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# Extension of inverse scattering method to nonlinear evolution equation in nonuniform medium

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By allowing the entire spectrum of certain linear eigenvalue problems to evolve with time a general type of nonlinear evolution equation in nonuniform medium which is exactly integrable by the inverse scattering method has been derived. The derivative nonlinear Schrödinger equation or the nonlinear Schrödinger equation with linear or parabolic density profiles are special cases of this generalized form.

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## I. INTRODUCTION

The method of inverse scattering has of late become a standard tool for solving initial value problems of nonlinear partial differential equations associated with the evolution of nonlinear waves. In this connection Ablowitz[1] *et al.* have shown that a general form of the nonlinear evolution equation whose exact solution can be determined by this technique from the inverse problem of the Zakharov–Shabat[2] type eigenvalue equation

$$(\partial_x - M)v = \begin{pmatrix} \partial_x & -q \\ -r & \partial_x \end{pmatrix} v = -i\rho \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v = -i\rho\sigma_3 v \quad (1)$$

is

$$(\sigma_3 \partial_t + 2\Omega(L_A)) \begin{pmatrix} r \\ q \end{pmatrix} = 0, \quad (2)$$

where

$$L_A = \frac{1}{2i} \begin{pmatrix} \partial_x - 2r \int_{-\infty}^{x} dy\, q & 2r \int_{-\infty}^{x} dy\, r \\ -2q \int_{-\infty}^{x} dy\, q & -\partial_x + 2q \int_{-\infty}^{x} dy\, r \end{pmatrix} \quad (3)$$

and $\Omega(\rho)$ is an entire function of $\rho$. Recently Kaup and Newell[3] have developed an elegant approach through application of which they have derived the general class of exactly integrable nonlinear evolution equations associated with Eq. (1). It is not imperative in their analysis that the eigenvalue remains time invariant. In fact they considered the case where the bound state eigenvalues are assumed to depend on time in a prescribed manner. However in such general situations the evolution equations are usually nonlocal and further cannot be given explicitly in terms of the potentials $r$ and $q$ alone. The present analysis pertains to the case where the entire spectrum evolves with time in accordance with

$$\frac{d\rho}{dt} = f(\rho;t). \quad (4)$$

$f(\rho,t)$ is an entire function $\rho$ with arbitrary functions of time $t$ occurring as coefficients of the different powers of $\rho$. Equation (4) is solvable and $\rho$ may therefore be expressed in terms of $t$ and the initial $\rho_0$. It is shown that the nonlinear evolution equation which can be exactly solved in this case is

$$[\sigma_3 \partial_t - 2if(L_A,t)x + 2\Omega(L_A)] \begin{pmatrix} r \\ q \end{pmatrix} = 0. \quad (5)$$

In obtaining (5) we have followed the AKNS[1] method. Alternatively one could derive the time evolution equation of the scattering data in a manner as shown by Kaup[4] and, by employing the closure properties of the complete set of eigenfunctions of $L_A$ and its adjoint[5], arrive at a more general form of evolution equation. The relationship between the two would be similar to that between the one determined by Kaup and Newell[3] and Eqs. (2) and (3). However our main purpose as exhibited by Eq. (5) is to show that the extension of the time dependence to the entire spectrum renders the initial value problem of nolinear wave propagation in certain types of nonuniform dispersive medium exactly integrable. The evolution equation though explicit in $r$ and $q$ may become nonlocal depending on the choice of $f(\rho;t)$. The nonlinear Schrödinger equation with linear[6] or parabolic density profiles[7] are special cases of the last equations. Further under restrictions similar to those on $\Omega(\rho)$[1,3] it is possible to extend $f(\rho;t)$ to suitable class of rational functions.

