2^2}{8\pi^2}\left[1+\frac{\chi_3+2\chi_2-\chi_1}{k_2}+\frac{1}{2}\cdot\frac{\chi_1+\chi_3}{k_1}-\frac{|\chi_1-\chi_3|}{k_2}\right]-\frac{3k_1^3r}{32\pi^2}\cdot\frac{m_2}{1+m_2}\left[1+\frac{5+3m_2}{4(1+m_2)}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_2}\right.$$
$$\left.+\frac{5+3m_1}{4(1+m_1)}\cdot\frac{\chi_3+\chi_1}{k_1}\right]+\frac{k_1k_2(k_1-k_2)r}{16\pi^2}\left[1+\frac{\chi_1-\chi_2}{k_1-k_2}+\frac{1-m_1}{4(1+m_1)}\cdot\frac{\chi_3+\chi_1}{k_1}+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_2}\right]$$
$$-\frac{k_1k_2(k_1-k_2)r_1}{8\pi^2}\left[1+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_2}+\frac{1-m_1}{4(1+m_1)}\cdot\frac{\chi_3+\chi_1}{k_1}+\frac{2(1+2m_{1-2})}{1+m_{1-2}}\cdot\frac{\chi_1-\chi_2}{k_1-k_2}\right.$$
$$\left.-\frac{d_1^{(-)}\chi_1-d_2^{(-)}\chi_2-d_3^{(-)}(\chi_3-\chi_1)}{k_1-k_2}\right]-\frac{k_1k_2(k_1+k_2)r_3}{8\pi^2}\left[1+\frac{\chi_2+\chi_3}{k_1+k_2}+\frac{3+5m_1}{4(1+m_1)}\cdot\frac{\chi_3+\chi_1}{k_1}\right.$$
$$\left.+\frac{3+5m_2}{4(1+m_2)}\cdot\frac{\chi_3+2\chi_2-\chi_1}{k_2}-\frac{d_1^{(+)}\chi_3+d_2^{(+)}\chi_2+d_3^{(+)}(\chi_3-\chi_1)}{k_1+k_2}\right]$$

$$T(\vec{k}_2+\vec{\chi}_2+\vec{\chi}_3-\vec{\chi}_1,\vec{k}_2+\vec{\chi}_1,\vec{k}_2+\vec{\chi}_2,\vec{k}_2+\vec{\chi}_3)$$
$$=\frac{k_2^3}{8\pi^2}\left[1+\frac{3}{2}\cdot\frac{\chi_2+\chi_3}{k_2}-\frac{|\chi_3-\chi_1|}{k_2}-\frac{|\chi_2-\chi_1|}{k_2}\right]-\frac{3k_2^3}{32\pi^2}\cdot\frac{m_2}{1+m_2}\left[1+\frac{5+3m_2}{2(1+m_2)}\cdot\frac{\chi_2+\chi_3}{k_2}\right]$$
$$+\frac{k_2^3}{8\pi^2}\cdot\frac{1+m_2}{1-2m_2}\left[1+\frac{3+15m_2+6m_2^2-24m_2^3}{2(1+m_2)(1+4m_2)(1-2m_2)}\cdot\frac{\chi_2+\chi_3}{k_2}\right].$$

APPENDIX B: COEFFICIENTS OF EVOLUTION EQUATIONS

Coefficients of the first evolution equation,

$$\beta_0^{(1)}=\frac{\kappa}{r}\cdot\frac{1+3m_1}{2(1+m_1)},$$

$$\beta_1^{(1)}=-\frac{\kappa^2}{r}\cdot\frac{1-6m_1-3m_1^2}{8(1+m_1)^2},$$

$$\beta_2^{(1)}=-\frac{\kappa^3}{r}\cdot\frac{1+5m_1-5m_1^2-m_1^3}{16(1+m_1)^3},$$

