Existence and stability of alternative ion-acoustic solitary wave solution of the combined MKdV-KdV-ZK equation in a magnetized nonthermal plasma consisting of warm adiabatic ions

Jayasree Das, Anup Bandyopadhyay, and K. P. Das

Citation: Physics of Plasmas (1994-present) **14**, 092304 (2007); doi: 10.1063/1.2772615
View online: http://dx.doi.org/10.1063/1.2772615
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/14/9?ver=pdfcov
Published by the AIP Publishing

Existence and stability of alternative ion-acoustic solitary wave solution of the combined MKdV-KdV-ZK equation in a magnetized nonthermal plasma consisting of warm adiabatic ions

Jayasree Das
Department of Mathematics, Chittaranjan College, 8A Beniatola Lane, Kolkata-700 009, India

Anup Bandyopadhyay
Department of Mathematics, Jadavpur University, Kolkata-700 032, India

K. P. Das
Department of Applied Mathematics, University of Calcutta, 92-Acharya Prfulla Chandra Road, Kolkata-700 009, India

(Received 29 April 2007; accepted 6 July 2007; published online 24 September 2007)

The purpose of this paper is to present the recent work of Das *et al.* [J. Plasma Phys. **72**, 587 (2006)] on the existence and stability of the alternative solitary wave solution of fixed width of the combined MKdV-KdV-ZK (Modified Korteweg-de Vries-Korteweg-de Vries-Zakharov-Kuznetsov) equation for the ion-acoustic wave in a magnetized nonthermal plasma consisting of warm adiabatic ions in a more generalized form. Here we derive the alternative solitary wave solution of variable width instead of fixed width of the combined MKdV-KdV-ZK equation along with the condition for its existence and find that this solution assumes the sech profile of the MKdV-ZK (Modified Korteweg-de Vries-Zakharov-Kuznetsov) equation, when the coefficient of the nonlinear term of the KdV-ZK (Korteweg-de Vries-Zakharov-Kuznetsov) equation tends to zero. The three-dimensional stability analysis of the alternative solitary wave solution of variable width of the combined MKdV-KdV-ZK equation shows that the instability condition and the first order growth rate of instability are exactly the same as those of the solitary wave solution (the sech profile) of the MKdV-ZK equation, when the coefficient of the nonlinear term of the KdV-ZK equation tends to zero. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2772615]

I. INTRODUCTION

The observations of solitary structures with density depletion made by the Freja Satellite,[2] influenced Cairns *et al.*[3–5] to investigate how the presence of nonthermal electrons change the properties of ion-acoustic waves for both positive and negative density perturbations. The model of the velocity distribution function of nonthermal electrons was considered for the first time by Cairns *et al.*[3] for the study of ion-acoustic solitary structures in the presence of the population of the fast energetic electrons together with a population of Maxwellian distributed electrons. This type of velocity distribution is termed as nonthermal distribution and was considered by many authors in various studies of different collective processes in plasmas and dusty plasmas.[3–19] The effect of ion temperature, external static magnetic field, and oblique propagation on the structure of these solitary waves have been carried out by Cairns *et al.*[5] The same problem for cold ions was considered by Mamun and Cairns,[6] in which they have not only shown the existence of compressive and rarefactive solitary waves but also investigated their stabilities. This paper of Mamun and Cairns[6] has been extended by Bandyopadhyay and Das[7] in the following two directions: (i) instead of taking ions as cold, the ion temperature has been included; and (ii) the case has been considered when the coefficient of the nonlinear term of the Korteweg-de Vries-Zakharov-Kuznetsov (KdV-ZK) equation derived in the first case vanishes. For case (i), Bandyopadhyay and Das[7] derived a KdV-ZK equation to describe the nonlinear behavior of the ion-acoustic wave. It is found that a factor B of the coefficient of the nonlinear term of the KdV-ZK equation derived for case (i) becomes a function of β and σ, where β is a parameter that determines the proportion of fast energetic electrons and σ is the ratio of the average temperatures of electrons and ions and this factor B becomes zero along the curve $\sigma=\sigma_\beta$ (Fig. 1) in the $\beta\sigma$-parametric plane, where σ_β is a function of β only. In this situation, a modified Korteweg-de Vries-Zakharov-Kuznetsov (MKdV-ZK) equation has been derived by Bandyopadhyay and Das[7] to investigate the nonlinear behavior of the ion-acoustic waves. In that paper they have also investigated the stability of these solitary waves by the small-k perturbation expansion method of Rowlands and Infeld.[20–26] When the coefficient of the nonlinear term of the KdV-ZK equation is of the order of zero, i.e., $B \approx \bigcirc(\varepsilon^{1/2})$, where ε is a small quantity that measures the weakness of dispersion, the KdV-ZK equation fails to describe the nonlinear behavior of the ion-acoustic wave. In this situation, i.e., for those points (β,σ) in the $\beta\sigma$ parametric plane lying in the small neighborhood of the curve $\sigma=\sigma_\beta$ (Fig. 1) in the $\beta\sigma$ parametric plane along which $B=0$, Bandyopadhyay and Das[8] derived a combined MKdV-KdV-ZK equation. This equation admits an alternative solitary wave solution of fixed width having a profile different from sech^2 or sech, propagating at an arbitrary angle δ with the uniform static magnetic field. It has also been shown in

