Discrete Mathematics 339 (2016) 1028–1051

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Forbidden substructure for interval digraphs/bigraphs

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ARTICLE INFO

Article history:
Received 6 June 2014
Received in revised form 30 September 2015
Accepted 1 October 2015
Available online 18 November 2015

Keywords:
Interval bigraphs/digraphs
Ferrers bigraphs/digraphs
Ferrers dimension
Associated graphs
Interior positions
ATE
Exobicliques
Edge-asteroids
Circular-arc graphs

ABSTRACT

An interval matrix is the adjacency matrix of an interval digraph or equivalently the biadjacency matrix of an interval bigraph. In this paper we investigate the forbidden substructures of an interval bigraph. Our method finds hitherto existing forbidden substructures for interval matrices, and via a more concise statement, as well as a new example showing that these substructures are not exhaustive.

1. Introduction

An *interval digraph* is a directed graph representable by assigning each vertex v an ordered pair (S_v, T_v) of closed intervals on a line so that uv is a (directed) edge if and only if S_u intersects T_v. An *interval bigraph* is a bipartite graph representable by assigning each vertex v an interval so that vertices in opposite partite sets are adjacent if and only if their intervals intersect. The *biadjacency matrix* (also called *reduced adjacency matrix*) of a bipartite graph is the submatrix of its adjacency matrix consisting of the rows indexed by one partite set and the columns indexed by the other.

Interval bigraphs and interval digraphs were introduced in [13] and [23] respectively. As observed in [18] and [25], the two concepts are equivalent. The point is that the adjacency matrix of an interval digraph is the biadjacency matrix of an interval bigraph and conversely the biadjacency matrix of an interval bigraph becomes the adjacency matrix of an interval digraph, by adding, if necessary, rows or columns of 0s to make it square.

Several characterizations of interval bigraph are known (see [14,22,23]). One characterization uses *Ferrers bigraphs* (introduced by Guttman [12] and by Riguet [20]), which are bigraphs satisfying any of the following equivalent conditions.

(A) The set of neighbors are linearly ordered by inclusion.

(B) The rows and the columns of the biadjacency matrix can be permuted (independently) so that the 1s cluster in the upper right (or, lower left) as a Ferrers diagram.

(C) The biadjacency matrix has no 2-by-2 permutation matrix as a submatrix.

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http://dx.doi.org/10.1016/j.disc.2015.10.010
0012-365X/

The condition in terms of Ferrers bigraphs similarly extends to arbitrary binary matrices. A binary matrix with no 2-by-2 permutation submatrix is a *Ferrers matrix*.

An example of a Ferrers matrix is

0	0	0	0	0	0
1	1	0	0	0	0
1	1	1	1	0	0
1	1	1	1	1	0

Theorem A characterizes interval bigraphs. We forbid multiple edges, so a bigraph is *complete* if every entry in its biadjacency matrix is 1. A *zero-partition* of a binary matrix is a coloring of each 0 with R or C in such a way that every R has only 0s colored R to its right and every C has only 0s colored C below it. A matrix that admits a zero partition after suitable row and column permutations is *zero-partitionable*.

Theorem A (*[23]*)**.** *The following conditions are equivalent*

(i) *B is an interval bigraph.*
(ii) *The rows and columns of its biadjacency matrix can be independently permuted such that the matrix admits a zero partition.*
(iii) *B is the intersection of two Ferrers bigraphs whose union is complete.*

We defined zero-partitionability for general binary matrices. Adding rows or columns of 0s does not affect zero-partitionability, so this property of the biadjacency matrix also characterizes interval bigraphs (our bigraphs are simple). We discuss interval bigraphs wholly in terms of the binary biadjacency matrices, so we call such a matrix an *interval matrix*. Our aim is to provide forbidden substructures of interval matrices.

In the present paper, we obtain five small matrix configurations, each of which give rise to a set of forbidden matrices for interval matrices. Interestingly, along the way, all the forbidden subgraphs for interval bigraphs, so far known, are subsumed in our approach; in addition, new forbidden substructures are obtained.

More results on interval digraphs/bigraphs and related topics appear in [2,16,25,27]. For a remarkably fine survey of the topic of the interval bigraphs and related areas, see Brown [1].

1.1. Background materials

A binary matrix with exactly one 0 is a Ferrers matrix. Hence every bigraph B is the intersection of finitely many Ferrers bigraphs. The minimum number of Ferrers bigraphs whose intersection is B is the *Ferrers dimension* of the bigraph B, written $fdim(B)$. More generally, the Ferrers dimension of a binary matrix A is the minimum number of Ferrers matrices whose intersection is A.

Ferrers bigraphs and Ferrers dimension were extensively studied, in the language of digraphs, by Cogis in [4–6]. For short proofs of equivalence of various characterizations of Ferrers bigraphs, see [27]. Mahadev and Peled in their book on Threshold Graphs [17] devoted half a chapter on Ferrers digraphs. Golumbic and Trenk [11] found it important to write a section on Ferrers dimension two in their book on Tolerance Graphs. Ferrers bigraphs and Ferrers dimension are also called (bipartite) chain graphs and chain dimension respectively. The term chain graph was first used by Yannakakis [28] in his paper on partial order dimension problem.

By Theorem A, every interval bigraph has Ferrers dimension at most 2. The converse is false [23]. Bigraphs with Ferrers dimension 2 were characterized by Cogis [4] and by Doignon, Ducamp, and Falmagne [9]. Cogis [4] introduced the *associated graph* $H(B)$ for a bigraph B. Its vertices are the 0s of the biadjacency matrix $A(B)$ of B, with two such vertices adjacent in $H(B)$ if and only if they are the 0s of a 2-by-2 permutation submatrix of $A(B)$. Such a submatrix is an *obstruction* and two zeros form obstructions with one another. Cogis [4] proved the following theorem.

Theorem B. *The Ferrers dimension of a bipartite graph B is at most 2 if and only if* $H(B)$ *is bipartite.*

This yields a fast algorithm for testing $fdim(B) \leq 2$.

The associated graph $H(A)$ is defined in the same way for a general binary matrix A. Entries in a row or column of 0s become isolated vertices in $H(A)$, which does not affect whether $H(A)$ is bipartite. Thus the characterization of $fdim(B) \leq 2$ extends to binary matrices. For more equivalent notions of Ferrers dimension two, and other related results, see [3,21,23,24].

Partial results on forbidden configuration for interval matrices were obtained in [7]. In Section 2, we introduce the necessary concepts and state our forbidden substructures, which strengthens the results of [7].

Müller [18] also considered this from a different approach. For this he defined asteroidal triple of edges (ATE) in a bigraph as follows : Three edges e_1, e_2, e_3 form an *asteroidal triple of edges* if there is a path joining two edges that avoids the neighbors of the third. Müller also proved that an interval bigraph is ATE-free. Das and Sen [8] showed that a bigraph B having Ferrers dimension less than equal to 2 is also ATE-free, which strengthens the Müller's result.

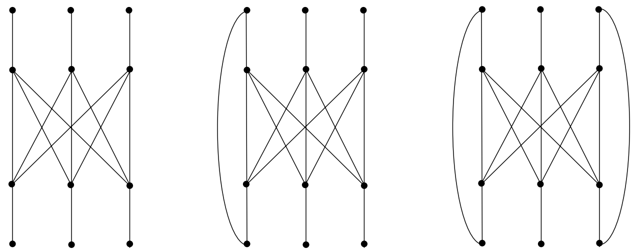

Fig. 1. The insects.

Circular arc graph is the intersection graph of a family of circular arcs of a host circle. If the vertices of the circular arc graph can be covered by two cliques then it is a *two-clique circular arc graph*. Trotter and Moore [26] characterized two-clique circular arc graph in terms of forbidden subgraphs. Their list of forbidden subgraphs contain several infinite families which are not described here (see also Brown [1]).

To unify these infinite families into a more compact form, Feder et al. [10] introduced the notion of edge-asteroid. An *edge-asteroid* is a set of edges $e_0, e_1, \dots, e_{2k}$ such that, for $i = 0, 1, \dots, 2k$, there is a path joining e_i and e_{i+1} and containing both e_i and e_{i+1}, that avoids the neighbors of e_{i+k+1}; the subscript addition is modulo $2k + 1$. In the same paper they characterized two-clique circular arc graphs as follows: A two-clique graph is a circular arc graph if and only if its complement does not contain induced cycles of length at least six and an edge-asteroid.

Two-clique circular arc graphs have also been characterized in [15] and [21]. It is proved that a bipartite graph B is of Ferrers dimension at most two if and only if its complement is a two clique circular arc graph. Now the results of Trotter and Moore [26] and Huang [15] combined together add to another characterization of bigraphs of Ferrers dimension at most two via six infinite families and three separate graphs. It is interesting to note that this is a nice example of a classical result viewed from a new perspective.

Müller [18] introduced a family of graphs called *insects* and conjectured that a bipartite graph is an interval bigraph if and only if it contains no ATE and no insects as an induced subgraph, which was disproved by Hell and Huang [14]. In the same paper, they also showed that the set of insects can be reduced to only three graphs they called "essential insects" [Fig. 1 in Section 3.1].

Hell and Huang [14] defined a bigraph called *exobicliques* as follows: An *exobiclique* in a bipartite graph B, with parts X and Y, is a biclique (i.e. a complete bipartite graph) with non empty parts $M \subseteq X$ and $N \subseteq Y$ such that each of $X - M$ and $Y - N$ contain three vertices with incomparable neighborhoods in the biclique. (Two neighborhoods are incomparable if one does not contain the other.) Then they proved that a bipartite graph containing an exobiclique is not an interval bigraph. Obviously the graphs of Fig. 1 are exobicliques. In the same paper they also provided another class of six bipartite graphs called *bugs* [Fig. 2 in Section 3.1] and called a graph an *extended insect* if it is either an insect or a bug. Finally, they showed that a bipartite graph contains an induced exobiclique if and only if it contains an induced extended insect. It might be worth mentioning that Müller's [18] ATEs are covered by the graphs of Trotter and Moore [26] and it is the insects of Müller that moved the research forward leading to the exobicliques of Hell and Huang [14].

To summarize, the forbidden subgraphs of an interval bigraph presently known are

(1) Induced cycles of length at least six or edge-asteroids.

(2) Extended insects (i.e. insects and bugs).

It is clear that the bipartite graphs of class (1) are of Ferrers dimension greater than 2 and accordingly these graphs are not interval bigraphs. It will be observed in Section 3.1 of the present paper that the class of "extended insects" of [14] is equivalent to the configuration M^0 of Section 3.1. (Configuration M^0 is obtained from the forbidden substructure F^0.)

In this paper, we start with a matrix of Ferrers dimension two, obtain the associated graph and use the notion of interior positions in its bicoloration to find out the isolated vertices which are forced to receive two colors prohibiting a zero-partition of the matrix. In this effort we have obtained the configuration F^0 and four other forbidden configurations which do not contain any of the forbidden subgraphs so far known as induced subgraphs.

2. Bicolorations and interior positions

Since the result of Cogis is valid for all binary matrices, we henceforth drop the notations for bigraphs and discuss simply a binary matrix A and the associated graph $H(A)$. Cogis [4] proved that $fdim(A) \leq 2$ if and only if $H(A)$ is bipartite. The graph $H(A)$ may be disconnected and may have isolated vertices corresponding to 0s in A that belong to no obstruction. Deleting the isolated vertices yields a graph H^b called the *bare graph* associated with A (see [9]).

Let A be a binary matrix with Ferrers dimension 2, so $H(A)$ is bipartite. Let $\mathbf{I}$ denote the set of all isolated vertices in $H(A)$. Let $(\mathbf{R}, \mathbf{C})$ denote a bicoloration of $H(A)$, where a *bicoloration* of a graph is an ordered pair of (possibly empty) stable sets whose union is the vertex set of H^b. Let $H_1, \dots, H_p$ be the components of H^b, with (R_i, C_i) denoting a bicoloration of H_i. Note that $(\bigcup R_i, \bigcup C_i)$ is a bicoloration of H^b. There are a total of 2^p bicolorations of H^b. We describe a bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b by writing R or C in A in place of the 0 for each vertex of H^b. Similarly, I, R_i, C_i designate the positions in $\mathbf{I}, \mathbf{R_i}, \mathbf{C_i}$ respectively.

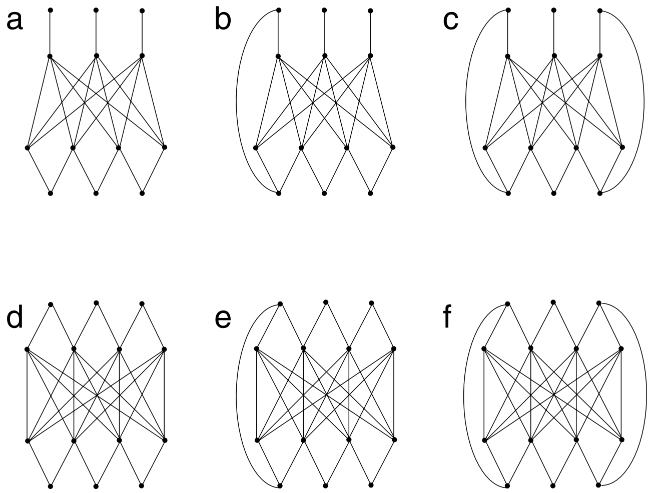

Fig. 2. The bugs.

Cogis showed that if $H(A)$ is bipartite, then H^b has a bicoloration $(\mathbf{R}, \mathbf{C})$ such that in A, each of $\boldsymbol{R} \bigcup \boldsymbol{I}$ and $\boldsymbol{C} \bigcup \boldsymbol{I}$ occupies the position of the 0s in a Ferrers matrix. Such a bicoloration is called a *satisfactory bicoloration*. It expresses A as the intersection of two Ferrers matrices, but for A to be an interval matrix this is not enough. The condition (iii) of Theorem A demands that the union of the two matrices have $1's$ in all its positions.

