Fourthorder nonlinear evolution equation for two Stokes wave trains in deep water

A. K. Dhar and K. P. Das

Citation: Physics of Fluids A: Fluid Dynamics (1989-1993) **3**, 3021 (1991); doi: 10.1063/1.858209
View online: http://dx.doi.org/10.1063/1.858209
View Table of Contents: http://scitation.aip.org/content/aip/journal/pofa/3/12?ver=pdfcov
Published by the AIP Publishing

Fourth-order nonlinear evolution equation for two Stokes wave trains in deep water

A. K. Dhar and K. P. Das
Department of Applied Mathematics, University of Calcutta, 92, Acharya Prafulla Chandra Road, Calcutta-700009, India

(Received 17 January 1991; accepted 12 July 1991)

Fourth-order nonlinear evolution equations, which are a good starting point for the study of nonlinear water waves as first pointed out by Dysthe [Proc. R. Soc. London Ser. A **369**, 105 (1979)] and later elaborated by Janssen [J. Fluid Mech. **126**, 1 (1983)], are derived for a deep-water surface gravity wave packet in the presence of a second wave packet. Here it is assumed that the space variation of the amplitude takes place only in a direction along which the group velocity projection of the two waves overlap. Stability analysis is made for a uniform Stokes wave train in the presence of a second wave train. Graphs are plotted for maximum growth rate of instability and for wave number at marginal stability against wave steepness. Significant deviations are noticed from the results obtained from the third-order evolution equations which consist of two coupled nonlinear Schrödinger equations.

I. INTRODUCTION

The most successful approach of studying the stability of finite-amplitude surface gravity waves in deep water is through the application of the lowest-order nonlinear evolution equation, which is the nonlinear Schrödinger equations. This analysis is suitable for small wave steepness and for long-wavelength perturbations. But for wave steepness greater than 0.15 predictions from the nonlinear Schrödinger equation do not agree with the exact result of Longuet-Higgins.[1,2] Dysthe[3] has shown that a stability analysis made from a fourth-order nonlinear evolution equation that is one order higher than the nonlinear Schrödinger equation gives results consistent with the exact results of Longuet-Higgins[1,2] and with the experimental results of Benjamin and Feir[4] for wave steepness up to 0.25. The fourth-order effects give a surprising improvement compared to ordinary nonlinear Schrödinger effects in many respects, and some of these points have been elaborated by Janssen.[5] The dominant new effect that comes in the fourth order is the influence of wave-induced mean flow and this produces a significant deviation in the stability character. From these it can be concluded that a fourth-order evolution equation is a good starting point for studying nonlinear effects in surface gravity waves. Deviations of fourth-order nonlinear evolution equations for deep-water surface gravity waves, including different effects and stability analyses made from them, were done by several authors.[5-8]

All these analyses made by the above-mentioned authors are for a single wave. Stability analysis of a surface gravity wave in deep water in the presence of a second wave has been made by Roskes[9] based on the lowest-order nonlinear evolution equation, which consists of two coupled nonlinear Schrödinger equations. In his investigation modulational perturbation is restricted to a direction along which group velocity projections of the two waves overlap and it is argued that the modulation will grow at a faster rate along this direction when $0<\theta<70.5°$, where θ is the angle between the two propagation directions of two waves.

The same analysis performed by Roskes[9] is made here starting from fourth-order nonlinear evolution equations. Two coupled fourth-order nonlinear evolution equations are derived for a surface gravity wave in deep water that propagates in the presence of a second wave. It is supposed that the space variation of the amplitude takes place in a direction along which group velocity projection of the two waves overlap. Stability analysis is made for the case when the amplitude of the two waves are the same. The maximum growth rate of instability and wave number at marginal stability are derived and graphs are drawn for these expressions against wave steepness. Significant deviations are noticed from the results obtained from the third-order evolution equations which consist of two coupled nonlinear Schrödinger equations.

In the evolution equations (19) and (20) the Hilbert transform terms vanish for $\theta = 180°$ and these Hilbert transform terms correspond to the wave-induced mean flow. So for this value of θ there is no fourth-order contribution in the expressions for maximum growth rate and wave number at marginal stability.