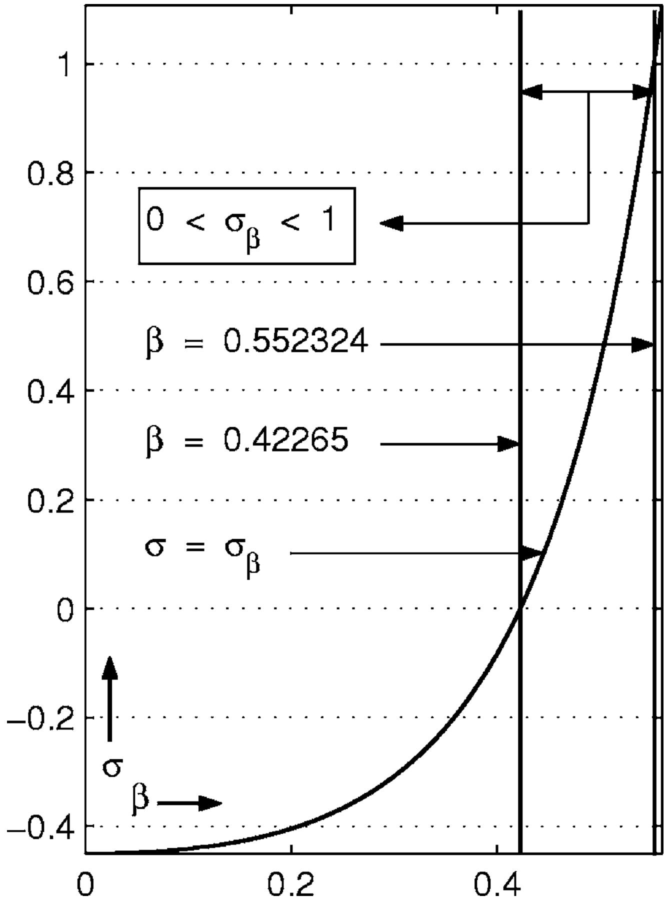

FIG. 1. σ_β is plotted against β.