By Theorem A, a binary matrix is an interval matrix if and only if it is zero partitionable. Vertices in opposite partite sets of a component of H^b must have different colors in a zero partition of the corresponding matrix A. Thus A is an interval matrix if and only if some R, C-bicoloration of H^b extends to $\mathbf{I}$ to complete a zero partition of A (after suitable permutations).

Given a particular bicoloration, some positions in $\mathbf{I}$ are immediately forced to receive R or C. This leads to the notion of "interior positions" introduced in [7]. We extend the definition from [7] to a more general context.

Definition 1. *A configuration or a substructure* is a matrix with entries in $\{0, 1, -\}$ (where "-" means "0 or 1") plus optional labels on 0s to indicate membership in various sets; permutations of rows and columns are allowed. With respect to a bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b, the sets $\mathbf{I_R}$ and $\mathbf{I_C}$ of R-interior and C-interior positions are the subsets of $\mathbf{I}$ defined as follows: a position in $\mathbf{I}$ belongs to $\mathbf{I_R}$ [respectively to $\mathbf{I_C}$] if it lies in a 2-by-2 configuration where the opposite corner is a 1 and the other diagonal is in $\mathbf{R}$ [respectively, in $\mathbf{C}$].

The set $\mathbf{I}$ is determined by the matrix A and is the same over all bicolorations of H^b, but $\mathbf{I_R}$ and $\mathbf{I_C}$ depend on the bicoloration. Interior positions were defined in [7] only for satisfactory bicolorations, emphasizing row and column permutations that express $\mathbf{R}\bigcup\mathbf{I}$ and $\mathbf{C}\bigcup\mathbf{I}$ as Ferrers matrices. Definition 1 extends the concept to all bicolorations and ignores the permutations used to record the matrix.

The sets $\mathbf{I_R}$ and $\mathbf{I_C}$ may intersect, and they may not exhaust $\mathbf{I}$. Our approach is motivated by the main result of [7], which we state here.

Theorem C *([7]). Let D be a digraph with Ferrers dimension 2.*
(A) If $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ for some satisfactory bicoloration of H^b, then $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ for every satisfactory bicoloration.
(B) If D is an interval digraph, then $\mathbf{I_R} \bigcap \mathbf{I_C} = \emptyset$ for every satisfactory bicoloration of H^b, but the converse is not true.

In the present paper we extend these conclusions to all bicolorations, dropping the word '*satisfactory*' from its appearances in Theorem C. Thus disjointness of $\mathbf{I_R}$ and $\mathbf{I_C}$ is independent of the bicoloration of H^b. Our results are motivated by Theorem C but do not require it; we use only Theorem A and the fact that H^b is bipartite when Ferrers dimension of the binary matrix is 2.

The relevance of disjointness of $\mathbf{R}$ and $\mathbf{C}$ is that a bicoloration of H^b with $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ cannot extend to a zero-partition of A, since a zero partition where all of $\mathbf{R}$ receives R and all of $\mathbf{C}$ receives C must also give R to all of $\mathbf{I_R}$ and C to all of $\mathbf{I_c}$.

Recall that "- " means "either 0 or 1". We will study the following configurations.

Definition 2. With respect to a bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b, the *configurations* F^0, F^1, F^2, F^3 and F^4 are

$$F^0 = \begin{pmatrix} - & 1 & R \\ 1 & - & C \\ C & R & I \end{pmatrix} \qquad F^1 = \begin{pmatrix} 1 & 1 & I \\ R & - & I_C \\ - & C & I_R \end{pmatrix} \qquad F^2 = \begin{pmatrix} - & - & 1 & C \\ - & I_C & I & I_R \\ 1 & I & 1 & - \\ C & I_R & - & - \end{pmatrix}$$

$$F^3=\begin{pmatrix}1 & - & I & - & 1\\ C & - & I_R & - & -\\ - & - & I & C & R\\ - & 1 & I & 1 & -\\ - & R & I_C & - & -\end{pmatrix}\qquad F^4=\begin{pmatrix}I & - & R & I & I_C\\ - & - & - & 1 & R\\ C & - & - & 1 & -\\ I & 1 & 1 & 1 & -\\ I_R & C & - & - & -\end{pmatrix}.$$

Here we can interchange the labels R and C.

The presence of configuration F^0 is simply the statement that $\mathbf{I_R}\bigcap\mathbf{I_C}\neq\emptyset$ (see [7]). The possibility that a non-interval bigraph may have a bicoloration with $\mathbf{I_R}\bigcap\mathbf{I_C}=\emptyset$ prevents F^0 alone from characterizing interval matrices (among matrices with Ferrers dimension 2). Our main result is the following.

Theorem 3. *If A is a binary matrix with $fdim(A)=2$, and* **(R, C)** *is a bicoloration of H^b giving rise to one of F^0, F^1, F^2, F^3, F^4 or their transposes, then* **(R, C)** *does not extend to a zero partition of A.*

The main work in proving our condition is showing that if one of F^0, F^1, F^2, F^3, F^4 occurs for some bicoloration of H^b, then it occurs for all bicolorations. In the next section, we show that when $fdim(A)=2$, configurations F^0, F^1, F^2, F^3, F^4, force the occurrence of special larger configurations M^0, M^1, M^2, M^3 and M^4 in A, respectively. These larger configurations are used in turn to show that all bicolorations contain the original forbidden configuration. For each of F^0, F^1, F^2, F^3, F^4, this last step uses the lemma below, which also serves as an introduction to our techniques.

Just as "—" can mean either 0 or 1, when we write "0" in a configuration it can mean any of R, C, I in a bicoloration. Our arguments about configurations require consideration of subconfigurations. We specify a subconfiguration by listing the names of its rows and columns, separated by a vertical bar. For example, in F^0 we have $(12|23)=\begin{pmatrix}1 & R\\ - & C\end{pmatrix}$ if the rows and columns are named 1, 2, 3 in order.

Lemma 4. *If A is a binary matrix with $fdim(A)\le 2$, and a bicoloration* **(R, C)** *of H^b contains the configuration P indexed as shown below, then position (i,j) in A belongs to $\mathbf{I_C}$. The statement remains true under transposition and interchange of the labels R and C.*

$$P=\begin{array}{c}\\ a\\ b\\ i\end{array}\begin{array}{c}\begin{array}{ccc}c & d & j\end{array}\\ \begin{pmatrix}- & 1 & I/R\\ 1 & 1 & C\\ C & R & -\end{pmatrix}\end{array}.$$

Proof. The reasons invoked remain true under transposition or under the complete interchange of R and C, so we need only consider P itself.

Since the 0s in an obstruction have different colors, $(b,i|c,j)\Rightarrow P_{i,j}=0$. By $(b,i|c,j)$, it now suffices to show that $(i,j)\in\mathbf{I}$. Suppose otherwise, with (i,j) forming an obstruction with some position (r,s). From this we derive restrictions on the enlargement of P to include row r and column s as shown below, and these lead to a contradiction.

$$\begin{array}{c}\\ a\\ b\\ i\\ r\end{array}\begin{array}{c}\begin{array}{cccc}c & d & j & s\end{array}\\ \begin{pmatrix}- & 1 & I/R & -\\ 1 & 1 & C & 1\\ C & R & 0 & 1\\ - & R & 1 & R\end{pmatrix}\end{array}.$$

Since $P_{r,j}=P_{i,s}=1$, we have r does not belong to $\{a,b,i\}$ and s does not belong to $\{c,d,j\}$. Since **(R, C)** is a bicoloration, a position cannot form obstructions with positions of both colors, so $(b,i|c,d,s)\Rightarrow P_{b,s}=1$. An obstruction uses both the colors, so $(b,r|j,s)\Rightarrow(r,s)\in\mathbf{R}$. Using the same reason, $(i,r|d,s)\Rightarrow P_{r,d}=0$, and then $(b,r|d,j)\Rightarrow(r,d)\in\mathbf{R}$, and then $(a,r|d,j)$ exhibits a contradiction. ■

3. Five forbidden configurations

This is the main section of the paper. Here we discuss each of the configurations F^i ($i=0,1,\ldots,4$) and see how they are developed into larger configurations M^i ($i=0,1,\ldots,4$). Lastly we prove the main result stated in Theorem 3 of Section 2.

Since sets in an interval representation can be repeated, a bigraph is an interval bigraph if and only if the matrix obtained from its biadjacency matrix by iteratively deleting a repeated row or column is an interval matrix. The resulting matrix is the *core* of the original matrix. Two rows (or columns) are *compatible* if they become identical for some instance of values in the "—" positions; otherwise they are *incompatible* and must be distinct. When we refer to "*a core of a configuration M*", we mean a matrix obtained by appropriately assigning 0 or 1 to the undetermined positions in M and then taking the core of the resulting matrix. Note that while the core of a binary matrix is unique, a configuration may have more than one core matrix.

3.1. F^0 and its enlarged form M^0

In the configuration M^0 below, rows 1 and 2 are incompatible, but rows 2 and 3 are compatible. If the "—" entries in M^0 are all 0, then column 2 and 3 are equal, as are rows 2 and 3. The core of M^0 then has six rows and columns. If this is the full matrix A, then $H = 6K_2 + 9K_1$, with a satisfactory bicoloration shown below.

$$
M^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & - & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & - \\
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & - & 0 & 0 & - & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & - & 1 & 0 & 0 & -
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{c} \\ 1 \\ 2.3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2.3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & R_1 & R_2 \\
1 & 1 & 1 & C_1 & 1 & R_3 \\
1 & 1 & 1 & C_2 & C_3 & 1 \\
1 & R_4 & R_5 & I & I & I \\
C_4 & 1 & R_6 & I & I & I \\
C_5 & C_6 & 1 & I & I & I
\end{pmatrix}
\end{array}.
$$

However, if the two unspecified diagonal entries of M^0 are 1 and other unspecified entries are 0, then associated graph H consists of the isolated vertex at $(6, 6)$ position plus one large component with 22 vertices. In both the cases $(1, 4, 6|1, 4, 6)$ is an instance of F^0, and position $(6, 6)$ belongs to $\mathbf{I_R} \bigcap \mathbf{I_C}$. We show in Lemma 5 that matrices such as these must arise in A when $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ for some bicoloration.

Now we shall show that the biadjacency matrices for all the insects and bugs can be obtained from the matrix M^0. In M^0 if the entries $(2, 5)$ and $(3, 7)$ are 0 then the rows 2 and 3 are equal. Similarly, if the entries $(5, 2)$ and $(7, 3)$ are 0 then the columns 2 and 3 are equal. Then the core of M^0 is the following matrix.

$$
\begin{array}{c} \\ 1 \\ 2.3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2.3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & - & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -
\end{pmatrix}
\end{array}.
$$

Now if the entries $(5, 5)$ and $(7, 7)$ are 0 then we have the biadjacency matrix of the first insect of Fig. 1. Next if one of the "—" entry is 1 and the other is 0 then we have the biadjacency matrix of the second insect. Lastly, if both the "—" entries are 1 then we have the biadjacency matrix of the third insect. It may be noted that the first insect is the smallest non-interval bigraph with Ferrers dimension 2.

Next if the entries $(5, 2)$ and $(7, 3)$ are 0 then the columns 2 and 3 are equal and the core of M^0 is the following matrix.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2.3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & - & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & - \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & - & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -
\end{pmatrix}
\end{array}.
$$

Now if the entries $(2, 5)$ and $(3, 7)$ are 1 and other "—" entries are 0 then we have the biadjacency matrix of the 1st bug of Fig. 2. Similarly in addition if one of the $(5, 5)$ and $(7, 7)$ entry is 1 and other is 0 we have the biadjacency matrix of second bug. And in the case when both the entries $(5, 5)$ and $(7, 7)$ are 1 we have the biadjacency matrix of the 3rd bug.

Lastly in M^0 if the entries $(2, 5)$, $(3, 7)$, $(5, 2)$, and $(7, 3)$ are 1 and the remaining "—" entries are 0 then we have the biadjacency matrix of the 4th bug. Similarly we have the biadjacency matrix of the 5th bug when one of the $(5, 5)$ or $(7, 7)$ entry is 0 and all other "—" entries are 1. We have the biadjacency matrix of last bug when all the "—" entries of M^0 are 1.

We therefore have that the biadjacency matrices of the *extended insects* of Hell and Huang [14] are represented via the configuration M^0.

