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Foundations of Vagueness : a Category-theoretic Approach

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Abstract

The idea that vagueness has its origin in indiscernibility is not new. In this paper, we have tried to establish this thesis on a category-theoretic basis. In the process some other related notions, e.g. property, have been clarified.

1 Introduction

So far as we can understand, the root of vagueness of a concept lies in a corresponding indistinguishability (indiscernibility / approximate identity) of objects in the universe of discourse. This view has been aired in the eighties by many (cf. [12] and the philosophy behind rough set theory). Without entering the debate whether vagueness resides in "reality" or not, everybody will, hopefully, accept that the language used for communication can, in no way, avoid it. Parikh [9] rightly observed that vagueness is an essential feature, not only of ordinary languages, but also of the "precise" "artificial" languages used in "physics and in fact, any science that attempts to correlate observation with words and numbers".

Indiscernibility leads to the age-old issue of identity and individuation. While indiscernibility and individuation have basically an epistemological content, identity is supposed to be an ontological notion. Yet "understanding"

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identity has remained eternally elusive. The Leibnizian principle is definitely pioneering in this regard :

$x = y$ if and only if x has every property that y has and conversely. (LP)

There are, however, quite a few problems with this principle. Firstly, the word "every" of "every property" in the definiens is inconceivable. Secondly, (LP) may characterize the identity of static objects only. And thirdly, understanding of properties and their identity then comes prior to that of identity of objects – which one is more transparent is debatable [13].

Because of the first two problems, it seems reasonable to modify the Leibnizian principle as follows :

$x = y$ if and only if x has P implies y has P and conversely. (MLP)

Here P belongs to a "specified" collection of properties.

Extensionally, "properties" can be considered as subsets of the universe of discourse within which identity should be understood, i.e. the elements of which are well-individuated. Without this, subsets cannot be defined at all. Let P be a property for objects in the universe X having an understood identity. Then

$x = y$ and x has P imply y has P. (S)

(S) is one wing of (MLP) and is called the "substitutivity principle" (substitutivity of identicals). A classical property understood extensionally by a Cantorian subset of X makes the above assertion a truism, viz.

$x = y$ and $x \in P$ imply $y \in P$.

But when the property (or concept) P is vague or/and the identity in X is approximate, (S) becomes non-trivial and extremely interesting, having multiple possibilities. P is then represented (extensionally, again) by fuzzy sets or rough sets or the like.

Our objective is to look into some kinds of indiscernibilities related with vague concepts and systematize them. The framework of study is categorial. A triple <Universe,Indiscernibility,Related Concept> will be a typical object of the defined categories.

2 The Identities

We shall adopt a uniform procedure to deal with all the approximate identities. $X, Y, Z, \ldots$ shall denote universes of discourse – individuation in them shall be assumed to be well-understood. The identity of objects in them shall be uniformly denoted by '='. An indiscernibility relation ϵ shall be taken on the universe. The general pattern of the indiscernibility shall be a mapping $\epsilon : X \times X \to (L, *, \leq)$ (a suitable structure) satisfying, for any $x, y, z \in X$, the conditions

(a) $\epsilon(< x, y >) \leq \epsilon(< x, x >)$,

(b) $\epsilon(< x, y >) = \epsilon(< y, x >)$ ('=' also denoting the identity in L), and

(c) $\epsilon(< x, y >) * \epsilon(< y, z >) \leq \epsilon(< x, z >)$.

$(L, *, \leq)$ is the "truth(-value) set" associated with the indiscernibility ϵ and domain X. It is a "residuated lattice", i.e. $(L, \leq)$ is a complete lattice, and '*' is a binary operation on L such that
(i) $(L, *, 1)$ is a commutative monoid (1 being the unit element of L), and
(ii) $(sup_i \ \alpha_i) * \beta = sup_i(\alpha_i * \beta)$, where $\alpha_i, \beta \in L$ and $i \in I$, an index set.
A binary operation '$\rightarrow$' can be defined in L as :

$$\alpha \rightarrow \beta \equiv sup\{\gamma \in L : \alpha * \gamma \leq \beta\}, \ \alpha, \beta \in L.$$

The indiscernibilities we shall be dealing with are I_C (classical identity), E_C, E_M [8], E_Z [15], E_R [11], E_T [14], E_{Ap}, E_f [4], E_H [7] and E.

Let $x, y, z \in X$.