With appropriate modification it is possible to obtain the general class of exactly integrable nonlinear evolution equation associated with the Newell–Kaup eigenvalue problem

$$(\partial_x - \rho M)v = -i\rho^2\sigma_3 v,$$

the entire spectrum $\rho$ being assumed to be time dependent. As shown by Newell and Kaup[8] the solution of the derivative nonlinear Schrödinger equation is related to the inverse scattering problem of this eigenvalue equation.

## II. EVOLUTION EQUATION CORRESPONDING TO ZAKHAROV–SHABAT PROBLEM

We assume that the time dependence of $v$ is given by

$$v_t = Nv = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v. \quad (6)$$

The eigenvalue $\rho$ being dependent on time cross differentiation of (1) and (6) leads to

$$M_t - N_x + [M - i\rho\sigma_3, N] - i\rho_t\sigma_3 = 0. \quad (7a)$$

In explicit form

$$A_x = qC - rB - i\rho_t\,,$$
$$B_x + 2i\rho B = q_t - 2Aq\,,$$
$$C_x - 2i\rho C = r_t + 2Ar\,. \qquad (7b)$$

The first equation of the set (7b) shows that for $q$ and $r \to 0$ as $|x| \to \infty$ the asymptotic behavior of $A$ is given by

$$A(x,t;\rho(t)) \underset{x\to-\infty}{\longrightarrow} A_0^{(-)} - i\rho_t x$$
$$\underset{x\to+\infty}{\longrightarrow} A_0^{(+)} - i\rho_t x\,. \qquad (8)$$

We now proceed to a formal solution of (7) and the time evolution equation of the scattering data for the inverse problem of (1). A sufficient condition for the existence of the solutions subject to (7) could yield the general form for the nonlinear evolution equation for the potential $q$ and $r$. This in essence is the procedure adopted by Ablowitz *et al.*[1]

Let $\phi,\bar\phi$ and $\psi,\bar\psi$ denote the pairs of linearly independent solutions of (1) with boundary conditions

$$\phi \underset{x\to-\infty}{\longrightarrow} \begin{pmatrix}1\\0\end{pmatrix} e^{-i\rho x}; \quad \bar\phi \underset{x\to-\infty}{\longrightarrow} \begin{pmatrix}0\\-1\end{pmatrix} e^{i\rho x},$$
$$\psi \underset{x\to+\infty}{\longrightarrow} \begin{pmatrix}0\\1\end{pmatrix} e^{i\rho x}; \quad \bar\psi \underset{x\to+\infty}{\longrightarrow} \begin{pmatrix}1\\0\end{pmatrix} e^{-i\rho x}. \qquad (9)$$

The time development of the scattering data defined through

$$\phi = a\bar\psi + b\psi; \quad \bar\phi = \bar b\bar\psi - \bar a\psi; \quad a\bar a + b\bar b = 1 \qquad (10)$$

are given by

$$a_t = (A_0^{(+)} - A_0^{(-)})a + B^{(+)}b;$$
$$b_t = C^{(+)}a - (A_0^{(+)} + A_0^{(-)})b\,,$$
$$\bar a_t = -(A_0^{(+)} - A_0^{(-)})\bar a - C^{(+)}\bar b; \qquad (11)$$
$$\bar b_t = -B^{(+}\bar a + (A_0^{(+)} + A_0^{(-)})\bar b\,,$$
$$B^{(+)} = \lim_{x\to\infty} Be^{2i\rho x}; \quad C^{(+)} = \lim_{x\to+\infty} Ce^{-2i\rho x}.$$

These follow easily on noting that the extra contributions proportional to $\pm i\rho_t x$ which arise from the time derivative of the asymptotic expressions for $\phi$ and $\bar\phi$ are exactly compensated by virtue of (8).