$$\Lambda_1^{(1)}=\frac{1}{r}\cdot\frac{8+m_1+2m_1^2}{16(1+m_1)(1-2m_1)},$$

$$\Lambda_2^{(1)}=-\frac{\kappa}{r}\cdot\frac{3(8-m_1+9m_1^2-4m_1^3-4m_1^4)}{16(1+m_1)^2(1-2m_1)^2},$$

$$\Lambda_3^{(1)}=-\frac{\kappa}{r}\cdot\frac{(1-m_1)(8+m_1+2m_1^2)}{32(1+m_1)^2(1-2m_1)},$$

$$\Lambda_4^{(1)}=\frac{\kappa}{r}\cdot\frac{1}{2},$$

$$\mu_1^{(1)}=\frac{1}{\kappa^3}\left[\frac{1}{2}\kappa^2-\frac{3}{8}rs_2-\frac{1}{4}\kappa(1-\kappa)r_2-\frac{1}{2}\kappa(1+\kappa)r_3\right],$$

$$\mu_2^{(1)}=\frac{1}{\kappa^2}\left[\kappa r_1(1+s_{1-2})+\frac{1}{8}\kappa(1-\kappa)r_2(1-2s_1)\right.$$
$$+\frac{1}{4}\kappa(1+\kappa)r_3(3+2s_1)+\frac{1}{2}\kappa r_3+\frac{3}{16}rs_2(5-2s_1)$$
$$\left.-\frac{1}{4}\kappa r-\frac{1}{2}\kappa^2-\frac{1}{2}\kappa r_1d_1^{(-)}-\frac{1}{2}\kappa r_3d_1^{(+)}\right],$$

$$\mu_3^{(1)}=\frac{1}{\kappa^2}\left[\frac{1}{8}\kappa(1-\kappa)r_2(1-2s_1)-\frac{1}{8}\kappa^2(3-2s_1)\right.$$
$$+\frac{3}{16}rs_2(3-2s_1)+\frac{1}{8}(1-\kappa)r_2(1+2s_2)$$
$$+\frac{1}{4}(1+\kappa)r_3(1+2s_2)-\frac{1}{8}\kappa(3+2s_2)+\frac{1}{2}\kappa(1+\kappa)r_3$$
$$+\frac{3}{8\kappa}rs_2+\frac{1}{2}\kappa r_3-\frac{1}{2}\kappa r_3d_2^{(+)}+\frac{1}{2}\kappa r_1d_3^{(-)}$$
$$\left.+\frac{1}{2}\kappa r_3d_3^{(+)}\right],$$

$$\mu_4^{(1)}=\frac{1}{\kappa^2}\left[\frac{1}{8}\kappa(1-\kappa)r_2(1-2s_1)\right.$$
$$-\frac{1}{8}\kappa^2(3-2s_1)+\frac{3}{16}rs_2(3-2s_1)+\frac{1}{8}\kappa(3+2s_2)$$
$$-\frac{1}{8}(1-\kappa)r_2(1+2s_2)-\frac{1}{4}(1+\kappa)r_3(1+2s_2)$$

$$+\kappa r_1(1+s_{1-2})+\frac{1}{2}\kappa(1+\kappa)r_3-\frac{3}{8\kappa}rs_2-\frac{1}{4}\kappa r$$
$$-\frac{1}{2}\kappa r_1 d_2^{(-)}+\frac{1}{2}\kappa r_1 d_3^{(-)}+\frac{1}{2}\kappa r_3 d_3^{(+)}\bigg],$$

$$\mu_5^{(1)}=\frac{1}{2\kappa}.$$

Coefficients of the second evolution equation,

$$\beta_0^{(2)}=\frac{1+3m_2}{2(1+m_2)},$$

$$\beta_1^{(2)}=-\frac{1-6m_2-3m_2^2}{8(1+m_2)^2},$$

$$\beta_2^{(2)}=-\frac{1+5m_2-5m_2^2-m_2^3}{16(1+m_2)^3},$$

$$\Lambda_1^{(2)}=\frac{8+m_2+2m_2^2}{16(1+m_2)(1-2m_2)},$$

$$\Lambda_2^{(2)}=-\frac{3(8-m_2+9m_2^2-4m_2^3-4m_2^4)}{16(1+m_2)^2(1-2m_2)^2},$$