• I_C
$I_C : X \times X \rightarrow \{0, 1\}$ such that
∘ $I_C(< x, y >) = 0$ for $x \neq y$.
An object is of the form $< X, I_C, A >$, where $A(x) = I_C(< x, x >), \ x \in X$. Here A denotes a subset of X such that $x \in A$ if and only if $I_C(x, x) = 1$. $I_C(x, x)$ may be 0 for some x. Thus the indiscernibility determines a subset of X.

• E_C
$E_C : X \times X \rightarrow \{0, 1\}$ such that

(i) $E_C(< x, y >) \leq E_C(< x, x >)$,

(ii) $E_C(< x, y >) = E_C(< y, x >)$,

(iii) $E_C(< x, y >) \wedge E_C(< y, z >) \leq E_C(< x, z >)$.

An object is of the form $< X, E_C, A >$, where $A(x) = E_C(< x, x >), \ x \in X$. Again E_C determines the subset A such that $x \in A$ if and only if $E_C(x, x) = 1$. But in this case, it is possible that $E_C(x, y) \neq 0$ for $x \neq y$. So, in effect, E_C determines a partition for the subset A of X. This idea has been extended to the follo wing general situations.

• E_T
$E_T : X \times X \rightarrow [0, 1]$ such that

(i) $E_T(< x, y >) \leq E_T(< x, x >)$, $E_T(< x, x >)$ is 1 or 0,

(ii) $E_T(< x, y >) = E_T(< y, x >)$,

(iii) $E_T(< x, y >) * E_T(< y, z >) \leq E_T(< x, z >)$.

An object is of the form $< X, E_T, A >$, where $A(x) = E_T(< x, x >), \ x \in X$.

E_M, E_Z and E_R are special cases of E_T, where particular '*' operations on L are considered.

• E_{Ap}

$E_{Ap} : X \times X \rightarrow L$ (a residuated lattice) such that

(i) $E_{Ap}(< x, y >) \leq E_{Ap}(< x, x >)$, $E_{Ap}(< x, x >)$ is 1 or 0,

(ii) $E_{Ap}(< x, y >) = E_{Ap}(< y, x >)$,

(iii) $E_{Ap}(< x, y >) * E_{Ap}(< y, z >) \leq E_{Ap}(< x, z >)$.

An object is of the form $< X, E_{Ap}, A >$, where $A(x) = E_{Ap}(< x, x >),\ x \in X$.

• E_f

$E_f : X \times X \to [0, 1]$ such that

(i) $E_f(< x, y >) \leq E_f(< x, x >)$,

(ii) $E_f(< x, y >) = 0$ for $x \neq y$.

An object is of the form $< X, E_f, A >$, where $A(x) = E_f(< x, x >),\ x \in X$.

• E_H

$E_H : X \times X \to L$ (a Heyting algebra) such that

(i) $E_H(< x, y >) \leq E_H(< x, x >)$,

(ii) $E_H(< x, y >) = E_H(< y, x >)$,

(iii) $E_H(< x, y >) \wedge E_H(< y, z >) \leq E_H(< x, z >)$.

An object is of the form $< X, E_H, A >$, where $A(x) = E_H(< x, x >),\ x \in X$.

• E

$E : X \times X \to L$ (a residuated lattice) such that

(i) $E(< x, y >) = E(< y, x >)$,

(ii) $E(< x, y >) * E(< y, z >) \leq E(< x, z >)$,

(iii) $E(< x, y >) * E(< x, x >) = E(< x, y >)$.

An object is of the form $< X, E, A >$, where $A(x) = E(< x, x >),\ x \in X$.

It may be remarked that E is a slightly modified form of an identity proposed in [1].

All the above indiscernibilities can be shown to satisfy the conditions (a), (b) and (c) with the respective modifications in accordance with their truth sets $(L, *, \leq)$. Also in all cases,

$$\epsilon(< x, y >) * A(x) \leq A(y),\ x, y \in X. \qquad (1)$$

The third component A is the set / fuzzy set emerging out of the indiscernibility by $A(x) = \epsilon(< x, x >),\ x \in X$. The degree of existence of an object x of the universe in the concept A is the degree to which x is indiscernible with itself.

The triple $< X, \epsilon, A >$ represents the universe X, the indiscernibility and a subset or a fuzzy subset A of X. The identity arrow in the defined category is the indiscernibility ϵ in each case. (1) shows that the subset (fuzzy or ordinary) sliced out of the universe using the indiscernibility ϵ satisfies the substitutivity principle relative to it. The converse, however, is not true. A necessary and sufficient condition when a subset satisfies the principle is given, for instance, by Pultr [10].