II. BASIC EQUATIONS

We take the free surface in the undisturbed state as the $z=0$ plane. We consider that the two waves move in the x-y plane with wave numbers $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. We take the x axis in a direction along which group velocity projections of the two waves overlap and consider the modulations only along this line. Let $z=\zeta(x,y,t)$ be the equation of the free surface at any time t in the perturbed state. We introduce the dimensionless quantities φ^*, ζ^*, (x^*,y^*,z^*), t^* which are, respectively, the perturbed velocity potential in water, elevation of the free surface, space coordinates and time. These dimensionless quantities are related to the corresponding dimensional quantities by the following relations:

$$\varphi^* = \sqrt{(k_0^3/g)}\,\varphi, \quad \zeta^* = k_0\zeta,$$
$$(x^*,y^*,z^*) = (k_0x,k_0y,k_0z), \quad t^* = \omega t, \qquad (1)$$

 0899-8213/91/123021-06$02.00

where k_0 is some characteristic wave number. In the future all these quantities will be written in their dimensionless form but with the asterisks dropped.

The perturbed velocity potential φ, from which perturbed velocity $\mathbf{u}$ of water can be obtained from the relation $\mathbf{u} = \nabla\varphi$, satisfies the following Laplace equation:

$$\nabla^2\varphi = 0 \quad \text{in} -\infty < z < \zeta. \tag{2}$$

The kinematic boundary condition to be satisfied at the free surface is

$$\frac{\partial\varphi}{\partial z} - \frac{\partial\zeta}{\partial t} = \frac{\partial\varphi}{\partial x}\cdot\frac{\partial\zeta}{\partial x} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\zeta}{\partial y}, \quad \text{where } z = \zeta. \tag{3}$$

The condition of continuity of pressure at the free surface gives

$$\frac{\partial\varphi}{\partial t} + \zeta = -\frac{1}{2}(\nabla\varphi)^2, \quad \text{when } z = \zeta. \tag{4}$$

Also, φ should satisfy the following condition:

$$\varphi \to 0 \quad \text{as } z \to -\infty. \tag{5}$$

We look for solutions of the above equations in the form

$$G = G_{00} + \sum_{\substack{m=0 \\ m=n\neq 0}}^{\infty}\sum_{n=0}^{\infty}[G_{mn}\exp i(m\Psi_1 + n\Psi_2) + G^*_{mn}\exp - i(m\Psi_1 + n\Psi_2)], \tag{6}$$

where

$$\Psi_1 = k_{1x}x + k_{1y}y - \omega_1 t, \quad \Psi_2 = k_{2x}x + k_{2y}y - \omega_2 t, \tag{7}$$

and G stands for φ and ζ. Here, φ_{00}, φ_{mn}, φ^*_{mn} are slowly varying functions of z, $x_1 = \epsilon x$, $y_1 = \epsilon y$, $t_1 = \epsilon t$; ζ_{00}, ζ_{mn}, ζ^*_{mn} are functions of x_1, y_1, t_1. Here * denotes complex conjugate, ϵ is a slow ordering parameter.

The linear dispersion relations are given by

$$\begin{aligned}\omega_1^2 - (k_{1x}^2 + k_{1y}^2)^{1/2} = 0,\\ \omega_2^2 - (k_{2x}^2 + k_{2y}^2)^{1/2} = 0.\end{aligned} \tag{8}$$

We consider the simplifying assumption that the wave numbers $|\mathbf{k}_1| = \sqrt{k_{1x}^2 + k_{1y}^2}$ and $|\mathbf{k}_2| = \sqrt{k_{2x}^2 + k_{2y}^2}$ are the same and let this common wave number be equal to k_0, the characteristic wave number. Thus we have $k \equiv |\mathbf{k}_1| = |\mathbf{k}_2| = k_0$ and the linear dispersion relation (8) then becomes

$$\omega^2 - k = 0. \tag{9}$$

Also, we have $k = 1$ and the linear dispersion relation determining $\omega > 0$ becomes

$$\lambda(\omega) \equiv \omega^2 - 1 = 0. \tag{10}$$

III. DERIVATION OF THE EVOLUTION EQUATIONS

On substituting the expansion (6) in (2) and equating the coefficients of $\exp i(m\Psi_1 + n\Psi_2)$ for $(m,n) = [(1,0),(0,1),(2,0),(0,2),(1,1),(1,-1)]$, we obtain the following equations:

$$\frac{d^2\varphi_{mn}}{dz^2} - \Delta_{mn}^2\varphi_{mn} = 0, \tag{11}$$

where Δ_{mn} is an operator given by

$$\Delta_{mn} = \left[\left(mk_{1x} + nk_{2x} - i\epsilon\frac{\partial}{\partial x_1}\right)^2 + \left(mk_{1y} + nk_{2y} - i\epsilon\frac{\partial}{\partial y_1}\right)^2\right]^{1/2}.$$

The solution of these equations satisfying boundary condition (5) can be put in the form

$$\varphi_{mn} = A_{mn}\exp(\Delta_{mn}z), \tag{12}$$

where A_{mn} are functions of x_1, y_1, and t_1.