Lemma 5. *Let A be a binary matrix with fdim*$(A) = 2$. *If* $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ *for some bicoloration of* H^b, *then A contains a core of* M^0 *as a submatrix.*

Proof. Consider a bicoloration of H^b such that $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$. As remarked in Section 2, a position in A is both R-interior and C-interior if and only if it belongs to 2-by-2 configurations that together form F^0. Below we index the rows and columns of F^0 to agree with those in the copy of M^0 we produce.

$$F^0 = \begin{array}{c} \\ 1 \\ 4 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 4 & 6 \end{array} \\ \begin{pmatrix} - & 1 & R \\ 1 & - & C \\ C & R & I \end{pmatrix} \end{array}.$$

The obstructions required to produce $F^0_{1,6} = R$ and $F^0_{4,6} = C$ are

$$\begin{array}{c} \\ 1 \\ - \end{array}\begin{array}{c} \begin{array}{cc} - & 6 \end{array} \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array} \qquad \textit{and} \qquad \begin{array}{c} \\ - \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} 6 & - \end{array} \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}$$

respectively, where we do not yet know the indices of rows and columns labeled "−". In each configuration, this row has a 1 in column 6 and hence is not in F^0. We call it row 3 in the first configuration and row 2 in the second, although they might be the same row in A. We let the column additions be column 5 in the first configuration and column 7 in the second; by construction they are distinct from column 6. We will see that these columns are incompatible with each other and with the columns 1 and 4. The combined configuration appeared below. We call the developing configuration F.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} - & 1 & 1 & R & - \\ - & - & - & 1 & R \\ - & - & C & 1 & - \\ 1 & - & - & C & 1 \\ C & R & - & I & - \end{pmatrix} \end{array}.$$

An obstruction cannot have two positions in **R** or two in **C** or one in **I**. Hence $F_{1,7} = F_{4,5} = F_{6,5} = F_{6,7} = 0$, as shown on the right above, using

$$(1, 2|6, 7) \Rightarrow F_{1,7} = 0 \qquad (3, 4|5, 6) \Rightarrow F_{4,5} = 0.$$
$$(2, 6|6, 7) \Rightarrow F_{6,7} = 0 \qquad (3, 6|5, 6) \Rightarrow F_{6,5} = 0.$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} - & 1 & 1 & R & 0 \\ - & - & - & 1 & R \\ - & - & C & 1 & - \\ 1 & - & 0 & C & 1 \\ C & R & 0 & I & 0 \end{pmatrix} \end{array}.$$

Transposing F^0 and interchanging the labels R and C yields F^0 again. Hence we can repeat the argument above for the obstructions yielding $F^0_{6,1} = C$ and $F^0_{6,4} = R$. This introduces columns 2 and 3 (possibly equal, but different from columns 1, 4, 6 due to row 6) and rows 5 and 7 (distinct and different from row 6). The resulting matrix, after the implications symmetric to those displayed above, is the first matrix below.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} - & - & - & 1 & 1 & R & 0 \\ - & - & - & - & - & 1 & R \\ - & - & - & - & C & 1 & - \\ 1 & - & - & - & 0 & C & 1 \\ 1 & - & R & 0 & - & 0 & - \\ C & 1 & 1 & R & 0 & I & 0 \\ 0 & C & - & 1 & - & 0 & - \end{pmatrix} \end{array}.$$

Using $F_{6,6}$ with 1s in row 6 and column 6, we obtain $F_{2,2} = F_{2,3} = F_{3,2} = F_{3,3} = 1$. This makes rows 5 and 7 incompatible with rows 2 and 3, and similarly for columns. Also now $(2, 5|3, 7) \Rightarrow (F_{5,7} = 0)$ and $(3, 7|2, 5) \Rightarrow (F_{7,5} = 0)$.

An entry that would form an obstruction both with an R and with a C if it were 0 must be a 1. This yields the remaining desired 1s. We list these implications as triples of positions: (location of R, location of C, location of new 1), using $i.j$ to denote (i, j) for clarity.

(5.3, 4.6, 2.1) (5.3, 4.6, 3.1) (6.4, 6.1, 4.2) (6.4, 6.1, 4.3) (5.3, 3.5, 1.1).
(6.4, 3.5, 1.2) (6.4, 3.5, 1.3) (1.6, 4.6, 2.4) (1.6, 4.6, 3.4) (2.7, 7.2, 4.4).

$$X^0 = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array}\begin{array}{c} \begin{array}{ccccccc} 1&2&3&4&5&6&7 \end{array} \\ \begin{pmatrix} 1&1&1&1&1&R&0\\ 1&1&1&1&-&1&R\\ 1&1&1&1&C&1&-\\ 1&1&1&1&0&C&1\\ 1&-&R&0&-&0&0\\ C&1&1&R&0&I&0\\ 0&C&-&1&0&0&- \end{pmatrix} \end{array}.$$

The 1s placed in the first row distinguish it from row 5 and 7 (similarly for columns), and then all rows and columns of F^0 must be distinct except for the possible collapsing of 2 with 3. Replacing all of $\{R, C, I\}$ with 0 now yields the claimed configuration M^0. ■

The proof above yields a stronger statement. A core of M^0 must occur when $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ and also there must be such a configuration colored as in the last matrix X^0 shown for F^0.

Proposition 6. *Let A be a binary matrix with fdim(A) = 2. If $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ for some bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b, then $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ for every bicoloration of H^b.*

Proof. By Lemma 5, we obtain M^0 or its core as a submatrix of A, with $(6, 6) \in \mathbf{I_R} \bigcap \mathbf{I_C}$. As noted above, we in fact have the configuration F_0 (or its core) reached at the end of the proof of Lemma 5. Since $(1, 4|5, 7)$ and $(5, 7|1, 4)$ are obstructions, positions $(1, 7), (4, 5), (5, 4)$, and $(7, 1)$ all are vertices in H^b. We start with the configuration shown below, except that R_2, C_2 may be switched, and R_5, C_5 may be switched.

We use indices to designate six obstructions in up to six components, but some of these components may coalesce due to positions (5, 5) or (7, 7) or positions outside M^0. Partition M^0 into four blocks UL (upper left), UR (upper right), LL (lower left), LR (lower right), as shown below. We ignore the unknown entries in UR and LL; all mentioned 0 or R or C in these submatrices refer to the known 0s.

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array}\begin{array}{c} \begin{array}{ccccccc} 1&2&3&4&5&6&7 \end{array} \\ \left(\begin{array}{cccc|ccc} 1&1&1&1&1&R_1&R_2\\ 1&1&1&1&-&1&R_3\\ 1&1&1&1&C_1&1&-\\ 1&1&1&1&C_2&C_3&1\\ \hline 1&-&R_4&R_5&-&0&0\\ C_4&1&1&R_6&0&I&0\\ C_5&C_6&-&1&0&0&- \end{array}\right) \end{array}.$$

The two 0 s in an obstruction have opposite colors in every bicoloration. Hence the labeled positions in UR and LL always have three Rs and three Cs. Since there are two of these six labels in each of the three columns of UR, always an odd number of columns in UR have both an R and a C in this set. Similarly, there are an odd number of such rows in LL.

Let i and j be a row in LL and a column in UR, respectively, that have both colors on their labeled 0s (there are an odd number of choices for each of these indices). It suffices to show that $(i, j) \in \mathbf{I}$. Let a and b be the rows having the R and C in column j of UR, and let c and d be the columns having the C and R in row i of LL. Now $(a, b, i|c, d, j)$ is an instance of configuration P in Lemma 4. By Lemma 4, $(i, j) \in \mathbf{I}$. ■

We now have the extension of most of Theorem C.

Theorem 7. *If B is a bigraph with Ferrers dimension 2, then $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ for some bicoloration of H^b if and only if this is true for every bicoloration of H^b. Also, if some bicoloration of H^b produces F^0, then B is not an interval bigraph.*

Proof. The first statement is immediate from Proposition 6. The second uses the first and the observation that a bicoloration containing F^0, is not extendible to a zero partition because presence of the configuration F^0 is equivalent to $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$. The result then follows from the first statement and Theorem C of Section 2. ■

3.2. Configurations F^1 and its enlarged form M^1

Configuration F^0 forces M^0; here we obtain another large configuration M^1 from F^1 when $\mathbf{I_R} \bigcap \mathbf{I_C} = \emptyset$.

Let M^1 be the configuration below, where "—" may denote 0 or 1. There are choices for "—" entries that yield Ferrers dimension at least 3. When M^1 arises in a matrix with Ferrers dimension 2 such instances are forbidden.

$$M^1 = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11 \end{array}\begin{array}{c} \begin{array}{cccccccccccc} 1&2&3&4&5&6&7&8&9&10&11 \end{array} \\ \left(\begin{array}{cccc|cccc|ccc} 1&1&1&1&1&1&1&1&0&0&0\\ 1&1&1&1&1&1&1&1&-&1&0\\ 1&1&1&1&1&1&1&1&0&1&-\\ \hline 1&-&0&0&-&-&-&-&-&0&0\\ 0&1&1&0&0&-&-&-&0&0&0\\ 1&1&1&1&0&-&-&-&0&0&1\\ 0&0&-&1&0&-&-&-&0&0&-\\ \hline -&-&-&0&1&-&0&0&-&0&0\\ -&-&-&0&0&1&1&0&0&0&0\\ -&-&-&0&1&1&1&1&1&0&0\\ -&-&-&-&0&0&-&1&0&0&- \end{array}\right) \end{array}.$$

When M^1 is all of A and every "—" in it becomes 0, there is a bicoloration of an induced subgraph of H^b as shown in the matrix below named X^1 (vertices labeled 0 or "—" here are dropped to form the subgraph). The subgraph consists of six isolated edges and one large component. No assignment of values to "—" entries can create an obstruction involving any of $\{X^1_{1,10}, X^1_{5,10}, X^1_{9,10}\}$. Thus $(1, 5, 9|4, 5, 10)$ is an instance of F^1.

$$X^1 = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11 \end{array}\begin{array}{c} \begin{array}{cccccccccccc} 1&2&3&4&5&6&7&8&9&10&11 \end{array} \\ \left(\begin{array}{cccc|cccc|ccc} 1&1&1&1&1&1&1&1&C_7&I&R_7\\ 1&1&1&1&1&1&1&1&-&1&R_7\\ 1&1&1&1&1&1&1&1&C_7&1&-\\ \hline 1&-&R_1&R_2&-&-&-&-&-&0&0\\ C_1&1&1&R_3&0&-&-&-&0&I_C&0\\ 1&1&1&1&C_7&-&-&-&C_7&C_7&1\\ C_2&C_3&-&1&C_7&-&-&-&C_7&0&-\\ \hline -&-&-&R_7&1&-&R_4&R_5&-&0&R_7\\ -&-&-&0&C_4&1&1&R_6&0&I_R&0\\ -&-&-&R_7&1&1&1&1&1&R_7&R_7\\ -&-&-&-&C_5&C_6&-&1&0&0&- \end{array}\right) \end{array}.$$

The large component H_7 in this bicoloration cannot be destroyed. Choices of "—" values can create additional edges in H but do not disturb the obstructions that put all of H_7 in the same component. Some of $H_1, \ldots, H_6$ can coalesce or join H_7.

Among columns each of $\{1, 2, 3\}$ is compatible with each of $\{6, 7, 8\}$. Also 2 and 3 are compatible, 4 is compatible with each of $\{8, 9, 10, 11\}$, 11 is compatible with each of $\{4, 5, 6, 7\}$, and 5 is compatible with 9.

When M^1 arises in a bicoloration of H^b with $\mathbf{I_R} \bigcap \mathbf{I_C} = \emptyset$, choices for "—" entries that put an element of $\mathbf{I_R}$ or $\mathbf{I_C}$ also into the other are forbidden. For example $M^1_{5,8} \neq R$ and $M^1_{9,1} \neq C$ under the requirement M^1 is all of A. This also makes rows 5 and 9 incompatible.

Lemma 8. *Let A be a binary matrix with $fdim(A) = 2$. If F^1 appears in a bicoloration of H^b with $\mathbf{I_R} \bigcap \mathbf{I_C} = \emptyset$, then A contains a core of M^1 as a submatrix.*

Proof. As in Lemma 5 we index rows and columns in F^1 by their eventual destinations in M^1. We start with

$$F^1 = \begin{array}{c} \\ 1\\5\\9 \end{array}\begin{array}{c} \begin{array}{ccc} 4&5&10 \end{array} \\ \left(\begin{array}{ccc} 1&1&I\\ R&-&I_C\\ -&C&I_R \end{array}\right) \end{array}.$$

The labels I_R and I_C in column 10 require the configurations

$$\begin{array}{c} \\ 9\\ - \end{array}\begin{array}{c} \begin{array}{cc} -&10 \end{array} \\ \left(\begin{array}{cc} R&I_R\\ 1&R \end{array}\right) \end{array} \quad \textit{and} \quad \begin{array}{c} \\ 5\\ - \end{array}\begin{array}{c} \begin{array}{cc} -&10 \end{array} \\ \left(\begin{array}{cc} C&I_C\\ 1&C \end{array}\right) \end{array}.$$

Since F^1 has no room for the labels R and C required in column 10, the added rows are distinct and not in $\{1, 5, 9\}$; we call them rows 6 and 10, respectively. Let X denote the developing configuration. Since $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ and $X_{10,10} = R$, we have

$(5, 10|4, 10) \Rightarrow (X_{10,4} = 0)$. Similarly, $(6, 9|5, 10) \Rightarrow (X_{6,5} = 0)$. (We use the hypothesis $\mathbf{I_R} \bigcap \mathbf{I_C} \neq \emptyset$ only here.) With this, the new columns are different from all the original columns, although they may be compatible with each other. We call them 8 and 1 respectively, and X is now as below.

$$\begin{array}{c} \\ 1 \\ 5 \\ 6 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 5 & 8 & 10 \end{array} \\ \begin{pmatrix} - & 1 & 1 & - & I \\ C & R & - & - & I_C \\ 1 & - & 0 & - & C \\ - & - & C & R & I_R \\ - & 0 & - & 1 & R \end{pmatrix} \end{array}.$$

At this point, X has three Rs and three Cs. These labels arise from obstructions. We introduce one new row label and column label for each obstruction. The obstructions for the Cs in $X_{9,5}$, $X_{6,10}$ and $X_{5,1}$ are

$$\begin{array}{c} \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cc} 5 & 7 \end{array} \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ 2 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 10 & 11 \end{array} \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array} \text{ and } \quad \begin{array}{c} \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}.$$

The obstructions for the R s in $X_{9,8}$, $X_{10,10}$ and $X_{5,4}$ are

$$\begin{array}{c} \\ 9 \\ 11 \end{array} \begin{array}{c} \begin{array}{cc} 6 & 8 \end{array} \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ 3 \\ 10 \end{array} \begin{array}{c} \begin{array}{cc} 9 & 10 \end{array} \\ \begin{pmatrix} C & 1 \\ 1 & R \end{pmatrix} \end{array} \text{ and } \quad \begin{array}{c} \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 4 \end{array} \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}.$$

The result of incorporating these new rows and columns is

$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{pmatrix} - & - & - & 1 & 1 & - & - & - & - & I & - \\ - & - & - & - & - & - & - & - & - & I & R \\ - & - & - & - & - & - & - & - & C & 1 & - \\ 1 & - & R & - & - & - & - & - & - & - & - \\ C & 1 & 1 & R & - & - & - & - & - & I_C & - \\ 1 & - & - & - & 0 & - & - & - & - & C & 1 \\ - & C & - & 1 & - & - & - & - & - & - & - \\ - & - & - & - & 1 & - & R & - & - & - & - \\ - & - & - & - & C & 1 & 1 & R & - & I_R & - \\ - & - & - & 0 & - & - & - & 1 & 1 & R & - \\ - & - & - & - & - & C & - & 1 & - & - & - \end{pmatrix} \end{array}.$$

We determine further entries in X from subconfigurations. When we cite two rows or two columns in a configuration, either they are not compatible, or collapsing them yields the same conclusion. This justifies keeping the labels distinct. At the end combining two compatible rows and columns produces either a matrix having no satisfactory bicoloration (contradicting $fdim(D) = 2$) or a matrix that is a core of M^1.