Theorem 2.1 *A fuzzy set $A : X \to L$ (a residuated lattice) satisfies the substitutivity principle (S) with respect to the indiscernibility ϵ if and only if* $\epsilon(< x, y >) \leq (A(x) \to A(y)) \wedge (A(y) \to A(x)),\ x, y \in X$.

Various forms of the substitutivity principle related with identities in different theories of uncertainty ,are discussed in [3].

The relationship among the defined identities is depicted in Fig. 1. An arrow indicates that the identity at its tail is a special instance of that at its head.

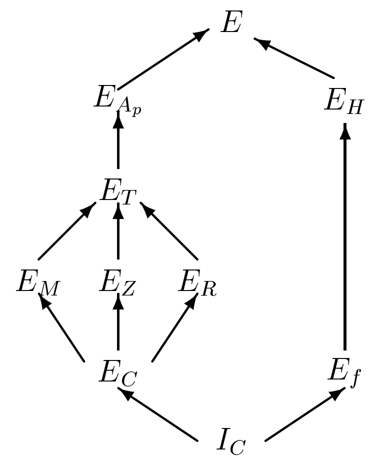

Fig. 1. The Tree of Indiscernibilities

T o accommodate rough (sub)sets in this framework, a slight modification is required. We define in the universe X, the indiscernibility (crisp equivalence) $E'_C : X \times X \to \{0,1\}$ satisfying

(i) $E'_C(< x, x >) = 1$, for all $x \in X$,

(ii) $E'_C(< x, y >) = E'_C(< y, x >)$, and

(iii) $E'_C(< x, y >) \wedge E'_C(< y, z >) \leq E'_C(< x, z >)$, for all $x, y, z \in X$.

The object in the corresponding category is $< X, E'_C, A >$, where $A(x) \leq E'_C(< x, x >)$, $x \in X$. Clearly, A is then any subset of X. Also, E'_C determines a partition of the whole domain X, and not of the subset A as in the case of E_C.

This will give rise to the category ROUGH (presented in the sequel). Here, the substitutivity principle (S) takes the form

$$E'_C(< x, y >) \wedge A(x) \leq \overline{A}(y), \ x, y \in X,$$

where $\overline{A}$ is the characteristic function of the upper approximation of A. In this case, the identity arrow is defined in a somewhat different manner.

Besides, there is another category Set(E'_C) with the identity arrow closer to what it is in the earlier cases, viz. the second component of the triple.

3 The Categories

We now present the categories that capture the indiscernibilities as identity arrows on their respective domains. As shown in Fig. 2 , the categories, in fact, preserve the relationship structure among the indiscernibilities as depicted in the previous section. An arrow in the diagram indicates that the category at its tail is identifiable with some subcategory of that at its head.

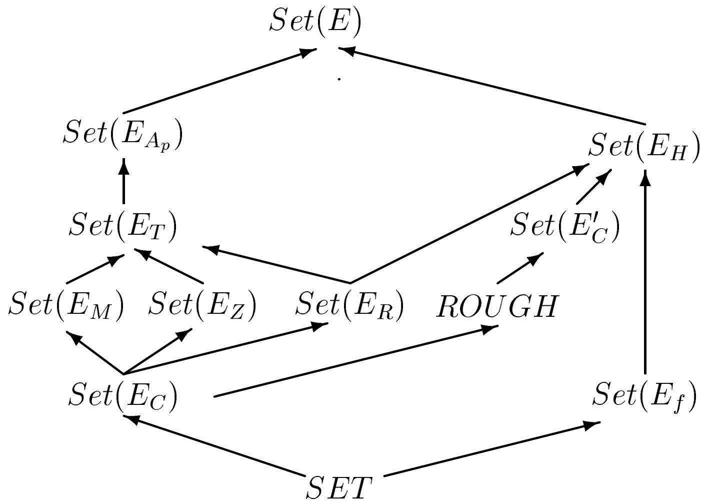

Fig. 2. The Tree of Categories

Henceforth, we shall use the same symbol to denote a set and its characteristic function. Let us begin at the bottom of the tree.

• *SET*

The most fundamental category, viz. the category SET, has objects of the form $< X, I_C, A >$, where $A(x) = I_C(x,x),\ x \in X$. A morphism with domain $< X, I_C^1, A >$ and codomain $< Y, I_C^2, B >$, is a function from A to B. Composites of morphisms are simply functional composites. The identity morphism on $< X, I_C, A >$ is I_C.