For the sake of convenience we take the Fourier transform of Eq. (2) for $(m,n) = (0,0)$. The solution of this transformed equation becomes

$$\overline{\varphi}_{00} = \overline{A}_{00}\exp(|\overline{k}|z), \tag{13}$$

where $\overline{\varphi}_{00}$ is the Fourier transform of φ_{00} defined by

$$\overline{\varphi}_{00} = \int\int_{-\infty}^{\infty}\int \varphi_{00}\exp[i(\overline{k}_x x_1 + \overline{k}_y y_1 - \overline{\omega}t_1)]dx_1\,dy_1\,dt_1, \tag{14}$$

where $\overline{k}^2 = \epsilon^2(\overline{k}_x^2 + \overline{k}_y^2)$ and $\overline{A}_{00}$ are functions of $\overline{k}_x$, $\overline{k}_y$, and $\overline{\omega}$.

On substituting the expansions (6) in the Taylor-expanded form of Eqs. (3) and (4) about $z = 0$ and then equating coefficients of $\exp i(m\Psi_1 + n\Psi_2)$ for $(m,n) = [(1,0),(0,1);\ (2,0),(0,2),\ (1,1),(1,-1);\ (0,0)]$ on both sides, we obtain the seven sets of equations, in which we substitute the solutions for φ_{mn} given by (12) and (13). For the sake of convenience we take the Fourier transform of the set of equations corresponding to $m = n = 0$. The sets of equations corresponding to $(m,n) = [(1,0),(0,1)]$, $(m,n) = [(2,0),(0,2),(1,1),(1,-1)]$, $(m,n) = (0,0)$ will be called, respectively, the first, second, and third sets. To solve the above three sets of equations we make the following perturbation expansion for the quantities A_{mn}, ζ_{mn} for above values of (m,n):

$$\begin{aligned}E_{mn} &= \sum_{p=1}^{\infty}\epsilon^p E_{mnp} \quad \text{for } (m,n) = [(1,0),(0,1)],\\ E_{mn} &= \sum_{p=2}^{\infty}\epsilon^p E_{mnp}\\ &\quad \text{for } (m,n) = [(2,0),(0,2)(1,1),(1,-1),(0,0)],\end{aligned} \tag{15}$$

where E_{mn} stands for A_{mn}, ζ_{mn}.

On substituting the expansions (15) in the above three sets of equations and then equating coefficients of various power of ϵ on both sides, we obtain a sequences of equations.

From the first-order (i.e., lowest order) and second-order equations corresponding to (3) of the first set of equations we obtain solutions for A_{101}, A_{102}, and A_{011}, A_{012}, respectively. Next, from the first-order and second-order equations corresponding to (3) and (4) of the second set of equations, we obtain solutions for A_{202}, A_{203}, ζ_{202}, ζ_{203}; A_{022}, A_{023}, ζ_{022}, ζ_{023}; A_{112}, A_{113}, ζ_{112}, ζ_{113}; A_{1-12}, A_{1-13}, ζ_{1-12}, ζ_{1-13}. Finally, from the first-order equations corresponding to Eqs. (3) and (4) of the third set of equations we obtain solutions for A_{002}, ζ_{002} and from the second-order equations corresponding to (4) of the third set of equations we obtain a solution for ζ_{003}. The equations corresponding to (4) of the first set of equations, which has not been used in obtaining the above perturbation solutions, can be put in the following convenient forms after eliminating A_{mn}:

$$\left[\left(\omega + i\epsilon \frac{\partial}{\partial t_1}\right)^2 - \Delta_{10}\right]\zeta_{10} = -i\left(\omega + i\epsilon \frac{\partial}{\partial t_1}\right) a_{10} - \Delta_{10} c_{10}, \tag{16}$$

$$\left[\left(\omega + i\epsilon \frac{\partial}{\partial t_1}\right)^2 - \Delta_{01}\right]\zeta_{01} = -i\left(\omega + i\epsilon \frac{\partial}{\partial t_1}\right) a_{01} - \Delta_{01} c_{01}, \tag{17}$$

where a_{10}, c_{10}, a_{01}, c_{01} are contributions from nonlinear terms.