Writing $\Phi = \begin{pmatrix}\phi_1 & \bar\phi_1\\ \phi_2 & \bar\phi_2\end{pmatrix}$ a formal solution for (7a) and (7b) is now expressed in the form

$$(\Phi^{-1}N\Phi)_{x=x'} - (\Phi^{-1}N\Phi)_{x=x''}$$
$$= -i\rho_t[x\Phi^{-1}\sigma_3\Phi]_{x''}^{x'} + \int_{x''}^{x'} dx\,\Phi^{-1}M_t\Phi$$
$$+ i\rho_t \int_{x''}^{x'} dx\,x\partial_x(\Phi^{-1}\sigma_3\Phi)\,. \qquad (12)$$

As $x' \to +\infty$ and $x'' \to -\infty$ the integrated part on the right-hand side of (12) exactly cancels the corresponding unbounded terms on the left-hand side arising from $A^{(\pm)} = \lim_{x\to\pm\infty} A(x,t;\rho)$. The last integral in (12) which completely takes into account the effect of the time dependence of $\rho$ may be written explicitly in the form ($x' \to +\infty$, $x'' \to -\infty$),

$$-2i\rho_t \int_{-\infty}^{+\infty} dx\,x \begin{pmatrix} q\phi_2\bar\phi_2 + r\phi_1\bar\phi_1 & q\bar\phi_2^2 + r\bar\phi_1^2 \\ -q\phi_2^2 - r\phi_1^2 & -q\phi_2\bar\phi_2 - r\phi_1\bar\phi_1 \end{pmatrix}. \qquad (13)$$

In deriving (13) we have used the eigenvalue equation (1) for $\phi$ and $\bar\phi$; it is also assumed that the behavior of $q$ and $r$ at $|x| \to \infty$ are such as to guarantee the convergence of the integrals.

Substituting (13) into (12) with $x' \to +\infty$ and $x'' \to -\infty$ we arrive at the expressions for $A_0^{(+)}, B^{(+)}, C^{(+)}$ in terms of $A_0^{(-)}$ and the integrals of the form

$$I(\phi,\bar\phi) = \int_{-\infty}^{+\infty} dx\,(-q_t\phi_2\bar\phi_2 + r_t\phi_1\bar\phi_1)\,,$$
$$J(\phi,\bar\phi) = \int_{-\infty}^{+\infty} dx\,x(-q\phi_2\bar\phi_2 + r\phi_1\bar\phi_1)\,, \qquad (14)$$

and finally with the help of (10) and (11) obtain

$$(\bar b|a)_t = \frac{K(\psi,\psi)}{a\bar b}\cdot(\bar b|a)\,, \qquad (15)$$
$$(b|\bar a)_t = \frac{K(\bar\psi,\bar\psi)}{\bar a b}\cdot(b|\bar a)\,, \qquad (16)$$
$$K(\psi,\psi) = I(\psi,\psi) - 2i\rho_t J(\psi,\psi)\,. \qquad (17)$$

$K(\psi,\psi)$ is the extension of $I(\psi,\psi)$[1] to the case $\rho_t \neq 0$. As in the analogous case of time independent $\rho$ we take

$$K(\psi,\psi) = 2\Omega(\rho)a\bar b = -2\Omega(\rho)\int_{-\infty}^{+\infty} dx\,(q\psi_2^2 + r\psi_1^2)\,,$$
$$K(\bar\psi,\bar\psi) = -2\Omega(\rho)\bar a b = 2\Omega(\rho)\int_{-\infty}^{+\infty} dx\,(q\bar\psi_2^2 + r\bar\psi_1^2)\,, \qquad (18)$$

where $\Omega(\rho)$ is an arbitrary entire function of $\rho$. The time development of the scattering data are thus determined by the set of linearized equations (15) and (16). We now assume that $\rho_t = f(\rho,t)$ is also an entire function of $\rho$, the coefficients of the different powers of $\rho$ being functions of $t$. Since $\Psi = \binom{\psi_1^2}{\psi_2^2}$ satisfies the equation $L\Psi = \rho\Psi$ where the operator $L$ is identical with that in the analogous case for $\rho_t = 0$, we have

$$\Omega(\rho)\Psi = \Omega(L)\Psi; \quad f(\rho,t)\Psi = f(L,t)\Psi\,. \qquad (19)$$