SET, of course, is an instance (and a motivation for the definition) of a "topos" [5] – the most "SET-like" category. Certain categories in the tree to be formed are topoi, at least one is not a topos, and the structures of some are yet to be studied. Some of the kinds shall be indicated as we proceed.

• $Set(E_C)$

Objects are of the form $< X, E_C, A >$, where $A(x) = E_C(x,x),\ x \in X$.
A morphism with domain $< X, E_C^1, A >$ and codomain $< Y, E_C^2, B >$, is a relation r from A to B such that
∘ if $x \in A$, there is $y \in B$ with $x\ r\ y$,
∘ if $x \in A,\ y, y' \in B,\ x\ r\ y$ and $y\ E_C^2\ y'$ in B then $x\ r\ y'$,
∘ if $x, x' \in A,\ y, y' \in B,\ x\ r\ y,\ x'\ r\ y'$ and $x\ E_C^1\ x'$ in A then $y\ E_C^2\ y'$ in B.
Composites of arrows are ordinary relational composites.
The identity on $< X, E_C, A >$ is E_C itself.

Theorem 3.1 *$Set(E_C)$ is a topos. In fact, it is equivalent to the topos SET.*

Let us look at the branch in the middle of the tree.

• *ROUGH* [2]

Objects of $ROUGH$, as mentioned in the previous section, are of the form $< X, E'_C, A >$, where $A(x) \leq E'_C(< x, x >),\ x \in X$.
For any $ROUGH$-object $< X, E'_C, A >$, let $\overline{\mathcal{A}}$ and $\underline{\mathcal{A}}$ denote the collections of equivalence classes of E'_C contained in the upper and lower approximations of A respectively. Then clearly, $\underline{\mathcal{A}} \subseteq \overline{\mathcal{A}}$.
An arrow in $ROUGH$ with domain $< X, E'^1_C, A >$ and codomain $< Y, E'^2_C, B >$ is a map $f : \overline{\mathcal{A}} \rightarrow \overline{\mathcal{B}}$ such that $f(\underline{\mathcal{A}}) \subseteq \underline{\mathcal{B}}$.
Composites of morphisms are again simply functional composites.
The identity arrow on $< X, E'_C, A >$ is the identity map $i : \overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}}$.

Theorem 3.2 *The cate goryROUGH is finitely c omplete. However, it is not a topos.*

• $Set(E'_C)$
Objects are those of $ROUGH$.
An arrow in $Set(E'_C)$ with domain $< X, E'^1_C, A >$ and codomain $< Y, E'^2_C, B >$ is a relation from A to B satisfying the same defining conditions that apply to arrows in $Set(E_C)$, with the indiscernibility replaced b y the crisp equivalence – restricted to the corresponding subsets A or B.
Composites of morphisms are ordinary relational composites.
The identity arrow on $< X, E'_C, A >$ is the restriction of E'_C to the subset A of X.

Theorem 3.3 *The c ate goriesSet(E_C) and Set(E'_C) ar e e quivalent. Hence Set(E'_C) is a topos.*

We also find the following relationship, pointing to the peculiar case when a category that is not a topos "lies between" two equivalent topoi. The notation $\mathcal{C} \subset \mathcal{D}$ for two categories $\mathcal{C},\ \mathcal{D}$ denotes that $\mathcal{C}$ is isomorphic to a subcategory of $\mathcal{D}$.

Theorem 3.4 $Set(E_C) \subset ROUGH \subset Set(E'_C)$.

• $Set(E_H)$ [7]
Objects of $Set(E_H)$ are of the form $< X, E_H, A >$, where
$A(x) = E_H(< x, x >),\ x \in X$.
A morphism with domain $< X, E^1_H, A >$ and codomain $< Y, E^2_H, B >$, is a map $g : X \times Y \rightarrow L$ satisfying, for $x, x' \in X,\ y, y' \in Y$,
∘ $g(< x, y >) \wedge E^1_H(< x, x' >) \leq g(< x', y >)$,
∘ $g(< x, y >) \wedge E^2_H(< y, y' >) \leq g(< x, y' >)$,
∘ $g(< x, y >) \wedge g(< x, y' >) \leq E^2_H(< y, y' >)$,
∘ $sup_{y \in Y}\ g(< x, y >) = A(x)$.
If $g :< X, E^1_H, A > \rightarrow < Y, E^2_H, B >$ and $h :< Y, E^2_H, B > \rightarrow < Z, E^3_H, C >$ are arrows, their composite is the map $h \circ g : X \times Z \rightarrow L$ defined as

$$h \circ g(< x, z >) = sup_{y \in Y}\ (g(< x, y >) \wedge h(< y, z >)),\ x \in X,\ z \in Z.$$

The identity arrow on $< X, E_H, A >$ is E_H itself.