We keep terms up to order ϵ^4 in Eqs. (16) and (17) and then substitute solutions for various perturbed quantities appearing on its right-hand sides and, finally, using the transformations

$$\xi = x_1 - v \cdot t_1, \quad \tau = \epsilon t_1, \tag{18}$$

where v is the component of group velocity of any one of the two waves along the x axis and is given by

$$v = \cos\left(\frac{\theta}{2}\right)\left(\frac{d\omega}{dk}\right)_{k=1} = \frac{\cos(\theta/2)}{2\omega}.$$

This $(d\omega/dk)_{k=1}$ has been obtained from Eq. (9).

Writing $\zeta_1 = \zeta_{101} + \epsilon\zeta_{102}, \zeta_2 = \zeta_{011} + \epsilon\zeta_{012}$ we obtain the following fourth-order evolution equations for ζ_1 and ζ_2:

$$i\frac{\partial \zeta_1}{\partial \tau} + \gamma_{11}\frac{\partial^2 \zeta_1}{\partial \xi^2} + i\gamma_{12}\frac{\partial^3 \zeta_1}{\partial \xi^3} = \zeta_1(\beta_{11}|\zeta_1|^2 + \beta_{12}|\zeta_2|^2) + i\alpha_{11}\zeta_1\zeta_1^*\frac{\partial \zeta_1}{\partial \xi} + i\alpha_{12}\zeta_1^2\frac{\partial \zeta_1^*}{\partial \xi} + i\lambda_{11}\zeta_2\zeta_2^*\frac{\partial \zeta_1}{\partial \xi}$$
$$+ i\lambda_{12}\zeta_1\zeta_2^*\frac{\partial \zeta_2}{\partial \xi} + i\lambda_{13}\zeta_1\zeta_2\frac{\partial \zeta_2^*}{\partial \xi} + \mu\zeta_1 H\frac{\partial}{\partial \xi}(\zeta_1\zeta_1^*) + \mu\zeta_1 H\frac{\partial}{\partial \xi}(\zeta_2\zeta_2^*) \tag{19}$$

and

$$i\frac{\partial \zeta_2}{\partial \tau} + \gamma_{11}\frac{\partial^2 \zeta_2}{\partial \xi^2} + i\gamma_{12}\frac{\partial^3 \zeta_2}{\partial \xi^3} = \zeta_2(\beta_{12}|\zeta_1|^2 + \beta_{11}|\zeta_2|^2) + i\alpha_{11}\zeta_2\zeta_2^*\frac{\partial \zeta_2}{\partial \xi} + i\alpha_{12}\zeta_2^2\frac{\partial \zeta_2^*}{\partial \xi} + i\lambda_{11}\zeta_1\zeta_1^*\frac{\partial \zeta_2}{\partial \xi}$$
$$+ i\lambda_{12}\zeta_2\zeta_1^*\frac{\partial \zeta_1}{\partial \xi} + i\lambda_{13}\zeta_2\zeta_1\frac{\partial \zeta_1^*}{\partial \xi} + \mu\zeta_2 H\frac{\partial}{\partial \xi}(\zeta_2\zeta_2^*) + \mu\zeta_2 H\frac{\partial}{\partial \xi}(\zeta_1\zeta_1^*), \tag{20}$$

where θ is the angle between $\mathbf{k}_1 = (k_{1x}, k_{1y})$ and $\mathbf{k}_2 = (k_{2x}, k_{2y})$; H is the Hilbert's transform given by

$$H(\Psi) = \frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{\Psi(\xi')d\xi'}{\xi' - \xi}, \tag{21}$$

and where the coefficients γ_{11}, γ_{12}, β_{11}, β_{12}, α_{11}, α_{12}, λ_{11}, λ_{12}, λ_{13}, and μ are given in the Appendix.

If we replace ζ_1 by $\zeta_1/2$ and ζ_2 by $\zeta_2/2$ in the evolution equations (19) and (20), then the coefficients of third-order terms become the same as those of the corresponding coefficients of Roskes.[9] Also, for $\theta = 0$ and in the absence of the second wave, the coupled equations reduce to a single equation that becomes equivalent to Eq. (2) of Janssen.[5]

IV. STABILITY ANALYSIS

Equations (19) and (20) admit the Stokes wave solution

$$\zeta_1 = (\alpha_1/2)\exp(i\Delta\omega_1\tau), \quad \zeta_2 = (\alpha_2/2)\exp(i\Delta\omega_2\tau), \tag{22}$$

where α_1 and α_2 are real constants and the nonlinear frequency shifts $\Delta\omega_1$ and $\Delta\omega_2$ are given by

$$\Delta\omega_1 = -\tfrac{1}{4}(\beta_{11}\alpha_1^2 + \beta_{12}\alpha_2^2), \tag{23}$$

$$\Delta\omega_2 = -\tfrac{1}{4}(\beta_{11}\alpha_2^2 + \beta_{12}\alpha_1^2). \tag{24}$$