$$\Lambda_3^{(2)}=-\frac{(1-m_2)(8+m_2+2m_2^2)}{32(1+m_2)^2(1-2m_2)},$$

$$\Lambda_4^{(2)}=\tfrac{1}{2},$$

$$\mu_1^{(2)}=\frac{1}{r}\left[\frac{1}{2}\kappa^2-\frac{3}{8}rs_2-\frac{1}{4}\kappa(1-\kappa)r_2-\frac{1}{2}\kappa(1+\kappa)r_3\right],$$

$$\mu_2^{(2)}=\frac{\kappa}{r}\bigg[-\kappa r_1(1+s_{1-2})+\frac{1}{8}(1-\kappa)r_2(3+2s_2)$$
$$+\frac{1}{4}(1+\kappa)r_3(3+2s_2)+\frac{3}{16\kappa}rs_2(5-2s_2)-\kappa$$
$$+\frac{1}{2}\kappa r_3+\frac{1}{4}\kappa r+\frac{1}{2}\kappa r_1 d_2^{(-)}-\frac{1}{2}\kappa r_3 d_2^{(+)}\bigg],$$

$$\mu_3^{(2)}=\frac{\kappa}{r}\bigg[\frac{1}{4}(1-\kappa)r_2+\frac{1}{2}(1+2\kappa)r_3-\frac{1}{8}\kappa^2(1+2s_1)$$
$$+\frac{1}{4}\kappa(1+\kappa)r_3(1+2s_1)+\frac{3rs_2}{8}-\frac{1}{8}\kappa(5-2s_2)$$
$$+\frac{3}{16\kappa}rs_2(3-2s_2)-\frac{1}{2}\kappa r_3 d_1^{(+)}+\frac{1}{2}\kappa r_1 d_3^{(-)}$$
$$-\frac{1}{2}\kappa r_3 d_3^{(+)}\bigg],$$

$$\mu_4^{(2)}=\frac{\kappa}{r}\bigg[\frac{1}{4}(1-\kappa)r_2+\frac{1}{2}(1+\kappa)r_3-\frac{1}{8}\kappa(5-2s_2)$$
$$+\frac{3}{16\kappa}rs_2(3-2s_2)+\frac{1}{8}\kappa^2(1+2s_1)$$
$$-\frac{1}{4}\kappa(1+\kappa)r_3(1+2s_1)-\kappa r_1(1+s_{1-2})-\frac{3}{8}rs_2$$
$$+\frac{1}{4}\kappa r+\frac{1}{2}\kappa r_1 d_1^{(-)}+\frac{1}{2}\kappa r_1 d_3^{(-)}-\frac{1}{2}\kappa r_3 d_3^{(+)}\bigg],$$

$$\mu_5^{(2)}=\frac{\kappa^2}{2r}.$$

APPENDIX C: CALCULATION OF $\mu_1^{(1)}$ BY THE MULTIPLE-SCALE METHOD

1. Basic equations

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad \text{for } -\infty<z<\zeta, \tag{C1}$$

$$\frac{\partial\phi}{\partial z}-\frac{\partial\zeta}{\partial t}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial\zeta}{\partial x} \quad \text{at } z=\zeta, \tag{C2}$$

$$\frac{\partial\zeta}{\partial t}+g\zeta=-\frac{1}{2}(\nabla\phi)^2+\nu\left[\frac{\partial^2\zeta}{\partial x^2}\right]\Bigg/\left[1+\left(\frac{\partial\zeta}{\partial x}\right)^2\right]^{3/2}$$
$$\text{at } z=\zeta, \tag{C3}$$

$$\frac{\partial\phi}{\partial z}\to 0 \quad \text{as } z\to-\infty. \tag{C4}$$

Here x and y are horizontal coordinates, and z is a vertical coordinate that is taken positive vertically upwards; $z=0$ is the undisturbed surface of water and $z=\zeta(x,t)$ represents wavy water surface elevation. $\phi=\phi(x,z,t)$ is the velocity potential.