Theorem 3.5 *$Set(E_H)$ is a topos. When $L \equiv \{0,1\}$ in particular, $Set(E'_C)$ is equivalent to $Set(E_H)$.*

Let us turn to the branch on the left of the tree.

• $Set(E_{Ap})$

Objects are of the form $< X, E_{Ap}, A >$, where $A(x) = E_{Ap}(< x, x >),\ x \in X$.
A morphism with domain $< X, E^1_{Ap}, A >$ and codomain $< Y, E^2_{Ap}, B >$, is a map $g : A \times B \to L$ satisfying, for $x, x' \in A,\ y, y' \in B$,

∘ $g(< x, y >) * E^1_{Ap}(< x, x' >) \leq g(< x', y >)$,
∘ $g(< x, y >) * E^2_{Ap}(< y, y' >) \leq g(< x, y' >)$,
∘ $g(< x, y >) * g(< x, y' >) \leq E^2_{Ap}(< y, y' >)$,
∘ $sup_{y \in B}\ g(< x, y >) = 1$.

If $g :< X, E^1_{Ap}, A > \to < Y, E^2_{Ap}, B >$ and $h :< Y, E^2_{Ap}, B > \to < Z, E^3_{Ap}, C >$ are arrows, their composite is the map $h \circ g : A \times C \to L$ defined as

$$h \circ g(< x, z >) = sup_{y \in B}\ (g(< x, y >) * h(< y, z >)),\ x \in A,\ z \in C.$$

The identity arrow on $< X, E_{Ap}, A >$ is E_{Ap} itself.

• $Set(E_T)$

This is the category obtained from $Set(E_{Ap})$ when $L \equiv [0, 1]$, and '*' reduces to a lower semi-continuous t-norm [6].

$Set(E_M)$, $Set(E_Z)$, $Set(E_R)$ are all instances of the category $Set(E_T)$, when the t-norm '*' denotes particular binary operations on $[0, 1]$.

Theorem 3.6 *$Set(E_C)$ is isomorphic to a subcategory of each of $Set(E_M)$, $Set(E_Z)$ and $Set(E_R)$. Also, $Set(E_Z) \subset Set(E_H)$.*

Next we come to the branch on the right of the tree.

• $Set(E_f)$ [4]

Objects are of the form $< X, E_f, A >$, where $A(x) = E_f(< x, x >),\ x \in X$.
A morphism with domain $< X, E^1_f, A >$ and codomain $< Y, E^2_f, B >$, is a map $g : X \times Y \to L$ satisfying

∘ $g(< x, y >) \leq A(x) \wedge B(y)$, for all $xinX,\ y \in Y$, and
∘ for any $x \in X$ with $A(x) > 0$, there is a unique $y \in Y$ with $B(y) > 0$, such that $g(< x, y >) = A(x)$ and $g(< x, y' >) = 0$ if $y' \neq y$ in Y.

If $g :< X, E^1_f, A > \to < Y, E^2_f, B >$ and $h :< Y, E^2_f, B > \to < Z, E^3_f, C >$ are arrows, their composite is the map $h \circ g : X \times Z \to L$ defined as

$$h \circ g(< x, z >) = sup_{y \in Y}\ (g(< x, y >) \wedge h(< y, z >)),\ x \in X,\ z \in Z.$$

The identity arrow on $< X, E_f, A >$ is E_f itself.

Theorem 3.7 *$Set(E_f)$ is isomorphic to a full subcategory of $Set(E_H)$.*

• $Set(E)$

Objects are of the form $< X, E, A >$, where $A(x) = E(< x, x >),\ x \in X$.
A morphism with domain $< X, E^1, A >$ and codomain $< Y, E^2, B >$, is a map $g : X \times Y \to L$ satisfying, for $x, x' \in X,\ y, y' \in Y$,

∘ $g(< x, y >) * E^1(< x, x >) = g(< x, y >) = g(< x, y >) * E^2(< y, y >)$,
∘ $g(< x, y >) * E^1(< x, x' >) \leq g(< x', y >)$,

$\circ\ g(<x,y>) * E^2(<y,y'>) \leq g(<x,y'>)$,
$\circ\ g(<x,y>) * g(<x,y'>) \leq E^2(<y,y'>)$,
$\circ\ sup_{y\in Y}\ g(<x,y>) = E^1(<x,x>)$.
If $g :< X, E^1, A > \to < Y, E^2, B >$ and $h :< Y, E^2, B > \to < Z, E^3, C >$ are arrows, their composite is the map $h \circ g : X \times Z \to L$ defined as

$$h \circ g(<x,z>) = sup_{y\in Y}\ (g(<x,y>) * h(<y,z>)),\ x \in X,\ z \in Z.$$

The identity arrow on $< X, E, A >$ is E itself.