These two expressions for frequency shifts become the same as that of Longuet-Higgins and Phillips[10] if in their equations we set $\sigma_1 = \sigma_2 = \omega, \beta = 180 - \theta/2$, and introduce the correction noted by Willebrand.[11]

To study modulational stability of these wave trains we introduce the following perturbations in the uniform solution:

$$\zeta_1 = (\alpha_1/2)(1 + \zeta_1')\exp i(\Delta\omega_1\tau + \theta_1'), \tag{25}$$

$$\zeta_2 = (\alpha_2/2)(1 + \zeta_2')\exp i(\Delta\omega_2\tau + \theta_2'), \tag{26}$$

where ζ_1', ζ_2', θ_1', and θ_2' are small perturbations of amplitudes and phases, respectively.

We insert (25) and (26) into (19) and (20) and then linearize with respect to ζ_1', θ_1', ζ_2', and θ_2' to obtain four linear equations for the four latter quantities. Next, assuming the time dependence of these quantities to be of the form $\exp(-i\Omega'\tau)$ and taking the Fourier transform of the four linear equations with respect to ξ, we obtain four algebraic equations for the four quantities $\bar{\zeta}_1'$, $\bar{\theta}_1'$, $\bar{\zeta}_2'$, and $\bar{\theta}_2'$, which are the Fourier transforms of ζ_1', θ_1', ζ_2', and θ_2', respectively, defined by

$$(\bar{\zeta}_1', \bar{\theta}_1', \bar{\zeta}_2', \bar{\theta}_2') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\zeta_1', \theta_1', \zeta_2', \theta_2')\exp(-i\lambda\xi)d\xi. \tag{27}$$

The condition for the existence of a nontrivial solution to the above four algebraic equations gives the following nonlinear

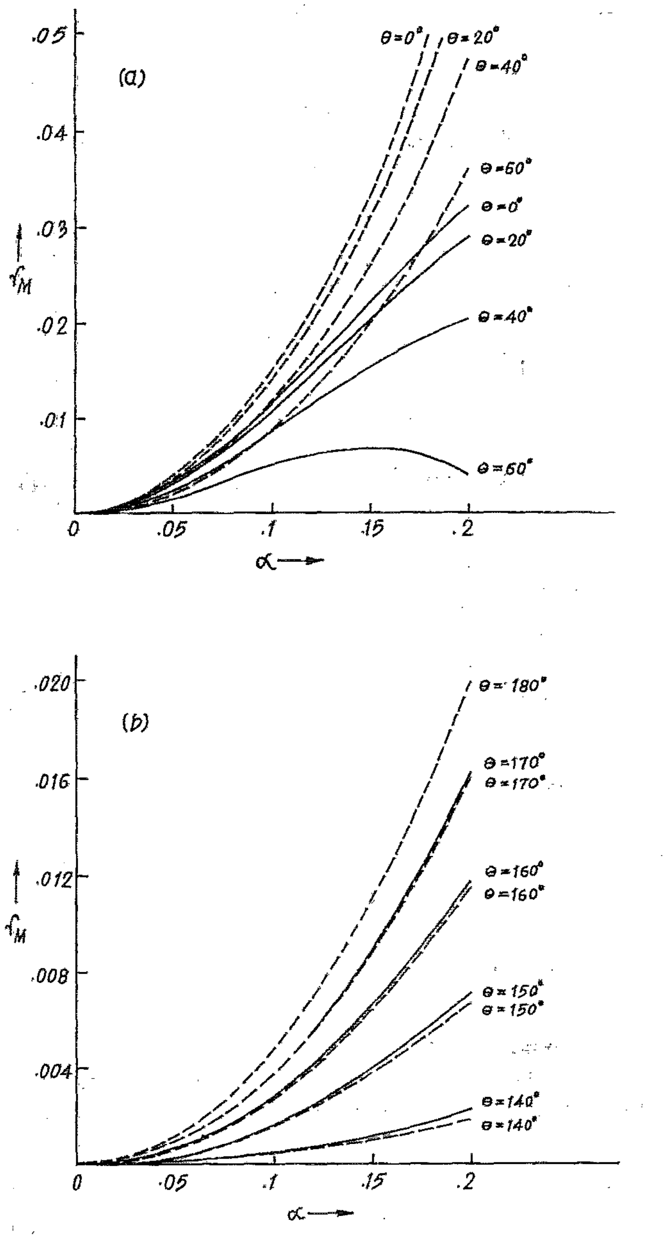

FIG. 1. Maximum growth rate γ_M as a function of dimensionless wave steepness α. ----: third-order result; ——: fourth-order result.