that paper of Bandyopadhyay and Das[8] that the alternative solitary wave solution of fixed width having a profile different from sech^2 or sech exists if and only if $L \equiv B^2 + 12B'(\cos^2\delta + D\sin^2\delta) > 0$, where B, B' are the coefficients (except for a common factor) of the two nonlinear terms and D is the coefficient (except for a common factor) of the perpendicular dispersive term of the combined MKdV-KdV-ZK equation. Bandyopadhyay and Das[8] have used the method of Malfliet and Hereman[27] to obtain the alternative solitary wave solution of fixed width. The stability of this alternative ion-acoustic solitary wave solution of fixed width of the combined MKdV-KdV-ZK equation has recently been investigated minutely by Das *et al.*,[1] where the three-dimensional stability analysis of the alternative solitary wave solution of fixed width having a profile different from sech^2 or sech has been made by the multiple-scale perturbation expansion method of Allen and Rowlands.[28–33] The instability condition and the growth rate of instability at the lowest order have been obtained. The correct expression of the growth rate of instability at the lowest order has been obtained for a limiting case, where the coefficient of the nonlinear term of the KdV-ZK equation approaches zero and the stability analysis has been carried out numerically from the model as presented in the paper of Das *et al.*[1] for arbitrary values of the parameters involved in the system. In that paper, it has been shown that the instability condition and the growth rate of instability of the alternative solitary wave solution of fixed width are exactly the same as those of the solitary waves (the sech-profile) of the MKdV-ZK equation, when the coefficient of the nonlinear term of the KdV-ZK equation tends to zero.

In the present paper, we consider the existence and stability of the alternative ion-acoustic solitary wave solution of variable width (instead of fixed width) of the same combined MKdV-KdV-ZK equation of Bandyopadhyay and Das.[8] Here we find that an alternative solitary wave solution of variable width of the combined MKdV-KdV-ZK equation having profile different from sech^2 or sech exists if and only if $L = MB^2 \equiv B^2 + 12p^2B'(\cos^2\delta + D\sin^2\delta) > 0$. Here p is a strictly positive real number which determines the width of the alternative solitary wave. The three-dimensional stability of the alternative solitary wave solution of variable width has been investigated by the multiple-scale perturbation expansion method of Allen and Rowlands.[28–33] The instability condition and the growth rate of instability at the lowest order have been derived. All the results of Das *et al.*[1] can be obtained from the present article if we set $p=1$. The methods or process and analysis of the present paper are exactly the same as Das *et al.*[1] So, in the present article, we shall present the final results only without showing the details of calculations involved in the method of analysis.

This paper is organized as follows: The basic equations are given in Sec. II. The evolution equations are given in Sec. III. The alternative solitary wave solution of the variable width of the combined MKdV-KdV-ZK equation is given in Sec. IV. The stability analysis of the alternative solitary wave solution of variable width of the combined MKdV-KdV-ZK equation is given in Sec. V. In Sec. VI, the exact expression of the growth rate of instability along with the instability condition are given for the limiting case where the coefficient of the nonlinear term of the KdV-ZK equation approaches zero. Finally, brief conclusions are given in Sec. VII.

II. BASIC EQUATIONS

The following are the governing equations describing the nonlinear behavior of ion-acoustic waves in fully ionized collisionless plasma consisting of warm adiabatic ions and nonthermal electrons immersed in a uniform static magnetic field directed along the z-axis. Here it is assumed that the plasma beta, i.e., the ratio of particle pressure to the magnetic pressure, is very small and the characteristic frequency is much smaller than ion cyclotron frequency (Cairns *et al.*[5]),

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\varphi + \omega_c(\mathbf{u}\times\hat{z}) - \frac{\sigma}{n}\nabla p, \tag{2}$$

$$\nabla^2\varphi = n_e - n, \tag{3}$$

$$n = p^r. \tag{4}$$

Here n, n_e, $\mathbf{u}$, p, φ, (x,y,z), and t are, respectively, the ion number density, electron number density, ion fluid velocity, ion pressure, electrostatic potential, spatial variables, and time, and they have been normalized by n_0 (unperturbed ion number density), n_0, $c_s(=\sqrt{K_BT_e/m})$ (ion-acoustic speed), $n_0K_BT_i$, K_BT_e/e, $\lambda_D(=\sqrt{K_BT_e/4\pi n_0e^2})$ (Debye length) and ω_p^{-1} (ion plasma period), where $\sigma=T_i/T_e$, ω_c is the ion cyclotron frequency normalized by $\omega_p(=\sqrt{4\pi n_0e^2/m})$ and $r(=5/3)$ is the ratio of two specific heats. Here K_B is the Boltzmann constant; T_e, T_i are, respectively, the electron and ion temperatures; m is the mass of an ion and e is the electronic charge. In Eq. (4), the adiabatic law has been taken to form a closed and consistent system of equations on the basis of the assumption that the effect of viscosity, thermal conductivity, and the energy transfer due to collision can be neglected.