As before we denote positions as decimals: $i.j$ instead of (i, j).

Positions 1.10, 5.10, 9.10 lie in $\mathbf{I}$ and hence belong to no obstruction. So we have 1 at {2.4, 2.5, 3.4, 3.5} to avoid obstruction with 1.10, at {2.2, 2.3, 3.2, 3.3} to avoid it with 5.10. Also we have 1 at {2.6, 2.7, 3.6, 3.7} to avoid it with 9.10. Similarly a 2-by-2 submatrix with 0 diagonal including a position in $\mathbf{I}$ cannot be an obstruction. This yields 0 at {1.9, 1.11} by comparing 1.10 with each of {3.9, 2.11}, at {5.9, 5.11, 4.10, 7.10} by comparing 5.10 with each of {3.9, 2.11, 4.3, 7.2}, and at {9.9, 9.11, 8.10, 11.10} by comparing 9.10 with each of {3.9, 2.11, 8.7, 11.6}.

No obstruction has same color on both 0s. Therefore in every 2-by-2 submatrix with diagonal in $\mathbf{C}$ or $\mathbf{R}$ that has a 1, the fourth entry is 0. This yields 0 at $X_{7,1}$ from $(5, 7|1, 2)$, at $X_{4,4}$ from $(4, 5|3, 4)$, at $X_{11,5}$ from $(9, 11|5, 6)$, and at $X_{8,8}$ from $(8, 9|7, 8)$. Similarly comparing 2.11 with {4.3, 8.7, 10.10} puts 0 at each of {4.11, 8.11, 10.11} and comparing 3.9 with {6.10, 7.2, 11.6} puts 0 at each of {6.9, 7.9, 11.9}.

In obstructions with one 0 colored (R or C), we give the opposite color to the other 0. Thus $(X_{2,11} = R) \Rightarrow (X_{6,5} = C)$ and then $X_{1,11} = X_{8,11} = R$ and also $(X_{3,9} = C) \Rightarrow (X_{10,4} = R)$ and then $X_{1,9} = X_{7,9} = C$.

An entry that would form an obstruction with an R and with a C if it were 0 must be a 1. This now yields numerous 1s. We list these triples of positions: (location of R, location of C, location of new 1).

(9.8, 3.9, 10.7), (9.8, 9.3, 10.6), (8.7, 9.3, 10.5)

(2.11, 5.1, 6.2), (2.11, 5.1, 6.3), (2.11, 7.2, 6.4)

(10.4, 9.5, 1.6), (10.4, 9.5, 1.7), (10.4, 11.6, 1.8), (10.4, 11.6, 2.8), (10.4, 11.6, 3.8)
(5.4, 6.5, 1.3), (5.4, 6.5, 1.2), (4.3, 6.5, 1.1), (4.3, 6.5, 2.1), (4.3, 6.5, 3.1).

By the exclusion of monochromatic obstructions, we now have 0 at {5.5, 7.5} by comparing 6.5 with each of {5.1, 7.2}, and we also have 0 at {8.4, 9.4} by comparing 10.4 with each of {8.7, 9.8}. At this point we have obtained M^1. ■

As in M^1, the associated graph H at the end of Proposition 9 has a large component. We can color the rest of it using the coloring of obstructions: $(X_{10,4} = R) \Rightarrow (X_{6,9} = C) \Rightarrow X_{10,11} = R$, and $(X_{6,5} = C) \Rightarrow (X_{8,4} = R) \Rightarrow X_{7,5} = C$. Also the two obstructions {4.4, 7.1} and {8.8, 11.5} must be properly colored; nothing we have introduced restricts the choice of coloring on either. We have produced not only M^1 but also the bicoloration of its subgraph shown in X^1, except that the colors may be switched on the obstructions assigned to H_2 and H_5.

Proposition 9. *Let A be a binary matrix with $fdim(A) = 2$. If F^1 occurs for a bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b such that $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$, then F^1 occurs for every bicoloration of H^b.*

Proof. By Proposition 6, $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$ for every bicoloration of H^b. By Lemma 8 we obtain a core of M^1 with the components of H^b colored as in X^1. It is convenient to name the nine blocks in the decomposition of M^1 by letting $B_{p,q}$ be the block formed by the pth set of rows and qth set of columns, for $p, q \in \{1, 2, 3\}$. The copy of F^1 that generates this copy of M^1 by Lemma 8 has one element in each block. We shall show that every bicoloration has a copy of F^1.

Consider another bicoloration f of H^b. By symmetry between R and C in the coloring of M^1, we may assume that the vertices in the large component have the same label under f as under **(R, C)**.

Now as in the proof of Proposition 6, f puts three Rs and three Cs in $B_{2,1}$ (on labeled positions). With two of these six in each of the rows 4, 5, 7 we can always choose $i \in \{4, 5, 7\}$ such that f has one R and one C in row i. Let c and d be the columns containing the C and R of row i in $B_{2,1}$, respectively. With $(a, b, i) = (1, 6, i)$ and $(c, d, j) = (c, d, 10)$ the configuration $(1, 6, i|c, d, 10)$ becomes an instance of P in Lemma 4. So by Lemma 4, we have $(i, 10) \in \mathbf{I}_C$ under f.

The argument for $(i, 10) \in \mathbf{I}_C$ makes no reference to the rows 8, 9, 10, 11. Hence it is valid regardless of whether any row in the bottom block collapses onto another row in the middle block. The analogous argument switching the roles of C and R produces $(i', 10)$ for some $i' \in \{8, 9, 11\}$, where f uses both C and R in row i' of block $B_{3,2}$. By Proposition 6, $\mathbf{I_R} \bigcap \mathbf{I_C} = \emptyset$ under f; therefore $i \neq i'$. We thus obtain F^1 as $(1, i, i'|d, d', 10)$, where d again is the column of the R in row i of $B_{2,1}$ under f, and d' is the column of the C in row i' of $B_{3,2}$ under f. ■

Recently Arash Rafiey [19] has provided an algorithm to recognize interval bigraphs. In that paper he has given an example [Fig. 4 of that paper] which is not an interval bigraph. One can easily check that biadjacency matrix of that graph contains the configuration F^1.

3.3. *Configurations F^2 and its enlarged form M^2*

Configurations F^0, F^1 respectively force M^0, M^1. Here we obtain another configuration M^2 from F^2 when $I_R \cap I_C = \emptyset$.

Let M^2 be the configuration below, where "—" may denote 0 or 1. There are choices for "—" entries that yield Ferrers dimension at least 3. When M^2 arises in a matrix with Ferrers dimension 2 such instances are forbidden.

$$
\begin{array}{c c c c c c c c c c c c c c c c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\end{array}
$$
$$
M^2 = \begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\end{array}
\begin{pmatrix}
- & - & - & - & - & 1 & 1 & 1 & - & - & 1 & 0 & 0 & 1 & - \\
- & - & - & - & - & 1 & 1 & 1 & - & - & 1 & 0 & 1 & - & - \\
- & - & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 0 & - \\
- & - & - & 0 & - & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
- & - & - & - & 0 & 0 & 0 & - & 1 & 0 & 0 & 0 & 0 & - & - \\
1 & 1 & 1 & 1 & 0 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & 1 & 0 & - & 0 & - & - & 0 & 0 & 0 & 0 & - & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 0 \\
- & - & - & 0 & 1 & 1 & - & 1 & 1 & 0 & 1 & - & - & - & 0 \\
- & - & - & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 \\
0 & 0 & - & 1 & 0 & - & 0 & - & - & 0 & 0 & 0 & 0 & - & - \\
0 & 1 & 1 & 0 & 0 & - & 0 & - & - & 0 & 0 & 0 & 0 & - & 0 \\
1 & - & 0 & 0 & - & - & - & - & - & 0 & - & - & - & - & 0 \\
- & - & - & 0 & - & - & 1 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0
\end{pmatrix}.
$$

When M^2 is all of A and every "—" in it becomes 0, the diagram below named X^2 illustrates a bicoloration of an induced subgraph of H^b (vertices labeled 0 or "—" here are dropped to form the subgraph). The subgraph consists of six isolated edges and one large component. No assignment of values to "—" entries can create an obstruction involving any of the positions $\{X^2_{10,10}, X^2_{10,11}, X^2_{11,10}, X^2_{13,10}\}$. Thus (4, 10, 11, 13|4, 10, 11, 13) is an instance of F^2.

$$X^2 = \begin{array}{c|ccccccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \hline 1 & - & - & - & - & - & 1 & 1 & 1 & - & - & 1 & R_7 & R_6 & 1 & - \\ 2 & - & - & - & - & - & 1 & 1 & 1 & - & - & 1 & R_5 & 1 & - & - \\ 3 & - & - & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & C_6 & - \\ 4 & - & - & - & C_1 & - & 1 & 1 & 1 & C_1 & 0 & 1 & 1 & C_5 & C_7 & C_1 \\ 5 & - & - & - & - & 0 & R_1 & R_1 & - & 1 & 0 & R_1 & R_1 & 0 & - & - \\ 6 & 1 & 1 & 1 & 1 & R_1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - \\ 7 & 1 & 1 & 1 & 1 & R_1 & - & R_1 & - & - & R_1 & R_1 & R_1 & 0 & - & 1 \\ 8 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & C_1 \\ 9 & - & - & - & C_1 & 1 & 1 & - & 1 & 1 & C_1 & 1 & - & - & - & C_1 \\ 10 & - & - & - & 0 & 0 & 1 & R_1 & 1 & C_1 & I_C & I & 0 & I_R & 0 & 0 \\ 11 & 1 & 1 & 1 & 1 & R_1 & 1 & R_1 & 1 & 1 & I & 1 & R_1 & 0 & - & C_1 \\ 12 & R_4 & R_2 & - & 1 & R_1 & - & R_1 & - & - & 0 & R_1 & R_1 & 0 & - & - \\ 13 & R_3 & 1 & 1 & C_2 & 0 & - & 0 & - & - & I_R & 0 & 0 & 0 & - & 0 \\ 14 & 1 & - & C_3 & C_4 & - & - & - & - & - & 0 & - & - & - & - & 0 \\ 15 & - & - & - & C_1 & - & - & 1 & C_1 & C_1 & 0 & C_1 & - & 0 & 0 & C_1 \end{array}.$$

Lemma 10. *Let A be a binary matrix with fdim(A) = 2. If F^2 appears in a bicoloration of H^b with $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$, then A contains a core of M^2 as a submatrix.*

Proof. As before, we index rows and columns in F^2 by their eventual destinations in M^2. We start with

$$F^2 = \begin{array}{c|cccc} & 4 & 10 & 11 & 13 \\ \hline 4 & - & - & 1 & C \\ 10 & - & I_C & I & I_R \\ 11 & 1 & I & 1 & - \\ 13 & C & I_R & - & - \end{array}.$$

The labels I_R and I_C require the configurations

$$\begin{array}{c|cc} & - & 13 \\ \hline - & 1 & R \\ 10 & R & I_R \end{array}, \qquad \begin{array}{c|cc} & - & 10 \\ \hline - & 1 & R \\ 13 & R & I_R \end{array}, \qquad \begin{array}{c|cc} & - & 10 \\ \hline - & 1 & C \\ 10 & C & I_C \end{array}.$$

The first configuration forces F^2 to add a new row we call 1 to accommodate the R in column 13. Similarly, the second configuration forces F^2 to add a new column we call 1. Let X denote the developing configuration. The extra column in the first configuration is different from 10 and 11 and will turn out to differ from 4; we call it column 7. Similarly the extra row in the second configuration is row 7. Similarly the third configuration forces F^2 to add a new row 9 and a new column 9. The comments about compatibility in Lemmata 5 and 8 that justify maintaining distinct indices apply here also. The combined configuration X is now as below.

$$X = \begin{array}{c|ccccccc} & 1 & 4 & 7 & 9 & 10 & 11 & 13 \\ \hline 1 & - & - & 1 & - & - & - & R \\ 4 & - & - & - & - & - & 1 & C \\ 7 & 1 & - & - & - & R & - & - \\ 9 & - & - & - & 1 & C & - & - \\ 10 & - & - & R & C & I_C & I & I_R \\ 11 & - & 1 & - & - & I & 1 & - \\ 13 & R & C & - & - & I_R & - & - \end{array}$$

At this point X has four Rs and four Cs. The label arises from obstruction. We introduce one new row label and column label for each obstruction. The obstructions for Cs in $X_{4,13}, X_{9,10}, X_{10,9}, X_{13,4}$ are

$$\begin{array}{c|cc} & 12 & 13 \\ \hline 2 & R & 1 \\ 4 & 1 & C \end{array}, \qquad \begin{array}{c|cc} & 5 & 10 \\ \hline 6 & R & 1 \\ 9 & 1 & C \end{array}, \qquad \begin{array}{c|cc} & 6 & 9 \\ \hline 5 & R & 1 \\ 10 & 1 & C \end{array} \text{ and } \begin{array}{c|cc} & 2 & 4 \\ \hline 12 & R & 1 \\ 13 & 1 & C \end{array}.$$

The obstructions for the Rs in $X_{1,13}, X_{7,10}, X_{10,7}, X_{13,1}$ are

$$\begin{array}{c|cc} & 13 & 14 \\ \hline 1 & R & 1 \\ 3 & 1 & C \end{array}, \qquad \begin{array}{c|cc} & 10 & 15 \\ \hline 7 & R & 1 \\ 8 & 1 & C \end{array}, \qquad \begin{array}{c|cc} & 7 & 8 \\ \hline 10 & R & 1 \\ 15 & 1 & C \end{array} \text{ and } \begin{array}{c|cc} & 1 & 3 \\ \hline 13 & R & 1 \\ 14 & 1 & C \end{array}.$$

The result of incorporating these new rows and columns is

$$X = \begin{array}{c|ccccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
1 & - & - & - & - & - & - & 1 & - & - & - & - & - & R & 1 & - \\
2 & - & - & - & - & - & - & - & - & - & - & - & R & 1 & - & - \\
3 & - & - & - & - & - & - & - & - & - & - & - & - & 1 & C & - \\
4 & - & - & - & - & - & - & - & - & - & - & - & 1 & 1 & C & - \\
5 & - & - & - & - & - & R & - & - & 1 & - & - & - & - & - & - \\
6 & - & - & - & - & R & - & - & - & - & 1 & - & - & - & - & - \\
7 & 1 & - & - & - & - & - & - & - & - & R & - & - & - & - & 1 \\
8 & - & - & - & - & - & - & - & - & - & 1 & - & - & - & - & C \\
9 & - & - & - & - & 1 & - & - & - & 1 & C & - & - & - & - & - \\
10 & - & - & - & - & - & 1 & R & 1 & C & I_C & I & - & I_R & - & - \\
11 & - & - & - & 1 & - & - & - & - & - & I & 1 & - & - & - & - \\
12 & - & R & - & 1 & - & - & - & - & - & - & - & - & - & - & - \\
13 & R & 1 & 1 & C & - & - & - & - & - & I_R & - & - & - & - & - \\
14 & 1 & - & C & - & - & - & - & - & - & - & - & - & - & - & - \\
15 & - & - & - & - & - & - & 1 & C & - & - & - & - & - & - & -
\end{array}.$$

We determine further entries in X from subconfigurations. At the end, combining two compatible rows and columns produces either a matrix having no satisfactory bicoloration (contradicting $fdim(D) = 2$) or a matrix that is a core of M^2.