A sufficient criterion for the validity of (17) then leads to the nonlinear evolution equation for the potentials $q$ and $r$,

$$[\sigma_3\partial_t - 2if(L_A,t)x + 2\Omega(L_A)]\begin{pmatrix}r\\q\end{pmatrix} = 0\,.$$

The operator $L_A$ denotes the adjoint of $L$ and is given by (3). Corresponding to the case where $\Omega(\rho), f(\rho)$ are the ratios of two entire functions we have the following generalizations:

$$[\Omega_2(L_A)f_2(L_A)\sigma_3\partial_t - 2i\Omega_2(L_A)f_1(L_A)x + 2\Omega_1(L_A)f_2(L_A)]$$
$$\times\begin{pmatrix}r\\q\end{pmatrix} = 0\,, \qquad (20)$$

where

$$\Omega(\rho) = \Omega_1(\rho)|\Omega_2(\rho); \quad f(\rho) = f_1(\rho)|f_2(\rho)\,.$$

If $A_0^{(+)} = A_0^{(-)}$ and $B^{(+)} = C^{(+)} = 0$, Eqs. (11) have the solutions

$$a(\rho(t),t) = a(\rho_0,0)\,,$$

$$b(\rho(t),t) = b(\rho_0,0)\exp\left(-2\int_0^t \Omega(\rho(t'))\,dt'\right),$$

$$\rho_0 = \rho(t=0). \qquad (21)$$

In the Marchenko equations determining $q$ and $r$ one has to make use of the above expressions for the time dependence of the scattering data. These last equations therefore enable us to obtain the solution of the nonlinear evolution equation (5) when the solutions of the corresponding homogeneous equations are known.

In particular with $f(\rho) = \epsilon$ and $\epsilon + \mu\rho^2$ and $\Omega(\rho) = i\rho^2$, Eq. (5) reduces to the nonlinear Schrödinger equation with linear and parabolic density profiles whose solutions have been obtained earlier.[6,7]

## III. EVOLUTION EQUATION CORRESPONDING TO NEWELL-KAUP EIGENVALUE PROBLEM

If (6) determines the time evolution of the solution of the eigenvalue problem

$$(\partial_x - \rho M)v = -i\rho^2\sigma_3 v, \qquad (22)$$

the Lax condition leads to

$$\rho M_t + \rho_t M - N_x + [\rho M - i\rho^2\sigma_3, N] - i(\rho^2)_t\sigma_3 = 0. \qquad (23)$$

The asymptotic behavior appropriate to this case is simply obtained by replacing $\rho$ and $\rho_t$ by $\rho^2$ and $(\rho^2)_t$ in Eqs. (8) and (9). With these modifications (22) yields in the usual manner the formal solution for $N$ in this case

$$[\Phi^{-1}N\Phi]_{x''}^{x'} = -i(\rho^2)_t[x\Phi^{-1}\sigma_3\Phi]_{x''}^{x'} + \int_{x''}^{x'} dx'\,[\rho\Phi^{-1}M_t\Phi + \rho_t\Phi^{-1}M\Phi + i(\rho^2)_t x\partial_x(\Phi^{-1}\sigma_3\Phi)]. \qquad (24)$$