Since the electrons are assumed to be nonthermally distributed, the velocity distribution function of electrons normalized by n_0 can be taken as follows (Cairns *et al.*[4]):

$$f_e(v,\varphi)=\frac{1}{(1+3\alpha)\sqrt{2\pi}}\{1+\alpha(v^2-2\varphi)^2\}\times\exp\left[-\frac{1}{2}(v^2-2\varphi)\right]. \tag{5}$$

Here v is the phase space velocity of electrons normalized by $v_e=\sqrt{K_BT_e/m_e}$ (average thermal velocity of electrons) with m_e, the mass of an electron. Therefore the electron number density n_e normalized by n_0 is given by the following equation:

$$n_e=\int_{-\infty}^{\infty}f_e(v,\varphi)dv=(1-\beta\varphi+\beta\varphi^2)e^{\varphi}, \tag{6}$$

where

$$\beta=\frac{4\alpha}{1+3\alpha} \tag{7}$$

is a parameter of the nonthermal electron distribution that determines the proportion of fast energetic electrons.

In view of Eq. (4), Eq. (2) can be written as

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla\varphi+\omega_c(\mathbf{u}\times\hat{z})-\frac{5\sigma}{3}n^{-1/3}\nabla n. \tag{8}$$

Substituting n_e as given by Eq. (6) into Eq. (3), we get the following equation:

$$\nabla^2\varphi=(1-\beta\varphi+\beta\varphi^2)e^{\varphi}-n. \tag{9}$$

Equations (1), (8), and (9) are the basic equations.

III. EVOLUTION EQUATIONS

Bandyopadhyay and Das[7,8] make the following stretching of coordinates and time to derive the different evolution equations:

$$\xi=\varepsilon^{1/2}x,\quad \eta=\varepsilon^{1/2}y,$$
$$\zeta=\varepsilon^{1/2}(z-Vt),\quad \tau=\varepsilon^{3/2}t, \tag{10}$$

where ε is a small parameter measuring the weakness of the dispersion and V is a constant. With the stretchings given by Eq. (10), Eqs. (1), (8), and (9), respectively, assume the following forms:

$$-\varepsilon^{1/2}V\frac{\partial n}{\partial\zeta}+\varepsilon^{3/2}\frac{\partial n}{\partial\tau}+\varepsilon^{1/2}\nabla_{\xi}\cdot(n\mathbf{u})=0, \tag{11}$$

$$-\varepsilon^{1/2}V\frac{\partial \mathbf{u}}{\partial\zeta}+\varepsilon^{3/2}\frac{\partial \mathbf{u}}{\partial\tau}+\varepsilon^{1/2}(\mathbf{u}\cdot\nabla_{\xi})\mathbf{u}+\varepsilon^{1/2}\nabla_{\xi}\varphi-\omega_c(\mathbf{u}\times\hat{z})+\frac{5}{3}\sigma\varepsilon^{1/2}n^{-1/3}\nabla_{\xi}n=0, \tag{12}$$

$$\varepsilon\nabla_{\xi}^2\varphi=(1-\beta\varphi+\beta\varphi^2)e^{\varphi}-n, \tag{13}$$

where

$$\nabla_{\xi}=\hat{x}\frac{\partial}{\partial\xi}+\hat{y}\frac{\partial}{\partial\eta}+\hat{z}\frac{\partial}{\partial\zeta},$$

$$\mathbf{u}=(u,v,w)=u\hat{x}+v\hat{y}+w\hat{z}. \tag{14}$$

A. KDV-ZK equation

Using Eqs. (11)–(13), Bandyopadhyay and Das[7] have derived the following KdV-ZK equation for ion-acoustic waves in a fully ionized, collisionless plasma consisting of warm adiabatic ions and nonthermal electrons immersed in a uniform static magnetic field,