As before we use decimals to denote positions. Positions in **I** belong to no obstruction. So we have 1 at {8.6, 6.6, 8.8, 6.8} to avoid obstructions with I at 10.10, at {2.6, 3.6, 2.8, 3.8} to avoid obstructions with 10.13. Also we have 1 at {6.4, 6.11, 8.4, 8.11} to avoid obstructions with 11.10, at {6.2, 6.3, 8.2, 8.3} to avoid obstructions with 13.10. Similarly a 2-by-2 submatrix with 0 diagonal including a position in **I** cannot be an obstruction. This yields 0 at {15.10, 10.15} by comparing 10.10 with each of 15.8, 8.15, at {14.10, 13.15} by comparing 13.10 with each of {14.3, 8.15}, at {10.14, 15.13} by comparing 10.13 with each of {3.14, 15.8}, at {11.15, 11.5, 5.10} by comparing 11.10 with each of {8.15, 6.5, 5.6}, at {15.11, 5.11, 10.5} by comparing 10.11 with each of {15.8, 5.6, 6.5}.

No obstruction has same color on both 0s. Therefore in every 2-by-2 submatrix with diagonal in C or R that has a 1, the fourth entry is 0. Using diagonals in C, we obtain 0 at 4.14 from (3, 4|13, 14), at 9.15 from (8, 9|10, 15), at 14.4 from (13, 14|3, 4), at 15.9 from (10, 15|8, 9), at 15.14 from (3, 15|8.14), at 14.15 from (8, 14|3, 15), at 15.15 from (8, 15|8, 15). Using diagonals in R, we obtain 0 at 1.12 from (1, 2|12, 13), at 5.12 from (2, 5|6, 12), at 5.5 from (5, 6|5, 6), at 12.5 from (6, 12|2, 5), at 12.1 from (12, 13|1, 2), at 7.5 from (6, 7|5, 10), at 5.7 from (5, 10|6, 7).

An entry that would form an obstruction with an R and with a C if it were 0 must be a 1. This now yields numerous 1s. We list these triples of positions: (location of R, location of C, location of new 1).

(10.7, 3.14, 1.6), (10.7, 3.14, 1.8), (1.13, 15.8, 2.7), (7.10, 14.3, 6.1), (5.6, 9.10, 6.9),

(1.13, 15.8, 3.7), (2.12, 15.8, 4.7), (13.1, 8.15, 7.2), (12.2, 8.15, 7.4), (13.1, 13.4, 7.3),

(7.10, 14.3, 8.1), (5.6, 9.10, 8.9), (6.5, 10.9, 9.6), (6.5, 10.9, 9.8).

If $X_{11,7} = 1$, then I at 11.10 is in conflict with both R and C and then we have the configuration F^1. So $X_{11,7} = 0$. Similarly from symmetry $X_{7,11} = 0$. Again $X_{7,7} = 0$, since otherwise I_C at 10.10 also becomes an I_R i.e. $I_R \cap I_C \neq \emptyset$. Also if $X_{4,9} = 1$ then I_R at 10.13 also becomes an I_C, so $X_{4,9} = 0$. And from symmetry $X_{9,4} = 0$.

In obstruction with one 0 colored (R or C), we give opposite color to the other 0. Thus $X_{6,5} = R \Rightarrow X_{9,4} = C$ and $X_{8,15} = C \Rightarrow X_{7,11} = R, X_{9,4} = C \Rightarrow X_{12,5} = X_{11,5} = X_{7,5} = R$. Also from symmetry $X_{5,6} = R \Rightarrow X_{4,9} = C$ and $X_{15,8} = C \Rightarrow X_{11,7} = R$. Again $X_{7,10} = C \Rightarrow X_{5,12} = X_{5,11} = X_{5,7} = R$ and $X_{7,11} = R \Rightarrow X_{11,15} = C, X_{11,7} = R \Rightarrow X_{15,11} = C$.

Again if $X_{2,11} = 0$ then $X_{4,13} = C \Rightarrow X_{2,11} = R$, but then I at 10.11 will become an I_R. So $X_{2,11} = 1$. From symmetry also $X_{11,2} = 1$. Similarly if $X_{11,9} = 0$ then $X_{5,6} = R \Rightarrow X_{11,9} = C$ but then I at 11.10 becomes an I_C, so position 11.9 is 1. From symmetry also the position 9.11 is 1.

Next if $X_{11,3} = 0$ then $X_{13,4} = C \Rightarrow X_{11,3} = R$ but then I at 11.10 becomes an I_R. So position 11.3 is 1. From symmetry position 3.11 is also 1.

No obstruction has same color on both 0s. So using diagonals in C, we obtain 0 at 4.15 from (4, 8|9, 15). From symmetry we also obtain 0 at 15.4. Now $X_{7,11} = R \Rightarrow X_{4,15} = C$ and then $X_{7,7} = R$. Similarly $X_{11,7} = R \Rightarrow X_{15,4} = C$. Again using diagonals in C, we obtain 0 at 4.4 from (4, 9|4, 9). Again $X_{7,7} = R \Rightarrow X_{4,4} = C$ and then using diagonals in R we have 0 at 12.7 from (11, 12|2, 7) and then $X_{15,4} = C \Rightarrow X_{12,7} = R$.

An entry that would form an obstruction with an R and with a C if it were 0 must be a 1. This yields 1s. As before we list these positions: (location of R, location of C, location of new 1) (11.7, 3.14, 1.11) and (7.11, 14.3, 11.1).

By the exclusion of monochromatic obstructions, we have 0 at {12.10, 12.11} by comparing 12.2 with each of {7.10, 7.11}, 0 at {7.12, 10.12, 11.12, 12.12} by comparing 2.12 with each of {7.7, 10.7, 11.7, 12.7}, also we have 0 at 12.13 from (1, 12|7, 13). For the same reason using diagonals in C we obtain 0 at 10.4 from (9, 10|4, 9) and 0 at 4.10 from symmetry of X. Next using diagonals in R we obtain 0 at 7.13 from (1, 7|11, 13), at 5.13 from (1, 5|7, 13), at 11.13 from (1, 11|7, 13). From symmetric structure of X we obtain 0 at {13.7, 13.5, 13.11}.

Again $X_{4,4} = C \Rightarrow X_{7,12} = R$ and $X_{11,12} = R$. Next using diagonals in R we obtain 0 at 13.12 from (7, 13|1, 12).

Finally if 13.13 position is 1 then the positions 11.13 and 13.11 are in obstruction. Thus position 13.11 is either an R or a C. This is not possible as can be seen by comparing 13.11 with the positions 1.13 and 4.13. Thus 13.13 position must be a 0. At this point we have obtained M^2. ■

Proposition 11. *Let A be a binary matrix with $fdim(A) = 2$. If F^2 occurs for a bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b such that $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$, then F^2 occurs for every bicoloration of H^b.*

Proof. By Proposition 6, $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$ for every bicoloration of H^b. By Lemma 10 we obtain a core of M^2 with the components of H^b colored as in X^2. In M^2 let the block B_1 be formed by 1st, 2nd, 3rd and 4th row and 12th, 13th and 14th column i.e. $B_1 = (1, 2, 3, 4|12, 13, 14)$. Similarly let $B_2 = (12, 13, 14|1, 2, 3, 4)$. Here (4, 10, 11, 13|4, 10, 11, 13) is an instance of F^2. We shall show that every bicoloration has a copy of F^2.

Consider another bicoloration f of H^b and suppose under f, H_1 does not change its color. Now as in the proof of Proposition 6, we can always choose a row i, $i \in \{12, 13, 14\}$ such that f has one C and one R in row i. Let c and d be the columns containing the C and R of row i in B_2 respectively. With $(a, b, i) = (11, 7, i)$ and $(c, d, j) = (c, d, 10)$ the configuration $(11, 7, i|c, d, 10)$ becomes an instance of P in Lemma 4. So by Lemma 4, we have $(i, 10) \in \mathbf{I}_R$ under f.

The argument for $(i, 10) \in \mathbf{I}_R$ makes no reference to the rows other than the rows of B_2 block. Hence it is valid regardless of where any of these rows collapse onto another row.

Let under f, column j, $j \in \{12, 13, 14\}$ is such that j has one R and one C. Also let a, b be the rows containing C and R of column j. Then analogously we conclude $(10, j) \in \mathbf{I}_R$ under f. Thus we obtain F^2 as $(a, 10, 11, i|c, 10, 11, j)$. On the other hand when H_1 also changes its color under f, similarly we obtain F^2 as $(b, 10, 11, i|d, 10, 11, j)$, where the labels R and C in F^2 are interchanged. Finally let under f only the colors of H_1 be interchanged. Then we obtain F^2 as (1, 10, 11, 13|1, 10, 11, 13), where as before R and C labels are interchanged. ■

3.4. Configurations F^3 and its enlarged form M^3

Here we develop the configuration M^3 from F^3. The argument follows that of the earlier sections.

Let M^3 be the configuration below, where "—" may denote 0 or 1. There are choices for "—" entries that yield Ferrers dimension at least 3. When M^3 arises in a matrix with Ferrers dimension 2, such instances are forbidden.

$$M^3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & - & 0 & - & - & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & 0 & - & - & 1 & 0 & 0 & - & - & - & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & - & 1 & 1 & 1 & 1 \\ 1 & 0 & - & 0 & - & 0 & - & - & - & 0 & 0 & - & - & - & 0 \\ 0 & 1 & 1 & 0 & - & 0 & - & - & 0 & 0 & 0 & - & - & - & 0 \\ 0 & - & 0 & 1 & - & - & - & - & 0 & 0 & - & - & - & - & - \\ - & - & - & - & - & 0 & - & - & - & 0 & 0 & 1 & 0 & - & 0 \\ 0 & - & - & - & - & 0 & - & - & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & - & - & - & - & - & - & - & 0 & 0 & - & 0 & - & 0 & 1 \\ 0 & - & - & - & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & - & - & - & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & - & - & - \\ - & - & - & - & - & 0 & 1 & 0 & - & 0 & 0 & - & - & - & - \\ 0 & - & - & - & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & - & - & - \\ 0 & - & - & - & 0 & 1 & 0 & - & 0 & 0 & - & 0 & - & - & - \end{pmatrix} \end{array}.$$

When M^3 is all of A and every "—" in it becomes 0, the diagram below named X^3 illustrates a bicoloration of an induced subgraph of H^b (vertices labeled 0 or "—" here are dropped to form the subgraph). The subgraph consists of nine isolated edges and one large component. No assignments of values to these "—" entries can create an obstruction involving any position labeled **I**. Thus (1, 5, 8, 10, 14|1, 6, 10, 12, 15) is an instance of F^3 when M^3 is all of A.

$$X^3 = \begin{array}{c|ccccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \hline
1 & 1 & 1 & 1 & 1 & - & R_1 & - & - & C_1 & I & R_1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & - & R_1 & - & - & 1 & R_1 & R_1 & - & - & - & R_1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & C_1 & 1 & - & 1 & 1 & 1 & 1 \\
4 & 1 & R_5 & - & R_7 & - & R_1 & - & - & - & 0 & R_1 & - & - & - & R_1 \\
5 & C_5 & 1 & 1 & R_6 & - & 0 & - & - & 0 & I_R & 0 & - & - & - & 0 \\
6 & C_7 & - & C_6 & 1 & - & - & - & - & 0 & 0 & - & - & - & - & - \\
7 & - & - & - & - & - & R_1 & - & - & - & 0 & R_1 & 1 & R_8 & - & R_{10} \\
8 & 0 & - & - & - & - & 0 & - & - & 0 & I & 0 & C_8 & 1 & 1 & R_9 \\
9 & C_1 & - & - & - & - & - & - & - & C_1 & 0 & - & C_{10} & - & C_9 & 1 \\
10 & C_1 & - & - & - & 1 & 1 & 1 & 1 & C_1 & I & R_1 & 1 & 1 & 1 & 1 \\
11 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & R_1 & 1 & 1 & 1 & 1 \\
12 & C_1 & - & - & - & 1 & 1 & 1 & 1 & C_1 & C_1 & 1 & C_1 & - & - & - \\
13 & - & - & - & - & - & R_4 & 1 & R_3 & - & 0 & 0 & - & - & - & - \\
14 & 0 & - & - & - & 1 & R_2 & C_3 & 1 & 0 & I_C & 0 & 0 & - & - & - \\
15 & C_1 & - & - & - & C_2 & 1 & C_4 & - & C_1 & 0 & - & C_1 & - & - & -
\end{array}.$$

Lemma 12. *Let A be a binary matrix with $fdim(A) = 2$. If F^3 appears in a bicoloration of H^b with $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$, then A contains a core of M^3 as a submatrix.*

Proof. As before, we index rows and columns in F^3 by their eventual destinations in M^3. We start with

$$\begin{array}{c|ccccc}
 & 1 & 6 & 10 & 12 & 15 \\ \hline
1 & 1 & - & I & - & 1 \\
5 & C & - & I_R & - & - \\
8 & - & - & I & C & R \\
10 & - & 1 & I & 1 & - \\
14 & - & R & I_C & - & -
\end{array}.$$

The labels I_R and I_C require the configurations

$$\begin{array}{c|cc}
 & - & 10 \\ \hline
- & 1 & R \\
5 & R & I_R
\end{array}, \quad \text{and} \quad \begin{array}{c|cc}
 & - & 10 \\ \hline
- & 1 & C \\
14 & C & I_C
\end{array}.$$

The first configuration forces F^3 to add a new column we call 4 to accommodate the R in the row 5. Similarly the second configuration forces to add a new row we call 12. Let X denote the developing configuration. The extra row in the first configuration is different from the rows 1, 5, 8, 10, 12 and 14 and wecall it 2. Similarly let the extra column in the second configuration be 7. The comments about the compatibility in Lemmata 5 and 8 justify maintaining distinct indices apply here also.