2. Calculation for determination of $\mu_1^{(1)}$

Following a standard procedure, we introduce the slow space-time variables,

$$x_1=\epsilon x, \quad t_1=\epsilon t, \tag{C5}$$

where ϵ is a small parameter measuring the weakness of wave steepness. We expand ϕ and ζ in the form

$$G=G_{00}-\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}[G_{ln}\exp i(l\psi_1+n\psi_2)$$
$$+G_{ln}^*\exp -i(l\psi_1+n\psi_2)], \tag{C6}$$

where $\psi_1=k_1x-\omega_1t$ and $\psi_2=k_2x-\omega_2t$. In the summation on the right-hand side of Eq. (C6), $(l,n)\neq(0,0)$. Here G stands for ϕ and ζ. The Fourier coefficients ϕ_{00}, ϕ_{ln}, and ϕ_{ln}^* are functions of x_1, z, t_1; and ζ_{00}, ζ_{ln}, and ζ_{ln}^* are functions of x_1 and t_1; ω and k satisfy the linear dispersion relation

$$D(\omega,k) \equiv \omega^2 - gk - \nu k^3 = 0 \tag{C7}$$

for capillary-gravity waves.

In order to derive evolution equations at the order ϵ^4, we require the quantities G_{00}, G_{10}, G_{01}, G_{20}, G_{02}, G_{11}, G_{-11}, and G_{1-1}. So we now determine these quantities as follows:

Substituting expansions (C6) in the governing equations (C1)–(C4) and then equating coefficients of $\exp \mathrm{i}(1\psi_1 + \mathrm{n}\psi_2)$ on both sides of each equation, we get the following:

$$\frac{\partial^2 \phi_{ln}}{\partial z^2} = 0 \quad \text{for} \; -\infty < z < -\zeta, \tag{C8}$$

$$\left(\frac{\partial \phi_{ln}}{\partial z}\right)_0 + iW_{ln}\zeta_{ln} = a_{ln}, \tag{C9}$$

$$-i(W_{ln}\phi_{ln})_0 + g\zeta_{ln} + \nu\triangle_{ln}^2\zeta_{ln} = b_{ln}, \tag{C10}$$

$$\frac{\partial \phi_{ln}}{\partial z} \to 0 \quad \text{as } z \to -\infty, \tag{C11}$$

where Δ_{ln} and W_{ln} are the following operators:

$$\triangle_{ln} = \left|(lk_1 + nk_2) - i\epsilon\frac{\partial}{\partial x_1}\right|, \tag{C12}$$

$$W_{ln} = \left\{(l\omega_1 + n\omega_2) + i\epsilon\frac{\partial}{\partial t_1}\right\}, \tag{C13}$$

and $(\;)_0$ implies the value of the quantity inside brackets at $z=0$, and a_{ln} and b_{ln} are contributions from nonlinear terms. The solution for ϕ_{ln} is

$$\phi_{ln} = \exp(z\triangle_l n)A_{ln}, \tag{C14}$$

where A_{ln} are functions of x_1 and t_1.

$$\triangle_{ln}A_{ln} + iW_{ln}\zeta_{ln} = a_{ln}, \tag{C15}$$

$$-iW_{ln}A_{ln} + g\zeta_{ln} + \nu\triangle_{ln}^2\zeta_{ln} = b_{ln}, \tag{C16}$$

$$[W_{ln}^2 - g\Delta_{ln} - \nu\Delta_{ln}^3]\zeta_{ln} = -\Delta_{ln}b_{ln} - iW_{ln}a_{ln}, \tag{C17}$$

$$A_{ln} = \Delta_{ln}^{-1}(a_{ln} - iW_{ln}\zeta_{ln}). \tag{C18}$$

For $l=1$ and $n=0$, Eq. (C17) gives the evolution equation for the first wave train.