$$\frac{\partial\varphi^{(1)}}{\partial\tau}+AB\varphi^{(1)}\frac{\partial\varphi^{(1)}}{\partial\zeta}+\frac{1}{2}A\frac{\partial^3\varphi^{(1)}}{\partial\zeta^3}+\frac{1}{2}AD\frac{\partial}{\partial\zeta}\times\left(\frac{\partial^2\varphi^{(1)}}{\partial\xi^2}+\frac{\partial^2\varphi^{(1)}}{\partial\eta^2}\right)=0, \tag{15}$$

where they have used the following perturbation expansion of the dependent variables:

$$(n,\varphi,w)=(1,0,0)+\sum_{i=1}^{\infty}\varepsilon^{i}(n^{(i)},\varphi^{(i)},w^{(i)}),$$

$$(u,v)=\sum_{i=1}^{\infty}\varepsilon^{(i+2)/2}(u^{(i)},v^{(i)}). \tag{16}$$

Here

$$A=\frac{1}{V}\left(V^2-\frac{5\sigma}{3}\right)^2, \tag{17}$$

$$B=\frac{1}{2}\left\{\frac{3V^2-\frac{5}{9}\sigma}{\left(V^2-\frac{5}{3}\sigma\right)^3}-1\right\}, \tag{18}$$

$$D=1+\frac{V^4}{\omega_c^2}\left(V^2-\frac{5}{3}\sigma\right)^{-2}, \tag{19}$$

and the constant V is given by

$$(1-\beta)\left(V^2-\tfrac{5}{3}\sigma\right)=1. \tag{20}$$

B. MKDV-ZK equation

Using Eq. (20), the expression for B given by Eq. (18) can be written as

$$B = \tfrac{20}{9}(1-\beta)^3(\sigma - \sigma_\beta), \tag{21}$$

where

$$\sigma_\beta = \frac{9}{40}\frac{1-3(1-\beta)^2}{(1-\beta)^3}. \tag{22}$$

From the above expression of B, we find that $B=0 \Leftrightarrow \sigma = \sigma_\beta$ and consequently the factor B becomes zero along the curve $\sigma=\sigma_\beta$ (Fig. 1) in the $\beta\sigma$-parametric plane. In Fig. 1, σ_β is plotted against β. As $0<\sigma<1$, the physically admissible value of σ_β is clearly pointed out in Fig. 1. From this figure we see that if $0.42265<\beta<0.552324$, the value of σ_β lies between 0 and 1. Therefore if $0.42265<\beta<0.552324$ and $\sigma=\sigma_\beta$ then only the factor B of the coefficient of the nonlinear term of the KdV-ZK Eq. (15) vanishes. In this situation, Bandyopadhyay and Das[7] derived a modified KdV-ZK (MKdV-ZK) equation. Along with the use of the critical condition $B=0$, the following is obtained from Eqs. (11)–(13):

$$\frac{\partial \varphi^{(1)}}{\partial \tau} + AB'(\varphi^{(1)})^2\frac{\partial \varphi^{(1)}}{\partial \zeta} + \frac{1}{2}A\frac{\partial^3 \varphi^{(1)}}{\partial \zeta^3} + \frac{1}{2}AD\frac{\partial}{\partial \zeta}\left(\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2}\right) = 0, \tag{23}$$

where they have used the following perturbation expansion of the dependent variables:

$$(n,\varphi,w) = (1,0,0) + \sum_{i=1}^{\infty} \varepsilon^{i/2}(n^{(i)},\varphi^{(i)},w^{(i)}),$$
$$(u,v) = \sum_{i=1}^{\infty} \varepsilon^{(i+1)/2}(u^{(i)},v^{(i)}). \tag{24}$$

Here A and D are the same as given by Eqs. (17) and (19), respectively, and B' is given by the following equation:

$$B' = \tfrac{1}{2}\Big[(1-\beta)^4\Big\{\tfrac{10}{27}\sigma - 6V^2 + \tfrac{3}{2}(1-\beta)\big(3V^2 - \tfrac{5}{9}\sigma\big)^2\Big\} - \tfrac{1}{2}(1+3\beta)\Big], \tag{25}$$

where V is given by Eq. (20).