The combined configuration X is now as below

$$\begin{array}{c|ccccccc}
 & 1 & 4 & 6 & 7 & 10 & 12 & 15 \\ \hline
1 & 1 & - & - & - & I & - & 1 \\
2 & - & 1 & - & - & R & - & - \\
5 & C & R & - & - & I_R & - & - \\
8 & - & - & - & - & I & C & R \\
10 & - & - & 1 & - & I & 1 & - \\
12 & - & - & - & 1 & C & - & - \\
14 & - & - & R & C & I_C & - & -
\end{array}.$$

At this point X has four Rs and four Cs. The labels arise from obstruction. We introduce one new row label and column label for each obstruction for the Cs in $X_{5,1}$, $X_{8,12}$, $X_{12,10}$, and $X_{14,7}$ as below

$$\begin{array}{c|cc}
 & 1 & 2 \\ \hline
4 & 1 & R \\
5 & C & 1
\end{array}, \quad \begin{array}{c|cc}
 & 12 & 13 \\ \hline
7 & 1 & R \\
8 & C & 1
\end{array}, \quad \begin{array}{c|cc}
 & 10 & 11 \\ \hline
11 & 1 & R \\
12 & C & 1
\end{array}, \quad \text{and} \quad \begin{array}{c|cc}
 & 7 & 8 \\ \hline
13 & 1 & R \\
14 & C & 1
\end{array}$$

The obstructions for the Rs in $X_{2,10}$, $X_{5,4}$, $X_{8,15}$ and $X_{14,6}$ are

$$\begin{array}{c|cc}
 & 9 & 10 \\ \hline
2 & 1 & R \\
3 & C & 1
\end{array}, \quad \begin{array}{c|cc}
 & 3 & 4 \\ \hline
5 & 1 & R \\
6 & C & 1
\end{array}, \quad \begin{array}{c|cc}
 & 14 & 15 \\ \hline
8 & 1 & R \\
9 & C & 1
\end{array}, \quad \text{and} \quad \begin{array}{c|cc}
 & 5 & 6 \\ \hline
14 & 1 & R \\
15 & C & 1
\end{array}$$

The result of incorporating these new rows and columns is

$$
X = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15 \end{array}\\
\left(\begin{array}{ccccccccccccccc}
1&-&-&-&-&-&-&-&-&I&-&-&-&-&1\\
-&-&-&1&-&-&-&-&1&R&-&-&-&-&-\\
-&-&-&-&-&-&-&-&C&1&-&-&-&-&-\\
1&R&-&-&-&-&-&-&-&-&-&-&-&-&-\\
C&1&1&R&-&-&-&-&-&I_R&-&-&-&-&-\\
-&-&C&1&-&-&-&-&-&-&-&-&-&-&-\\
-&-&-&-&-&-&-&-&-&-&-&1&R&-&-\\
-&-&-&-&-&-&-&-&-&I&-&C&1&1&R\\
-&-&-&-&-&-&-&-&-&-&-&-&-&C&1\\
-&-&-&-&-&1&-&-&-&I&-&1&-&-&-\\
-&-&-&-&-&-&-&-&-&1&R&-&-&-&-\\
-&-&-&-&-&-&1&-&-&C&1&-&-&-&-\\
-&-&-&-&-&-&1&R&-&-&-&-&-&-&-\\
-&-&-&-&1&R&C&1&-&I_C&-&-&-&-&-\\
-&-&-&-&C&1&-&-&-&-&-&-&-&-&-
\end{array}\right)
\end{array}.
$$

We determine further entries in X from the subconfiguration. At the end, combining two compatible rows or columns produces either a matrix having no satisfactory bicoloration (contradicting $fdim(D) = 2$) or a matrix that is core of X. As usual we use decimals to list positions.

Positions 1.10, 8.10 and 10.10 are I and hence belong to no obstruction. We have 1 at {3.1, 3.15, 11.1, 11.15} to avoid obstruction with 1.10, at {3.13, 3.14, 11.13, 11.14} to avoid obstruction with 8.10, at {3.6, 3.12, 11.6, 11.12} to avoid obstruction with 10.10. Again positions 5.10 and 14.10 are I. So we have 1 at {3.2, 3.5, 11.2, 11.3} to avoid obstruction with 5.10, at {3.3, 3.8, 11.5, 11.8} to avoid obstruction with 14.10. Similarly a 2-by-2 submatrix with 0 at diagonals including a position in $\mathbf{I}$ cannot be an obstruction. This yields 0 at {1.9, 1.11} by comparing 1.10 with each of {3.9, 11.11}, 0 at {4.10, 6.10, 5.9, 5.11} by comparing 5.10 with each of {4.2, 6.5, 3.9, 11.11}, 0 at {7.10, 9.10} by comparing 8.10 with each of {7.13, 9.14}, 0 at {10.9, 10.11} by comparing 10.10 with each of {3.9, 11.11}, 0 at {13.10, 15.10, 14.9, 14.11} by comparing 14.10 with each of {13.8, 15.5, 3.9, 11.11}.

No obstruction has the same color on both 0s. Therefore, if every 2-by-2 submatrix with diagonals in C or R has a 1, the fourth entry is 0. Using diagonals in C, we obtain 0 at 6.1 from (5, 6|1, 3), at 6.9 from (3, 6|3, 9), at 8.9 from (3, 8|9, 12), at 9.9 from (3, 9|9, 14), at 9.12 from (8, 9|12, 14), at 12.9 from (3, 12|9, 10), at 15.7 from (14, 15|5, 7), at 15.9 from (3, 15|5, 9). Again using diagonals in R, we obtain 0 at 4.4 from (4, 5|2, 4), at 4.11 from (4, 11|2, 11), at 7.11 from (7, 11|11, 13), at 7.15 from (7, 8|13, 15), at 8.11 from (8, 11|11, 15), at 13.6 from (13, 14|6, 8), at 13.11 from (11, 13|8, 11), at 2.11 from (2, 11|10, 11).

An entry that would form an obstruction both with an R and with a C if it were 0 must be a 1. This yields numerous 1s. We list these positions: (location of R, location of C, location of new 1).

(5.4, 3.9, 2.2), (5.4, 3.9, 2.3), (2.10, 6.5, 3.4), (13.8, 12.10, 3.7), (2.10, 6.3, 11.4),

(13.8, 12.10, 11.7), (11.11, 14.7, 12.5), (11.11, 14.7, 12.8), (4.2, 3.9, 2.1), (11.11, 11.5, 12.6).

Since I at 1.10 is in conflict with C at 5.1, it will not be in conflict with R at 14.6. So position 1.6 is 0. Similarly as I at 10.10 must not be in conflict with C at 5.1 so position 10.1 is a 0. Again I at 8.10 must not be in conflict with C at 5.1 and R at 14.6, thus the positions 8.1 and 8.6 are 0.

Next if the position 2.15 is a 1 then I at 8.10 is I_R (then it should not be C-colored) so the position 2.15 is a 0. Similarly the position 8.10 should not be I_C, so the position 2.12 is a 0. Again if the position 12.1 is a 1 then because of (5, 12|1, 10) the position 5.10 is also I_C. Therefore $\mathbf{I}_R \cap \mathbf{I}_C \neq \emptyset$ which contradicts our assumption. So position 12.1 is a 0. Similarly if position 2.6 is a 1 then $\mathbf{I}_R \cap \mathbf{I}_C \neq \emptyset$. Thus position 2.6 is a 0.

In obstruction with one 0 colored (R or C), we give opposite color to the other 0. Thus $X_{3,9} = C \Rightarrow X_{2,6} = X_{2,15} = R$ and then $X_{1,9} = X_{9,9} = X_{10,1} = X_{12,1} = X_{12,9} = C$. Again $X_{11,11} = R \Rightarrow X_{12,12} = C$ and then $X_{10,11} = X_{7,11} = R$. Also $X_{2,6} = R \Rightarrow X_{15,9} = C, X_{12,1} = C \Rightarrow X_{1,6} = X_{2,11} = X_{1,11} = X_{4,11} = R$.

No obstruction has same color on both 0s. Therefore, as before using diagonals in C, we obtain 0 at 15.1 from (12, 15|1, 5), at 14.12 from (12, 14|7, 12), at 14.1 from (12, 14|1, 7). Next using diagonals in R, we obtain 0 at 4.15 from (2, 4|2, 15), at 4.6 from (2, 4|2, 6), at 5.6 from (2, 5|4, 6), at 5.15 from (2, 5|4, 15).

Now if the position 7.6 is 1 then (7, 8|6, 13) implies that the position 8.6 is a C and then I at 8.10 is I_C. So as before position 7.6 is a 0. Next $X_{12,12} = C \Rightarrow X_{7,6} = R$ and then $X_{15,12} = C$. Also $X_{2,6} = R \Rightarrow X_{15,1} = C, X_{10,1} = C \Rightarrow X_{4,6} = R$. Similarly if $X_{9,1} = 1$ then $(8, 9|1, 14)$ implies that the position 8.1 is an R and then I at 8.10 is in I_R. So the position 9.1 is a 0. Then $X_{2,15} = R \Rightarrow X_{9,1} = C$ and $X_{2,6} = R \Rightarrow X_{10,9} = C$. Also $X_{9,1} = C \Rightarrow X_{14,15} = R$.

Next if the position 10.13 is 0 then $X_{8,12} = C \Rightarrow X_{10,13} = R$ and then $X_{8,6} = C$. But then $(8, 12|6, 10)$ implies that the position 10.13 is 1. Similarly the position 10.14 is also 1.

Finally as before an entry that would form an obstruction with an R and with a C if it were 0 must be 1. We list such implications as triple of positions: (location of R, location of C, location of new 1).

(2.15, 5.1, 1.3), (2.15, 6.3, 1.4), (14.6, 12.12, 10.5), (1.6, 9.14, 10.15), (8.15, 10.1, 1.13), (8.15, 10.1, 1.14), (7.13, 10.1, 1.12), (2.15, 5.1, 1.2), (14.6, 12.12, 10.8), (13.8, 12.12, 10.7).

At this point we have obtained M^3. ∎

To prove the next proposition we need the following lemma.

Lemma 13. *If A is a binary matrix with fdim$(A) = 2$ and a bicoloration $(\boldsymbol{R},\boldsymbol{C})$ of H^b contains the configuration P' indexed as shown below, then the position (i, j) in A belongs to* **I**.

$$P' = \begin{array}{c} \\ a \\ i \end{array} \begin{array}{c} \begin{array}{ccc} c & d & j \end{array} \\ \begin{pmatrix} 1 & 1 & I \\ C & R & - \end{pmatrix} \end{array}.$$

The statement remains true under transposition and/or interchange of the labels R and C.

Proof. To show that $(i, j) \in \mathbf{I}$, on contrary we suppose that (i, j) forming an obstruction with some position (r, s). From this we derive restriction on the enlargement of P' to include row and columns as shown below, and this leads to a contradiction.

$$\begin{array}{c} \\ a \\ i \\ r \end{array} \begin{array}{c} \begin{array}{cccc} c & d & j & s \end{array} \\ \begin{pmatrix} 1 & 1 & I & - \\ C & R & 0 & 1 \\ - & - & 1 & 0 \end{pmatrix} \end{array}.$$

Since I belongs to no obstruction, $P'_{r,c} = P'_{r,d} = 1$. Since $(\mathbf{R}, \mathbf{C})$ is a bicoloration, the position (r, s) cannot form an obstruction with both the colors. This contradiction proves the lemma. ∎

Proposition 14. *Let A be a binary matrix with fdim$(A) = 2$. If F^3 occurs for a bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b such that $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$, then F^3 occurs for every bicoloration of H^b.*

Proof. By Proposition 6, $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$, for every bicoloration of H^b. By Lemma 12, we obtain a core of M^3 having a bicoloration of H^b as in X^3. In M^3 let B_1 be the block formed by 4th, 5th, 6th row and 1st, 2nd, 3rd and 4th column i.e. $B_1 = (4, 5, 6|1, 2, 3, 4)$. Similarly let $B_2 = (13, 14, 15|5, 6, 7, 8)$ and $B_3 = (7, 8, 9|12, 13, 14, 15)$. Here $(1, 5, 8, 10, 14|1, 6, 10, 12, 15)$ is an instance of F^3. We will show that every bicoloration has a copy of F^3.