Here as well we find that the unbounded terms in (24) are compensated as $x' \to +\infty$ and $x'' \to -\infty$ leading to

$$(a\bar{a} - b\bar{b})A_0^{(+)} + \bar{a}bB^{(+)} + a\bar{b}C^{(+)} = A_0^{(-)} + \widetilde{K}(\phi,\bar{\phi}),$$

$$2abA_0^{(+)} + b^2B^{(+)} - a^2C^{(+)} = -\widetilde{K}(\phi,\phi),$$

$$2\bar{a}\bar{b}A_0^{(+)} - \bar{a}^2B^{(+)} + \bar{b}^2C^{(+)} = \widetilde{K}(\bar{\phi},\bar{\phi}), \qquad (25)$$

where,

$$\widetilde{K}(\phi,\bar{\phi}) = \rho I(\phi,\bar{\phi}) - 4i\rho^2\rho_t J(\phi,\bar{\phi}) + \rho_t H(\phi,\bar{\phi}),$$

$$H(\phi,\bar{\phi}) = \int_{-\infty}^{+\infty} (-q\phi_2\bar{\phi}_2 + r\phi_1\bar{\phi}_1)\,dx. \qquad (26)$$

The time development equation for the scattering data reduce to

$$(\bar{b}|a)_t = \widetilde{K}(\psi,\psi)|a^2; \quad (b|\bar{a})_t = \widetilde{K}(\psi,\psi)|\bar{a}^2. \qquad (27)$$

If $q,r \to 0$ as $x \to \infty$, then it can be shown from (22) that $\Psi = \binom{\psi_1^2}{\psi_2^2}$ satisfies

$$\Lambda\Psi = \rho^2\Psi, \qquad (28)$$

where

$$\Lambda = \frac{1}{2}\begin{pmatrix} i\partial_x + qr + q\int_x^\infty dy\, r_y & -q^2 - q\int_x^\infty dy\, q_y \\ -r^2 - r\int_x^\infty dy\, r_y & -i\partial_x + qr + r\int_x^\infty dy\, q_y \end{pmatrix} \qquad (29)$$

Equation (27) suggests the *ansatz*

$$\widetilde{K}(\psi,\psi) = 2\Omega(\rho^2)a\bar{b} = -2\rho\Omega(\rho^2)\int_{-\infty}^{+\infty}(q\psi_2^2 + r\psi_1^2)\,dx, \qquad (30)$$

where $\Omega(\rho^2)$ is an entire function of $\rho^2$. We next assume that the explicit time dependence of $\rho$ is of the form

$$(\ln\rho^2)_t = f(\rho^2), \qquad (31)$$

where $f(\rho^2)$ is also an entire function of $\rho^2$. Substituting for $\widetilde{K}$ from (26) into (30) we obtain after the usual transformation the nonlinear evolution equation for $\binom{r}{q}$ as a sufficient criterion for the integrability of this scattering data equations:

$$\sigma_3\begin{pmatrix} r \\ q \end{pmatrix}_t + [\tfrac{1}{2}f(\Lambda_A)\sigma_3 - 2i\Lambda_A f(\Lambda_A)x + 2\Omega(L_A)]\begin{pmatrix} r \\ q \end{pmatrix} = 0, \qquad (32)$$

where $\Lambda_A$ is the adjoint operator

$$\Lambda_A = \frac{1}{2}\begin{pmatrix} -i\partial_x + qr + r_x\int_{-\infty}^x dy\, q & -r^2 - r_x\int_{-\infty}^x dy\, r \\ -q^2 - q_x\int_{-\infty}^x dy\, q & i\partial_x + qr + q_x\int_{-\infty}^x dy\, r \end{pmatrix}. \qquad (33)$$

With $\Omega(\rho^2) = -2i\rho^4$ and $f(\rho^2) = \alpha$, equation (32) reduces to

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t + \tfrac{1}{2}\alpha\begin{pmatrix} r \\ -q \end{pmatrix} + \alpha\begin{pmatrix} (-xr)_x \\ (xq)_x \end{pmatrix} + \begin{pmatrix} ir_{xx} - (r^2q)_x \\ iq_{xx} + (q^2r)_x \end{pmatrix} = 0, \qquad (34)$$

which describes approximately propagation of circularly polarized waves in a magnetoplasma in the presence of inhomogeneities. Clearly its exact solution can be obtained by the inverse scattering method.

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