C. Combined MKDV-KDV-ZK equation

When $B\neq 0$ but $B\to 0$, it can be easily checked from Eq. (23) along with Eq. (25a) of Bandyopadhyay and Das[7] that the KdV-ZK equation gives an unbounded solitary wave solution for ion-acoustic wave and consequently the KdV-ZK Eq. (15) fails to describe the nonlinear behavior of the ion-acoustic wave even when $B\neq 0$ but B lies in a small neighborhood of zero. In this situation, i.e., when $B\approx \bigcirc(\varepsilon^{1/2})$, Bandyopadhyay and Das[8] derived a combined MKdV-KdV-ZK equation for ion-acoustic waves in a fully ionized collisionless plasma consisting of warm adiabatic ions and nonthermal electrons immersed in a uniform static magnetic field which remains valid near the points in the $\beta\sigma$-parametric plane where $B=0$. This combined MKdV-KdV-ZK equation is the following:

$$\frac{\partial \varphi^{(1)}}{\partial \tau} + AB\varphi^{(1)}\frac{\partial \varphi^{(1)}}{\partial \zeta} + AB'(\varphi^{(1)})^2\frac{\partial \varphi^{(1)}}{\partial \zeta} + \frac{1}{2}A\frac{\partial^3 \varphi^{(1)}}{\partial \zeta^3} + \frac{1}{2}AD\frac{\partial}{\partial \zeta}\left(\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2}\right) = 0, \tag{26}$$

where Bandyopadhyay and Das[8] have used the same perturbation expansion of the dependent variables as given by Eq. (24) along with the condition $B\approx \bigcirc(\varepsilon^{1/2})$.

Here A, B, D, B' are the same as those given, respectively, by Eqs. (17)–(19) and (25), and V is given by Eq. (20).

IV. SOLITARY WAVE SOLUTION OF THE DIFFERENT EVOLUTION EQUATIONS

For solitary wave solutions of Eqs. (15), (23), and (26) propagating at an angle δ with the uniform static magnetic field, Bandyopadhyay and Das[7,8] make the following change of variables:

$$\xi' = \xi\cos\delta - \zeta\sin\delta, \quad \eta' = \eta,$$
$$\zeta' = \xi\sin\delta + \zeta\cos\delta, \quad \tau' = \tau, \tag{27}$$
$$Z = \zeta' - U\tau'.$$

Under these changes of variables, Eqs. (15), (23), and (26) assume, respectively, the following forms, in which we drop the primes from the independent variables:

$$-U\frac{\partial \varphi^{(1)}}{\partial Z} + \frac{\partial \varphi^{(1)}}{\partial \tau} + a_1\varphi^{(1)}\frac{\partial \varphi^{(1)}}{\partial Z} + \bar{a}_1\varphi^{(1)}\frac{\partial \varphi^{(1)}}{\partial \xi} + a_3\frac{\partial^3 \varphi^{(1)}}{\partial Z^3} + a_4\frac{\partial^3 \varphi^{(1)}}{\partial \xi\,\partial Z^2} + a_5\frac{\partial^3 \varphi^{(1)}}{\partial \xi^2\,\partial Z} + a_6\frac{\partial^3 \varphi^{(1)}}{\partial \eta^2\,\partial Z} + a_7\frac{\partial^3 \varphi^{(1)}}{\partial \xi^3} + a_8\frac{\partial^3 \varphi^{(1)}}{\partial \xi\,\partial \eta^2} = 0, \tag{28}$$