It may be noted that $Set(E_H)$ is simply a special case of $Set(E)$, viz. when L is a complete Heyting algebra. We also have

Theorem 3.8 $Set(E_{Ap}) \subset Set(E)$.

Thus the tree of categories is completed at the top.

One may interpret the tree, thus formed, in the following line. The fact that the category *ROUGH* occupies a position in between the categories $Set(E_C)$ and $Set(E_H)$ could signify that *ROUGH* gives a finer description of a concept than the first and a coarser one than the second. There may be other nodes on this path, e.g. $Set(E'_C)$. These may represent the same concept with finer/coarser indiscernibility granules.

4 Conclusions

The paper is an attempt to substantiate, with the help of a category-theoretic basis, the thesis that vagueness arises out of indiscernibility. The triple <Universe,Indiscernibility,Related Concept> that is taken as an object in each of the defined categories, and in which the related concept (whether crisp or vague) is defined in terms of the indiscernibility, represents this thesis.

The study indicates a way for introducing new kinds of indiscernibilities and corresponding objects, and further, for doing mathematics therein – the last being the reason behind adopting the categorial approach. The structure of a category would entail the strength of the mathematics that can be built on its objects. So, the possibility of developing a good mathematics on an object with an indiscernibility depends upon whether the object is included in a "good" category. In this respect, the categories presented here need further investigation. In particular, the category *ROUGH* demands a detailed study of its structure, occupying as it does, a kind of unique position in the presented tree of categories – not being a topos itself, but lying between equivalent topoi.

References

[1] Banerjee, M., "A Categorial Approach to the Algebra and Logic of the Indiscernible", Ph.D. Thesis, University of Calcutta, India, 1995.

[2] Banerjee, M. and M.K. Chakraborty, *A category for rough sets*, Foundations of Computing and Decision Sciences, **18(3-4)** (1993), 167–180.

[3] Chakraborty, M.K. and E. Orłowska, *Substitutivity principles in some theories of uncertainty*, Fundamenta Informaticae, **32** (1997), 107–120.

[4] Eytan, M., *Fuzzy sets : a topos-logical view*, Fuzzy Sets and Systems, **5** (1981), 47–67.

[5] Goldblatt, R., "Topoi, the Categorial Analysis of Logic", North Holland, 1984.

[6] Gottwald, S., *Fuzzy set theory with t-norms and ϕ-operators*, In : "The Mathematics of Fuzzy Systems", Eds. A. Di Nola and A.G.S. Ventre, D. Reidel, 1984, 119–135.

[7] Higgs, D., *A categorial approach to Boolean-valued set theory*, preprint, 1973.

[8] Menger, K., *Probabilistic theory of relations*, Proc. Nat. Acad. Sci. U.S.A., **37** (1951), 178–180.

[9] Parikh, R., *The problem of vague predicates*, In : "Language, Logic and Method", Eds. R.S. Cohen and M.W. Wartofeny, D. Reidel, 1983, 241–261.

[10] Pultr, A., *Fuzziness and fuzzy equality*, In : "Aspects of Vagueness", Eds. H.J. Skala, S. Termini and E. Trillas, D. Reidel, 1984, 119–135.

[11] Ruspini, E., *Recent developments in fuzzy clustering*, In : "Fuzzy Set and Probability Theory : Recent Developments", Ed. R.R. Yager, Pergamon, 1982, 133–147.

[12] Skala, H.J., Termini, S. and E. Trillas (Eds.), "Aspects of Vagueness", D. Reidel, 1984.

[13] Tarski, A., "Introduction to Logic and to the Methodology of Deductive Sciences", Oxford, 1965.

[14] Trillas, E. and L. Valverde, *An inquiry into indistinguishability operators*, In : "Aspects of Vagueness", Eds. H.J. Skala, S. Termini and E. Trillas, D. Reidel, 1984, 231–256.

[15] Zadeh, L.A., *Similarity relations and fuzzy orderings*, Inf. Sci., **3** (1971), 177–200.