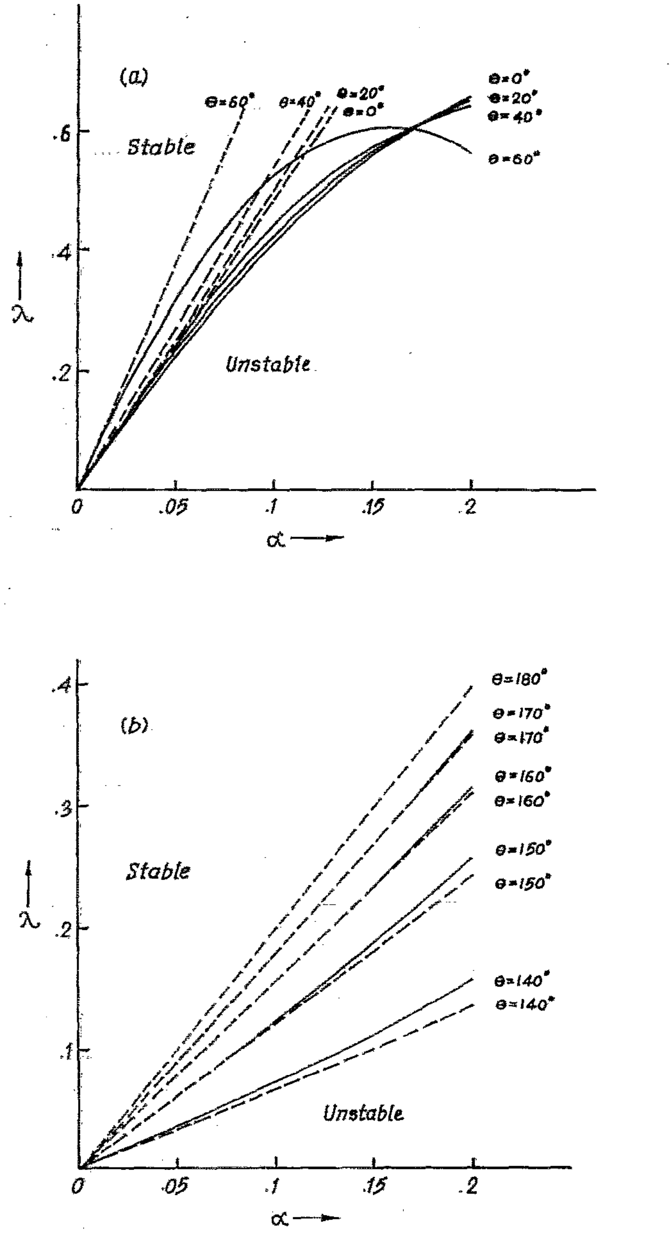

FIG. 2. Plot of perturbed wave number λ at marginal stability against dimensionless wave steepness α. ----: third-order result; ——: fourth-order result.

dispersion relation, where we assume that the amplitudes of the two modes are equal ($\alpha_1 = \alpha_2 = \alpha$):

$$[\overline{P}_1 + (\alpha^2/4)(\alpha_{11} + \alpha_{12} + \lambda_{11} + \lambda_{12} + \lambda_{13})\lambda\,] \times [\overline{P}_1 + (\alpha^2/4)(\alpha_{11} - \alpha_{12} + \lambda_{11} + \lambda_{12} - \lambda_{13})\lambda\,] = \overline{P}_2\,[\overline{P}_2 + (\alpha^2/2)(\beta_{11} + \beta_{12} - 2\mu\lambda)], \quad (28)$$

where

$$\overline{P}_1 = \Omega - \vartheta\lambda + \gamma_{12}\lambda^3, \quad (29)$$

$$\overline{P}_2 = \gamma_{11}\lambda^2, \quad (30)$$

and $\Omega = \Omega' + \vartheta\lambda$. Solving (28) for $\overline{P}_1$ we obtain

$$\overline{P}_1 = -[\alpha^2(\alpha_{11} + \lambda_{11} + \lambda_{12})\lambda/4] \pm \{\overline{P}_2\,[\overline{P}_2 + (\alpha^2/2)(\beta_{11} + \beta_{12} - 2\mu\lambda)]\}^{1/2}. \quad (31)$$