$$-U\frac{\partial \varphi^{(1)}}{\partial Z} + \frac{\partial \varphi^{(1)}}{\partial \tau} + a_2(\varphi^{(1)})^2\frac{\partial \varphi^{(1)}}{\partial Z} + \bar{a}_2(\varphi^{(1)})^2\frac{\partial \varphi^{(1)}}{\partial \xi} + a_3\frac{\partial^3 \varphi^{(1)}}{\partial Z^3} + a_4\frac{\partial^3 \varphi^{(1)}}{\partial \xi\,\partial Z^2} + a_5\frac{\partial^3 \varphi^{(1)}}{\partial \xi^2\,\partial Z} + a_6\frac{\partial^3 \varphi^{(1)}}{\partial \eta^2\,\partial Z} + a_7\frac{\partial^3 \varphi^{(1)}}{\partial \xi^3} + a_8\frac{\partial^3 \varphi^{(1)}}{\partial \xi\,\partial \eta^2} = 0, \tag{29}$$

$$-U\frac{\partial \varphi^{(1)}}{\partial Z} + \frac{\partial \varphi^{(1)}}{\partial \tau} + a_1\varphi^{(1)}\frac{\partial \varphi^{(1)}}{\partial Z} + a_2(\varphi^{(1)})^2\frac{\partial \varphi^{(1)}}{\partial Z} + \bar{a}_1\varphi^{(1)}\frac{\partial \varphi^{(1)}}{\partial \xi} + \bar{a}_2(\varphi^{(1)})^2\frac{\partial \varphi^{(1)}}{\partial \xi} + a_3\frac{\partial^3 \varphi^{(1)}}{\partial Z^3} + a_4\frac{\partial^3 \varphi^{(1)}}{\partial \xi\,\partial Z^2} + a_5\frac{\partial^3 \varphi^{(1)}}{\partial \xi^2\,\partial Z} + a_6\frac{\partial^3 \varphi^{(1)}}{\partial \eta^2\,\partial Z} + a_7\frac{\partial^3 \varphi^{(1)}}{\partial \xi^3} + a_8\frac{\partial^3 \varphi^{(1)}}{\partial \xi\,\partial \eta^2} = 0. \tag{30}$$

The coefficients a_1, $\bar{a}_1$, a_2, $\bar{a}_2$, a_3, a_4, a_5, a_6, a_7, a_8 appearing in the above equations are given in Appendix A of Das *et al.*[1]

For travelling wave solutions of Eqs. (28)–(30) we set

$$\varphi^{(1)} = \varphi_0(Z). \tag{31}$$

Then Eqs. (28)–(30) assume, respectively, the following forms:

$$-U\frac{d\varphi_0}{dZ} + a_1\varphi_0\frac{d\varphi_0}{dZ} + a_3\frac{d^3\varphi_0}{dZ^3} = 0, \tag{32}$$

$$-U\frac{d\varphi_0}{dZ} + a_2(\varphi_0)^2\frac{d\varphi_0}{dZ} + a_3\frac{d^3\varphi_0}{dZ^3} = 0, \tag{33}$$

$$-U\frac{d\varphi_0}{dZ} + a_1\varphi_0\frac{d\varphi_0}{dZ} + a_2(\varphi_0)^2\frac{d\varphi_0}{dZ} + a_3\frac{d^3\varphi_0}{dZ^3} = 0. \tag{34}$$

Using the boundary condition,

$$\varphi_0, \frac{d\varphi_0}{dZ}, \frac{d^2\varphi_0}{dZ^2} \to 0 \quad \text{as } |Z| \to \infty, \tag{35}$$

the solitary wave solutions of Eqs. (32) and (33) can, respectively, be put in the following forms:

$$\varphi_0 = \frac{3U}{a_1}\,\text{sech}^2\, pZ \quad \text{with } U = 4p^2 a_3 \tag{36}$$

and

$$\varphi_0 = \pm\sqrt{\frac{6U}{a_2}}\,\text{sech}\, pZ \quad \text{with } U = p^2 a_3. \tag{37}$$

From Eqs. (36) and (37), we find that there exists a relation between U and p in each case. Any one of them can be made free and other can be determined from the relation. Now as p gives the inverse measure of the width of the solitary wave, the linear velocity U of the solitary wave increases as the width of the solitary wave decreases.