Consider a bicoloration f of H^b and suppose under f, H_1 does not change its color. Now as earlier we can always choose a row i in the block B_1, i.e. $i \in \{4, 5, 6\}$) such that f has one C and one R in the row i. Let c, d be the columns containing C and R of row i in B_1 respectively. With $(a, b, i) = (1, 2, i)$ and $(c, d, j) = (c, d, 10)$, the configuration $(1, 2, i|c, d, 10)$ becomes an instances of P in Lemma 4. By that lemma we have $(i, 10) \in \mathbf{I}_R$.

Similarly let under f the row i' in block B_2 (i.e. $i' \in \{13, 14, 15\}$) have one C and R. Also let c', d' be the columns containing C and R of row i' in B_2 respectively. Then as before the configuration $(10, 12, i'|c', d', 10)$ is an instance of P in Lemma 4. So we have $(i', 10) \in \mathbf{I}_C$.

Next let under f the row i'' the block B_3 (i.e. $i'' \in \{7, 8, 9\}$) have one C and one R. Also suppose c'', d'' be the columns containing C and R of row i'' in B_3 block respectively. Then the configuration $(1, i''|c'', d'', 10)$ is an instance of Lemma 13. So we have $(i'', 10) \in \mathbf{I}$. Thus we obtain F^3 as $(1, i, i'', 10, i'|c, d', 10, c'', d'')$.

On the other hand if H_1 also changes its color under f then we obtain F^3 as $(1, i, i'', 10, i'|d, c', 10, c'', d'')$. Finally if the colors of H_1 are only changed under f then we obtain F^3 as $(1, 5, 8, 10, 14|4, 7, 10, 12, 15)$. ∎

3.5. *Configurations F^4 and its enlarged form M^4*

Here we develop the configuration M^4 from F^4. The argument follows that of the earlier sections.

Let M^4 be the configuration below, where "−" may denote 0 or 1. There are choices for "−" entries that yield Ferrers dimension at least 3. When M^4 arises in a matrix with Ferrers dimension 2, such instances are forbidden.

$$
M^4 = \begin{array}{c|ccccccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 0 \\
2 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
3 & - & - & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
4 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 0 & 1 & - \\
5 & - & - & 0 & - & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
6 & - & - & - & - & 0 & - & 0 & 0 & - & 0 & 0 & 1 & - & 0 & 0 \\
7 & - & - & 0 & - & 0 & 0 & - & 1 & 0 & 0 & - & 0 & 0 & 0 & - \\
8 & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & - \\
9 & 1 & 1 & 1 & 1 & 1 & - & 0 & 1 & 1 & 1 & 1 & - & - & - & - \\
10 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & - & - & - \\
11 & 1 & 1 & 1 & 1 & 1 & 0 & - & 1 & 1 & 1 & 1 & - & - & - & - \\
12 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & - & - & - & - & - & - & - \\
13 & - & 0 & 1 & 0 & 0 & - & 0 & 0 & - & - & - & - & - & - & - \\
14 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & - & - & - & - & - & - & - \\
15 & 0 & 1 & 0 & - & 0 & 0 & - & - & - & - & - & - & - & - & -
\end{array}.
$$

When M^4 is all of A and every "$-$" in it becomes 0, the diagram below named X^4 illustrates a bicoloration of an induced subgraph of H^b (vertices labeled 0 of "$-$" here are dropped to form the subgraph). The subgraph consists of six isolated edges and two components. No assignment of values to these "$-$" entries can create an obstruction involving any position labeled I. Thus (3, 5, 8, 10, 14|3, 5, 8, 10, 14) is an instance of F^4 when M^4 is all of A.

$$
X^4 = \begin{array}{c|ccccccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & R_3 \\
2 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & C_4 & C_3 & 1 \\
3 & - & - & - & - & 0 & - & R_1 & R_1 & 1 & 1 & 1 & 1 & 1 & R_2 & R_4 \\
4 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & C_2 & 1 & - \\
5 & - & - & 0 & - & I & 0 & 0 & R_1 & 1 & I & 1 & C_1 & 0 & I_C & 0 \\
6 & - & - & - & - & 0 & - & R_1 & R_1 & - & R_1 & R_1 & 1 & - & 0 & 0 \\
7 & - & - & C_5 & - & 0 & C_5 & - & 1 & C_1 & C_1 & - & C_1 & C_1 & 0 & - \\
8 & - & - & C_5 & - & C_5 & C_5 & 1 & 1 & 1 & 1 & 1 & C_1 & C_1 & 0 & - \\
9 & 1 & 1 & 1 & 1 & 1 & - & R_5 & 1 & 1 & 1 & 1 & - & - & - & - \\
10 & 1 & 1 & 1 & 1 & I & C_5 & R_5 & 1 & 1 & 1 & 1 & - & - & - & - \\
11 & 1 & 1 & 1 & 1 & 1 & C_5 & - & 1 & 1 & 1 & 1 & - & - & - & - \\
12 & 1 & 1 & 1 & 1 & R_5 & 1 & R_5 & R_5 & - & - & - & - & - & - & - \\
13 & - & R_8 & 1 & R_6 & 0 & - & R_5 & R_5 & - & - & - & - & - & - & - \\
14 & 1 & R_7 & C_6 & 1 & I_R & 0 & 0 & 0 & - & - & - & - & - & - & - \\
15 & C_7 & 1 & C_8 & - & 0 & 0 & - & - & - & - & - & - & - & - & -
\end{array}.
$$

Lemma 15. *Let A be a binary matrix with $fdim(A) = 2$. If F^4 appears in a bicoloration of H^b with $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$, then A contains a core of M^4 as a submatrix.*

Proof. As before, we index rows and columns in F^4 by their eventual destinations in M^4. We start with

$$
F^4 = \begin{array}{c|ccccc|}
 & 3 & 5 & 8 & 10 & 14 \\
3 & - & - & - & 1 & R \\
5 & - & I & R & I & I_C \\
8 & - & C & - & 1 & - \\
10 & 1 & I & 1 & 1 & - \\
14 & C & I_R & - & - & -
\end{array}.
$$

The labels I_R and I_C require the configurations

$$
\begin{array}{c|cc|}
 & - & 5 \\
- & 1 & R \\
14 & R & I_R
\end{array}
\quad \text{and} \quad
\begin{array}{c|cc|}
 & - & 14 \\
- & 1 & C \\
5 & C & I_C
\end{array}.
$$

The first configuration forces F^4 to add a new column we call 2 to accommodate the R in the row 14. Similarly the second configuration forces F^4 to add a new row we call 2. Let X denote the developing configuration. The extra row in the first configuration is different from the rows 14, 10, 8 and 5 and will turn out to differ from 3; we call it row 12. Similarly the extra column in the second configuration is column 12. The comments about the compatibility in Lemmata 5 and 8 justify maintaining distinct indices apply here also.

The combined configuration X is now as below

$$\begin{array}{c} \\ 2\\3\\5\\8\\10\\12\\14 \end{array}\begin{array}{c} \begin{array}{ccccccc} 2&3&5&8&10&12&14 \end{array} \\ \left(\begin{array}{ccccccc} - & - & - & - & - & 1 & C \\ - & - & - & - & 1 & - & R \\ - & - & I & R & I & C & I_C \\ - & - & C & - & 1 & - & - \\ - & 1 & I & 1 & 1 & - & - \\ 1 & - & R & - & - & - & - \\ R & C & I_R & - & - & - & - \end{array}\right) \end{array}.$$

At this point X has four Rs and four Cs. The labels arise from obstructions. We introduce one new row label and column label for each obstruction. The obstruction for the Cs in $X_{2,14}$, $X_{5,12}$, $X_{8,5}$, and $X_{14,3}$ are

$$\begin{array}{c} \\ 1\\2 \end{array}\begin{array}{c} 14\quad 15 \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ 5\\6 \end{array}\begin{array}{c} 11\quad 12 \\ \begin{pmatrix} 1 & C \\ R & 1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ 8\\9 \end{array}\begin{array}{c} 5\quad 7 \\ \begin{pmatrix} C & 1 \\ 1 & R \end{pmatrix} \end{array} \text{ and } \begin{array}{c} \\ 13\\14 \end{array}\begin{array}{c} 3\quad 4 \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}.$$

The obstructions for the Rs in $X_{5,8}$, $X_{3,14}$, $X_{12,5}$ and $X_{14,2}$ are

$$\begin{array}{c} \\ 5\\7 \end{array}\begin{array}{c} 8\quad 9 \\ \begin{pmatrix} R & 1 \\ 1 & C \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ 3\\4 \end{array}\begin{array}{c} 13\quad 14 \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ 11\\12 \end{array}\begin{array}{c} 5\quad 6 \\ \begin{pmatrix} 1 & C \\ R & 1 \end{pmatrix} \end{array} \text{ and } \begin{array}{c} \\ 14\\15 \end{array}\begin{array}{c} 1\quad 2 \\ \begin{pmatrix} 1 & R \\ C & 1 \end{pmatrix} \end{array}.$$

The result of incorporating these new rows and columns is

$$X = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array}\begin{array}{c} \begin{array}{ccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15 \end{array} \\ \left(\begin{array}{ccccccccccccccc} - & - & - & - & - & - & - & - & - & - & - & - & - & 1 & R \\ - & - & - & - & - & - & - & - & - & - & - & 1 & - & C & 1 \\ - & - & - & - & - & - & - & - & - & 1 & - & - & 1 & R & - \\ - & - & - & - & - & - & - & - & - & - & - & - & C & 1 & - \\ - & - & - & - & I & - & - & R & 1 & I & 1 & C & - & I_C & - \\ - & - & - & - & - & - & - & - & - & - & R & 1 & - & - & - \\ - & - & - & - & - & - & - & 1 & C & - & - & - & - & - & - \\ - & - & - & - & C & - & 1 & - & - & 1 & - & - & - & - & - \\ - & - & - & - & 1 & - & R & - & - & - & - & - & - & - & - \\ - & - & 1 & - & I & - & - & 1 & - & 1 & - & - & - & - & - \\ - & - & - & - & 1 & C & - & - & - & - & - & - & - & - & - \\ - & 1 & - & - & R & 1 & - & - & - & - & - & - & - & - & - \\ - & - & 1 & R & - & - & - & - & - & - & - & - & - & - & - \\ 1 & R & C & 1 & I_R & - & - & - & - & - & - & - & - & - & - \\ C & 1 & - & - & - & - & - & - & - & - & - & - & - & - & - \end{array}\right) \end{array}.$$

We determine further entries in X from subconfigurations. At the end, combining two compatible rows or columns produces either a matrix having no satisfactory bicoloration (contradicting $fdim(D) = 2$) or a matrix that is a core of M^4. As usual we use decimals to list positions.

Positions 5.5, 5.10 and 10.5 are I and hence belong to no obstruction. We have 1 at {9.9, 9.11, 11.9, 11.11} to avoid obstruction with 5.5, at {3.9, 3.11, 8.9, 8.11, 10.9, 10.11} to avoid obstruction with 5.10, at {9.3, 9.8, 9.10, 11.3, 11.8, 11.10} to avoid obstruction with 10.5. Similarly we have 1 at {4.9, 4.11, 1.9, 1.11} to avoid obstruction with 5.14 and at {9.4, 11.4, 9.1, 11.1} to avoid obstruction with 14.5. Similarly, a 2-by-2 submatrix with 0 diagonal including a position in **I** cannot be an obstruction. This yields 0 at {6.5, 7.5, 5.6, 5.7} by comparing 5.5 with each of {6.11, 7.9, 11.6, 9.7}, at {6.10, 7.10} by comparing 5.10 with each of {6.11, 7.9}, at {10.6, 10.7} by comparing 10.5 with each of {11.6, 9.7}, at {6.14, 5.15, 7.14} by comparing 5.14 with each of {6.11, 1.15, 7.9}, at {14.6, 14.7, 15.5} by comparing 14.5 with each of {11.6, 9.7, 15.1}.

No obstruction has the same color on both 0s. Therefore in every 2-by-2 submatrix with diagonal in C or R that has a 1, the fourth entry is 0. This yields 0 at $X_{2,13}$ from (2, 4|13, 14), at $X_{7,13}$ from (7, 4|9, 13), at $X_{7,6}$ from (7, 11|6, 9), at $X_{7,12}$ from (5, 7|9, 12), at $X_{8,6}$ from (8, 11|5, 6), at $X_{15,3}$ from (14, 15|1, 3), at $X_{15,6}$ from (11, 15|1, 6).

Similarly considering diagonals in R, we obtain 0 at $X_{6,7}$ from (6, 9|7, 11), at $X_{6,8}$ from (5, 6|8, 11), at $X_{3,15}$ from (3, 1|14, 15), at $X_{13,2}$ from (13, 14|2, 4), at $X_{13,7}$ from (9, 13|4, 7), at $X_{12,7}$ from (9, 12|5, 7), at $X_{6,15}$ from (1, 6|11, 15).

An entry that would form an obstruction with an R and with a C if it were 0 must be a 1. We list such implications as triple of position: (location of R, location of C, location of new 1).

(12.5, 15.1, 11.2), (14.2, 11.6, 12.4), (12.5, 15.1, 9.2), (14.2, 11.6, 12.1), (1.15, 5.12, 2.11),
(1.15, 5.12, 2.9), (6.11, 2.14, 1.12), (9.7, 7.9, 8.8), (13.4, 11.6, 12.3), (6.11, 2.14, 4.12), (6.11, 4.13, 3.12).

I at 5.5 is not in conflict with R at 3.14 and also with C at 14.3. This implies that the positions 3.5 and 5.3 are 0.

Again if the position 8.3 is an 1 then (8, 14|3, 5) puts 14.5 into I_C. But our hypothesis is $\mathbf{I}_R \cap \mathbf{I}_C = \phi$. Hence $X_{8,3} = 0$. Similarly if $X_{3,8} = 1$ then (3, 5|8, 14) puts 5.14 into I_R. So as before $X_{3,8} = 0$.

Consider the position 10.1. If $X_{10,1} = 0$ then $X_{14,3} = C \Rightarrow X_{10,1} = R$. But then I at 10.5 becomes an I_R. Hence $X_{10,1} = 1$. Similarly if $X_{10,2} = 0$ then $X_{15,1} = C \Rightarrow X_{10,2} = R$. Then again position 10.5 becomes an I_R. So $X_{10,2} = 1$. Next consider the position 10.4. If $X_{10,4} = 0$ then it would form an obstruction with an R at 14.2 and with a C at 14.3. Hence $X_{10,4} = 1$.