For instability

$$\overline{P}_2\,[\overline{P}_2 + (\alpha^2/2)(\beta_{11} + \beta_{12} - 2\mu\lambda)] < 0. \quad (32)$$

This condition is met when θ lies either in $0 \leqslant \theta < 70.5°$ or in $136.1° < \theta \leqslant 180°$. The maximum growth rate of instability γ_M for θ lying within $0 \leqslant \theta < 70.5°$ is

$$\gamma_M = [(|\beta_{11}| + |\beta_{12}|)\alpha^2/4] \times \{1 - [\mu\alpha/\sqrt{\gamma(|\beta_{11}| + |\beta_{12}|)}\,]\}, \quad (33)$$

where $\gamma = -\gamma_{11} > 0$ and the same for θ lying within $136.1° < \theta < 180°$ is

$$\gamma_M = [(|\beta_{12}| - |\beta_{11}|)\alpha/4][1 + (\mu\alpha/\sqrt{\delta\gamma_{11}})], \quad (34)$$

where $\delta = |\beta_{12}| - |\beta_{11}| > 0$.

At marginal stability we have

$$\bar{P}_2[\bar{P}_2 + (\alpha^2/2)(\beta_{11} + \beta_{12} - 2\mu\lambda)] = 0$$

and the wave number λ at marginal stability is given by

$$\lambda = \sqrt{(|\beta_{11}| + |\beta_{12}|)/2\gamma}\,\alpha \times \{1 - [\mu\alpha/\sqrt{2\gamma(|\beta_{11}| + |\beta_{12}|)}]\} \quad \text{for } 0 < \theta < 70.5° \quad (35)$$

and

$$\lambda = \sqrt{(\delta/2\gamma_{11})}\,\alpha[1 + (\mu\alpha/\sqrt{2\gamma_{11}\delta})] \quad \text{for } 136.1° < \theta < 180°. \quad (36)$$

If we put $\mu = 0$, then Eqs. (33) and (34) reduce to the corresponding equations of Roskes.[9]

In Figs. 1(a) and 1(b) the maximum growth rate γ_M of instability and in Figs. 2(a) and 2(b) the wave number λ at marginal stability have been plotted against dimensionless wave steepness α for some different values of θ. These graphs show deviations from the results obtained from the third-order evolution equations and the deviations become very significant when θ lies in the range $0 < \theta < 70.5°$.

In the evolution equations (19) and (20) the Hilbert transform terms vanish for $\theta = 180°$ and so for this value of θ there is no fourth-order contribution in the expressions for maximum growth rate and wave number at marginal stability.

V. CONCLUSION

In the Introduction it has been mentioned that a fourth-order evolution equation is a good starting point for studying nonlinear effects in surface gravity waves. In view of this we have made a stability analysis of a surface gravity wave packet in the presence of a second wave packet, starting from fourth-order evolution equations for this system. Our results show considerable deviation from the results obtained from the lowest-order evolution equation, which has been considered by Roskes.[9] This deviation is due to the effect of wave-induced mean flow, which comes at the fourth order. This wave-induced mean flow corresponds to the terms involving Hilbert transforms in Eqs. (19) and (20). These two terms vanish for $\theta = 180°$ and so for this value of θ there is no fourth-order contribution in stability character.

ACKNOWLEDGMENT

The authors acknowledge the helpful comments of the referees.

APPENDIX: THE COEFFICIENTS OF EQS. (19) and (20)

$$\gamma_{11} = \tfrac{1}{8}[2 - 3\cos^2(\theta/2)],$$

$$\gamma_{12} = \frac{\cos(\theta/2)[6\sin^2(\theta/2) - \cos^2(\theta/2)]}{16},$$

$$\beta_{11} = 2,$$

$$\beta_{12} = \frac{-4[\cos^5(\theta/2) + 2\cos^4(\theta/2) - 5\cos^3(\theta/2) + 2\cos^2(\theta/2) + 3\cos(\theta/2) - 2]}{[\cos(\theta/2) - 2]},$$