Keeping terms up to $O(\epsilon^2)$, we get

$$\zeta_{20} = -\frac{2k_1\omega_1^2}{D_{1+1}}\zeta_{10}^2, \tag{C19}$$

$$\zeta_{02} = -\frac{2k_2\omega_2^2}{D_{2+2}}\zeta_{01}^2, \tag{C20}$$

$$\zeta_{11} = \frac{2(k_1+k_2)\omega_1\omega_2}{D_{1+2}}\zeta_{10}\zeta_{01}, \tag{C21}$$

$$\zeta_{1-1} = \frac{2(k_1-k_2)\omega_2(\omega_1-\omega_2)}{D_{1-2}}\zeta_{10}\zeta_{01}^*, \tag{C22}$$

$$\zeta_{-11} = \frac{2(k_1-k_2)\omega_1(\omega_1-\omega_2)}{D_{1-2}}\zeta_{10}^*\zeta_{01} \tag{C23}$$

correct to $o(\epsilon^2)$ terms. Here $D_{i\pm j} \equiv D(\omega_i \pm \omega_j, k_i \pm k_j)$.

Equations for $(l,n)=(0,0)$ are

$$\left(\frac{\partial^2}{\partial z^2} + \epsilon^2\frac{\partial^2}{\partial x_1^2}\right)\phi_{00} = 0, \tag{C24}$$

$$\left(\frac{\partial \phi_{00}}{\partial z}\right)_0 - \epsilon\frac{\partial \zeta_{00}}{\partial t_1} = a_{00}, \tag{C25}$$

$$\epsilon\left(\frac{\partial \phi_{00}}{\partial z}\right)_0 + g\zeta_{00} - \epsilon^2\nu\left(\frac{\partial^2}{\partial x_1^2}\right)\zeta_{00} = b_{00}, \tag{C26}$$

$$\frac{\partial \phi_{00}}{\partial z} \to 0 \quad \text{as } z \to -\infty. \tag{C27}$$

It is found that a_{00} is of $O(\epsilon^3)$ and b_{00} is of $O(\epsilon^4)$. Hence ζ_{00} and ϕ_{00} will have no contribution to the coefficient $\zeta_{01}\zeta_{01}^*\zeta_{10}$.

For $l=1$ and $n=0$, Eq. (C17) gives the evolution equation for the first wave train as follows:

$$[W_{10}^2 - g\Delta_{10} - \nu\Delta_{10}^3]\zeta_{10} = -\Delta_{10}b_{10} - iW_{10}a_{10}. \tag{C28}$$

We then calculate the terms in a_{10} and b_{10} that contribute to the coefficient of $\zeta_{10}\zeta_{01}\zeta_{01}^*$ in the first evolution equation. It is found that the coefficient of $\zeta_{01}\zeta_{01}^*\zeta_{10}$ on the right-hand side of Eq. (C28) is as follows:

$$-2k_1k_2\omega_2^2 + 2k_1^2\omega_2^2 + 4k_1k_2\omega_1\omega_2 - 3\nu k_1^3k_2^2 + \frac{4k_1(k_1+k_2)\omega_1^2\omega_2^2}{D_{1+2}} + \frac{4k_1(k_1-k_2)\omega_2^2(\omega_1-\omega_2)^2}{D_{1-2}}. \tag{C29}$$

Finally, introducing the dimensionless variables

$$\zeta_{10}' = k_1\zeta_{10}, \quad \zeta_{01}' = k_2\zeta_{01}, \quad x' = k_2x, \quad t' = \omega_2 t \tag{C30}$$

in Eqs. (C28) and (C29), we get the coefficient $\mu_1^{(1)}$ of $\zeta_{01}\zeta_{01}^*\zeta_{10}$. It is found that this coefficient matches $\mu_1^{(1)}$ as given in Appendix B.

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