Similarly the symmetric arguments put 1 in positions 1.10, 2.10, and 4.10. If $X_{7,3} = 1$ then $X_{7,9} = C \Rightarrow X_{5,3} = R$. But then (5, 12|3, 5) puts 5.5 into I_R. So $X_{7,3} = 0$. The symmetric arguments puts 0 in position 3.7. Also if $X_{8,12} = 1$, then (5, 8|5, 12) puts 5.5 into I_C. So $X_{8,12} = 0$. Also from symmetry $X_{12,8} = 0$.

Next consider the position 14.8. If $X_{14,8} = 1$ then 0 at 12.8 form an obstruction with R at 14.2 and C at 14.3. Hence $X_{14,8} = 0$. And from symmetry $X_{8,14} = 0$. Now $X_{7,9} = C \Rightarrow X_{3,8} = R$ and then $X_{7,10} = X_{7,13} = X_{7,12} = C$. Similarly $X_{9,7} = R \Rightarrow X_{8,3} = C$ and then $X_{10,7} = X_{12,7} = X_{13,7} = R$.

Also $X_{11,6} = C \Rightarrow X_{12,8} = R$ and $X_{6,11} = R \Rightarrow X_{8,12} = C$.

By the exclusion of monochromatic obstruction we have 0 at 5.13 by comparing the two Cs at 4.13 and 5.12. Also we have 0 at 13.5 by comparing the two Rs at 12.5 and 13.4. For the similar reason we have 0 at 8.13 by comparing two Cs at 4.13 and 8.12. Also we have 0 at 13.8 by comparing two Rs at 13.4 and 12.8.

At this point we have obtained M^4. ■

Proposition 16. *Let A be a binary matrix with $fdim(A) = 2$. If F^4 occurs for a bicoloration* (**R,C**) *of H^b such that $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$, then F^4 or F^2 occurs for every bicoloration of H^b.*

Proof. By Proposition 6, $\mathbf{I}_R \cap \mathbf{I}_C = \emptyset$ for every bicoloration of H^b. By Lemma 15, we obtain a core of M^4. In M^4, let B_1 be the block formed by 1st, 2nd, 3rd, and 4th row and 13th, 14th and 15th column i.e. $B_1 = (1, 2, 3, 4|13, 14, 15)$. Similarly $B_2 = (13, 14, 15|1, 2, 3, 4)$. Here (3, 5, 8, 10, 14|3, 5, 8, 10, 14) is an instance of F^4. Let f be any bicoloration of H^b. There are two cases.

Case 1: Under f none of H_1 and H_5 changes its color or else H_1 and H_5 both change their colors.
Case 2: Under f one of H_1 and H_5 changes its color.

We will show that in Case 1, every bicoloration has a copy of F^4 and in Case 2, every bicoloration has a copy of F^2.

Consider Case 1. As before, we can always choose a row $i \in \{13, 14, 15\}$ such that f has one C and R in row i. Let c and d be the columns containing the C and R of row i in B_2 respectively. With $(a, b, i) = (10, 12, i)$ and $(c, d, j) = (c, d, 5)$ the configuration $(10, 12, i|c, d, 5)$ becomes an instance of P in Lemma 4. So by Lemma 4, we have $(i, 5) \in \mathbf{I}_R$ or $(i, 5) \in \mathbf{I}_C$ according as H_5 does not change its color or H_5 does change its color under f.

Let under f column $j \in \{13, 14, 15\}$ be such that j has one R and C. Also let a, b be the rows containing C and R of column j in B_1 respectively. Then analogously we conclude that $(5, j) \in \mathbf{I}_C$ or $\mathbf{I}_R$ according as H_1 does not change its color or H_1 does change its color under f. Thus we obtain F^4 as $(b, 5, 8, 10, i|c, 5, 8, 10, j)$ or $(a, 5, 8, 10, i|d, 5, 8, 10, j)$.

Next consider Case 2. Let under f, H_1 changes its color. Then (5, 8|5, 8) puts 5.5 into $\mathbf{I}_C$. Again let under f column $j \in \{13, 14, 15\}$ has one R and one C. As before let a, b be the rows containing C and R of column j in B_1 respectively. Then as before we conclude that $(5, j) \in \mathbf{I}_R$. Also let under f the row $i \in \{13, 14, 15\}$ has one R and one C. Let c and d be the columns containing the C and R of row i in B_2 respectively. Then by previous arguments we conclude that $(i, 5) \in \mathbf{I}_R$. Now we have F^2 as $(a, 5, 10, i|d, 5, 10, j)$. Here we can see that H_1 and H_5 coalesce. ■

3.6. The main result

Now we restate the main result mentioned in Section 2.

Theorem 3. *If A is a binary matrix with $fdim(A) = 2$, and* (**R**, **C**) *is a bicoloration of H^b giving rise to one of F^0, F^1, F^2, F^3, F^4 or their transposes, then* (**R**, **C**) *does not extend to a zero partition of A.*

Proof. A zero partition of A is an R, C-coloring of the 0s in A so that each color is used on the positions of the 0s in a Ferrers matrix. If (**R**, **C**) extends to a zero-partition of A, then all of $\mathbf{R} \bigcup I_R$ must have color R in the extension, and all of $\mathbf{C} \bigcup I_C$ must have color C. Hence there is no such extension when $\mathbf{I}_R \cap \mathbf{I}_C \neq \emptyset$, which is equivalent to the occurrence of F^0.

If the bicoloration (**R,C**) of H^b give rise to F^1, then F^1 contains two configurations $\begin{pmatrix} 1 & I \\ R & I_C \end{pmatrix}$ and $\begin{pmatrix} 1 & I \\ C & I_R \end{pmatrix}$ (or their transposes), in which the entry labeled I is the same position. We must give colors C and R to the positions labeled I_C and I_R, respectively. Thus choosing a color for the position labeled I creates an obstruction in one of the supposed Ferrers matrices being produced. Hence there is no extension to a zero-partition.

Again let the bicoloration (**R**, **C**) of H^b give rise to

$$F^2 = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} i \\ j \\ k \\ l \end{matrix} & \begin{pmatrix} - & - & 1 & C \\ - & I_C & I & I_R \\ 1 & I & 1 & - \\ C & I_R & - & - \end{pmatrix} \end{matrix}.$$

As before, the configuration

$$\begin{pmatrix} 1 & C \\ I & I_R \end{pmatrix}$$

implies that the positions (j, c) and (k, b) labeled I must be R-colored. But then the configuration

$$\begin{array}{c} \\ j \\ k \end{array} \begin{array}{c} \begin{array}{cc} b & c \end{array} \\ \begin{pmatrix} I_C & I \\ I & 1 \end{pmatrix} \end{array}$$

is a contradiction since all the positions colored R form a Ferrers matrix. Next let a bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b give rise to

$$F^3 = \begin{array}{c} \\ i \\ j \\ k \\ l \\ m \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & - & I & - & 1 \\ C & - & I_R & - & - \\ - & - & I & C & R \\ - & 1 & I & 1 & - \\ - & R & I_C & - & - \end{pmatrix} \end{array}.$$

The configuration

$$\begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cc} a & c \end{array} \\ \begin{pmatrix} 1 & I \\ C & I_R \end{pmatrix} \end{array}$$

implies that the position (i, c) labeled I must be R-colored. Again the configuration

$$\begin{array}{c} \\ l \\ m \end{array} \begin{array}{c} \begin{array}{cc} b & c \end{array} \\ \begin{pmatrix} 1 & I \\ R & I_C \end{pmatrix} \end{array}$$

implies that the position (l, c) labeled I must be C-colored. Next the configuration

$$\begin{array}{c} \\ i \\ k \end{array} \begin{array}{c} \begin{array}{cc} c & e \end{array} \\ \begin{pmatrix} I & 1 \\ I & R \end{pmatrix} \end{array}$$

implies that the position (k, c) labeled I must be R-colored.

But then the configuration

$$\begin{array}{c} \\ k \\ l \end{array} \begin{array}{c} \begin{array}{cc} c & d \end{array} \\ \begin{pmatrix} I & C \\ I & 1 \end{pmatrix} \end{array}$$

is a contradiction since two C-colors create an obstruction in a Ferrers matrix.

Finally let a bicoloration $(\mathbf{R}, \mathbf{C})$ of H^b give rise to

$$F^4 = \begin{array}{c} \\ i \\ j \\ k \\ l \\ m \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} I & - & R & I & I_C \\ - & - & - & 1 & R \\ C & - & - & 1 & - \\ I & 1 & 1 & 1 & - \\ I_R & C & - & - & - \end{pmatrix} \end{array}.$$

The configuration

$$\begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cc} d & e \end{array} \\ \begin{pmatrix} I & I_C \\ 1 & R \end{pmatrix} \end{array}$$

implies that the position (i, d) labeled I must be C-colored. Also the configuration

$$\begin{array}{c} \\ l \\ m \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} I & 1 \\ I_R & C \end{pmatrix} \end{array}$$

implies that the position (l, a) labeled I must be R-colored. Then the configuration

$$\begin{array}{c} \\ i \\ k \end{array}\begin{array}{c} \begin{array}{cc} a & d \end{array} \\ \begin{pmatrix} I & I \\ C & 1 \end{pmatrix} \end{array}$$

implies that the position (i, a) must be C-colored. But then as before the configuration

$$\begin{array}{c} \\ i \\ l \end{array}\begin{array}{c} \begin{array}{cc} a & c \end{array} \\ \begin{pmatrix} I & R \\ I & 1 \end{pmatrix} \end{array}$$

is a contradiction. ■

4. A counterexample

The following example shows that the list of forbidden configurations given in Section 3 is not exhaustive. Consider the binary matrix A in Fig. 3 which is of Ferrers dimension 2.

$$A = \begin{array}{r|cccccccccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 7 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 9 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 10 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 11 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 12 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Fig. 3.

A satisfactory bicoloration of the above matrix is shown below in Fig. 4.

$$A = \begin{array}{r|cccccccccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & R & R & R & R & R & R & R & R & R \\ 2 & 1 & 1 & 1 & 1 & C & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & R \\ 3 & 1 & 1 & 1 & 1 & C & 1 & 1 & 1 & R & R & R & R & I & R \\ 4 & 1 & 1 & R & R & I & R & R & R & R & R & R & R & I_R & R \\ 5 & 1 & C & 1 & R & I_C & I_R & R & I_R & R & I_R & I_R & R & I_R & I_R \\ 6 & 1 & C & 1 & 1 & C & 1 & 1 & 1 & 1 & 1 & 1 & 1 & I & R \\ 7 & 1 & C & 1 & 1 & C & 1 & R & R & R & R & R & R & I & R \\ 8 & 1 & C & 1 & 1 & C & C & 1 & 1 & 1 & 1 & 1 & 1 & C & 1 \\ 9 & 1 & C & 1 & 1 & C & C & 1 & 1 & 1 & 1 & R & R & I & I_R \\ 10 & 1 & C & 1 & 1 & C & C & 1 & 1 & 1 & C & 1 & R & I_C & I_R \\ 11 & 1 & C & 1 & 1 & C & C & 1 & 1 & 1 & C & C & 1 & I_C & I \\ 12 & 1 & C & 1 & 1 & C & C & 1 & I & R & I & I_R & R & I & I_R \\ 13 & C & C & 1 & 1 & C & C & 1 & C & 1 & C & C & 1 & I_C & I \end{array}$$

Fig. 4.

Here H^b consists of only one component. But it is not possible to extend the colors (**R**, **C**) to the set **I** so that A admits a zero partition as shown below (see Figs. 5 and 6).

Zero partition of A is not possible because (12, 13) entry receives both colors R and C.

It is clear that the graph of the above matrix is not included in any of the infinite families of forbidden subgraphs already known. Also it can be verified that this graph lies outside the class of graphs generated from the configurations in the present paper.

	5	1	2	3	4	6	7	8	10	9	13	11	12	14
4				R	R	R	R	R	R	R	I_R	R	R	R
5					R	I_R	R	I_R	I_R	R	I_R	I_R	R	I_R
1						R	R	R	R	R	R	R	R	R
7							R	R	R	R	I	R	R	R
12								I	I	R	I	I_R	R	I_R
3									R	R	I	R	R	R
9												R	R	I_R
10													R	I_R
6														R
2														R
11														I
13														I
8														

Fig. 5. Ferrers matrix from Fig.4 where all the entries may be given colors R.

	5	2	13	6	10	11	1	8	3	4	7	9	12	14
1														
2	C													
3	C													
4	I													
5	I_C	C												
7	C	C												
6	C	C	I											
8	C	C	C	C										
9	C	C	I	C										
12	C	C	I	C										
10	C	C	I_C	C	C									
11	C	C	I_C	C	C	C								
13	C	C	I_C	C	C	C	C	C						

Fig. 6. Ferrers matrix from Fig.4 where all the entries may be given colors C.

5. Conclusion

In this paper, we have obtained five forbidden substructures of interval matrices. Unfortunately they do not exhaust all possibilities. Nevertheless, we hope this paper will be a motivating factor to obtain a complete list of forbidden substructures and thereby find out more efficient recognition algorithm for an interval matrix than that is presently available. The idea that is prevalent in our paper is that the isolated vertices I which receive the color R for a zero partition are the vertices I_R and those in the configuration $(1, C|I, I_R)$ or their transposes (and similarly for those receiving C). It now remains open to find out if there are any other such I's being forced to receive a specific color for a zero partition. The answer to this question, it is hoped, may help one resolving the main problem.

Acknowledgments

Authors are grateful to Douglas West. He had taken an active interest in the paper, so much so that he had drafted the manuscript of the paper till midway and provided insightful suggestion for a qualitative improvement of the paper. The authors are also thankful to all the referees for their patient reading and nice comments leading to an overall reorganization of the paper.

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