$$\alpha_{11} = -6\cos(\theta/2), \quad \alpha_{12} = -\cos(\theta/2),$$

$$\lambda_{11} = \frac{1}{2}\Bigg\{-\Bigg[\left(2\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2} - 1\right)\delta_1 + 2\left(2\cos^2\frac{\theta}{2} - 1\right)\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} + \frac{2[\cos(\theta/2) + 2]\cos(\theta/2)[1 + \sin^2(\theta/2) - 2\cos(\theta/2)]}{[\cos(\theta/2) - 2]} + 2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} + 5\cos\frac{\theta}{2} - 2\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} - 7\cos^2\frac{\theta}{2} - 2\cos^3\frac{\theta}{2}\sin\frac{\theta}{2}\Bigg] + \frac{\cos(\theta/2)}{2}\Bigg[\left(2\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2} - 1\right)\delta_2 + 2\left(2\cos^2\frac{\theta}{2} - 1\right)\sin^2\frac{\theta}{2} - \frac{2\cos(\theta/2)[1 + \sin^2(\theta/2) - 2\cos(\theta/2)]}{[\cos(\theta/2) - 2]} + 2\cos^3\frac{\theta}{2} - 1\Bigg]\Bigg\},$$

$$\lambda_{12} = \frac{1}{2}\Bigg\{-\Bigg[\left(2\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2} - 1\right)\delta_1 + \frac{2\cos(\theta/2)[\cos(\theta/2) + 2][1 + \sin^2(\theta/2) - 2\cos(\theta/2)]}{[\cos(\theta/2) - 2]} + 2\sin^2\frac{\theta}{2} \times\left(3\cos\frac{\theta}{2} - 2\cos^3\frac{\theta}{2}\right) + 5\cos\frac{\theta}{2} - 2\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\Bigg] + \frac{\cos(\theta/2)}{2}\Bigg[\left(2\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2} - 1\right)\delta_2 - \frac{2\cos^2(\theta/2)[1 + \sin^2(\theta/2) - 2\cos(\theta/2)]}{\cos(\theta/2) - 2} - 2\sin\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right) + 2\cos^3\frac{\theta}{2} + 1 - 2\cos\frac{\theta}{2} - 2\cos\frac{\theta}{2}\Bigg]\Bigg\},$$

$$\lambda_{13} = \frac{1}{2}\Bigg[-\Bigg(2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\cos\theta + \frac{2\cos(\theta/2)[2\cos^3(\theta/2) - \cos(\theta/2) + 2][1 + \sin^2(\theta/2) - 2\cos(\theta/2)]}{[\cos(\theta/2) - 2]}$$
$$+ 2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} + 2\cos^3\frac{\theta}{2}\sin\frac{\theta}{2} - 4\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\Bigg)$$
$$+ \frac{\cos(\theta/2)}{2}\Bigg(2\cos\theta\sin^2\frac{\theta}{2} - 4\sin^3\frac{\theta}{2} + \frac{2\cos\theta\cos(\theta/2)[1 + \sin^2(\theta/2) - 2\cos(\theta/2)]}{[\cos(\theta/2) - 2]} + \cos\theta\Bigg) - 2\cos\frac{\theta}{2}\Bigg],$$
$$\mu = 2\cos^2(\theta/2),$$

where

$$\delta_1 = \frac{1}{[\cos(\theta/2) - 2]}\Bigg(2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} + 2\cos\frac{\theta}{2} + 2\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} - 1 - \sin^2\frac{\theta}{2}$$
$$+ \frac{\cos(\theta/2)[1 + \sin^2(\theta/2) - 2\cos(\theta/2)]}{[\cos(\theta/2) - 2]}\Bigg),$$

$$\delta_2 = \frac{2\cos(\theta/2)[1 + \sin^2(\theta/2) - \cos(\theta/2)]}{[\cos(\theta/2) - 2]} + \frac{4\cos(\theta/2)[1 + \sin^2(\theta/2) - 2\cos(\theta/2)]}{[\cos(\theta/2) - 2]^2}.$$

[1] M. S. Longuet-Higgins, Proc. R. Soc. London Ser. A **360**, 471 (1978).
[2] M. S. Longuet-Higgins, Proc. R. Soc. London Ser. A **360**, 489 (1978).
[3] K. B. Dysthe, Proc. R. Soc. London Ser. A **369**, 105 (1979).
[4] T. B. Benjamin and J. E. Feir, J. Fluid Mech. **27**, 417 (1967).
[5] P. A. E. M. Janssen, J. Fluid Mech. **126**, 1 (1983).
[6] M. Stiassnie, Wave Motion **6**, 431 (1984).
[7] S. J. Hogan, Proc. R. Soc. London Ser. A **402**, 359 (1985).
[8] A. K. Dhar and K. P. Das, Phys. Fluids A **2**, 778 (1990).
[9] G. J. Roskes, Phys. Fluids **19**, 1253 (1976).
[10] M. S. Longuet-Higgins and O. M. Phillips, J. Fluid Mech. **12**, 333 (1962).
[11] J. Willebrand, J. Fluid. Mech. **70**, 113 (1975).