Using the boundary condition (35), we can write Eq. (34) as follows:

$$-U\varphi_0 + \frac{1}{2}a_1\varphi_0^2 + \frac{1}{3}a_2\varphi_0^3 + a_3\frac{d^2\varphi_0}{dZ^2} = 0. \tag{38}$$

According to Malfliet and Hereman,[27] we take

$$\varphi_0 = a_0\frac{\text{sech}^2\, pZ}{b_0 + c_0\,\text{sech}^2\, pZ} \tag{39}$$

as a solution of Eq. (38). Substituting Eq. (39) into Eq. (38) and following the same method as given in Sec. 4.2 of Das *et al.*,[34] the alternative solitary wave solution (39) of Eq. (38) can be put in the form

$$\varphi_0 = a\frac{S}{\Psi}, \tag{40}$$

where

$$S = \text{sech}[2pZ], \tag{41}$$

$$\Psi = S + \lambda\sqrt{M}, \tag{42}$$

$$a = \frac{12p^2(\cos^2\delta + D\sin^2\delta)}{B}, \tag{43}$$

$$M = 1 + \frac{12B' p^2(\cos^2\delta + D\sin^2\delta)}{B^2}, \tag{44}$$

$$\lambda = \pm 1. \tag{45}$$

The solution (40) exists if and only if

$$L = MB^2 > 0, \tag{46}$$

with

$$L = B^2 + 12B' p^2(\cos^2\delta + D\sin^2\delta),$$

and if this condition holds good, U is given by

$$U = 4p^2 a_3. \tag{47}$$

Equations (40)–(47) are exactly the same as Eqs. (22)–(29) of Das *et al.*[1] if we set $p=1$. The condition for existence [inequality (46)] of the alternative solitary wave solution (40) for $p=1$ has been extensively investigated by Bandyopadhyay and Das.[8] Now for any arbitrary value of p and for the other parameters involved in the system, there always exists a finite range of the nonthermal parameter β such that the condition for existence [inequality (46)] of the alternative solitary wave solution (40) holds good because first of all we note the following facts:

- From the expression of D as given by (19), we see that $D>1$ for any values of the parameters involved in the system and consequently $\cos^2\delta + D\sin^2\delta > 1$ and of course p^2 is strictly positive for any real nonzero value of p.
- As $B \approx O(\epsilon^{1/2})$, B actually approaches zero, i.e., $B \neq 0$ but $B \to 0$.
- $B \to 0 \Leftrightarrow \sigma \to \sigma_\beta$.

Therefore the sign of L as $B \to 0$ ($\Leftrightarrow \sigma \to \sigma_\beta$) mainly depends on the sign of B' as $B \to 0$ ($\Leftrightarrow \sigma \to \sigma_\beta$). Now one can easily check that

$$\lim_{B\to 0} B' = \lim_{\sigma \to \sigma_\beta} B' = B'_\beta, \tag{48}$$

where

$$B'_\beta = \frac{3\beta^4 - 12\beta^3 + 14\beta^2 + 8\beta - 4}{12(1-\beta)}. \tag{49}$$

Therefore the sign of L depends on the sign of B'_β. In Fig. 2, B'_β is plotted against β. This graph clearly shows that if $0.346724 < \beta < 0.552324$, $B'_\beta > 0$. In Fig. 3, σ_β and B'_β both are plotted against β. From Fig. 3, we see that for $0.346724 < \beta < 0.42265$, $\sigma_\beta < 0$, which is not an admissible value of σ. This graph clearly shows that if $0.42265 < \beta < 0.552324$, then $0 < \sigma_\beta < 1$ as well as $B'_\beta > 0$. Therefore if $0.42265 < \beta < 0.552324$ and σ lies in a small neighborhood of σ_β, then $L > 0$ and consequently the alternative solitary wave of variable width exists for any value of the parameters involved in the system and for a definite (fixed) range of the nonthermal parameter β.

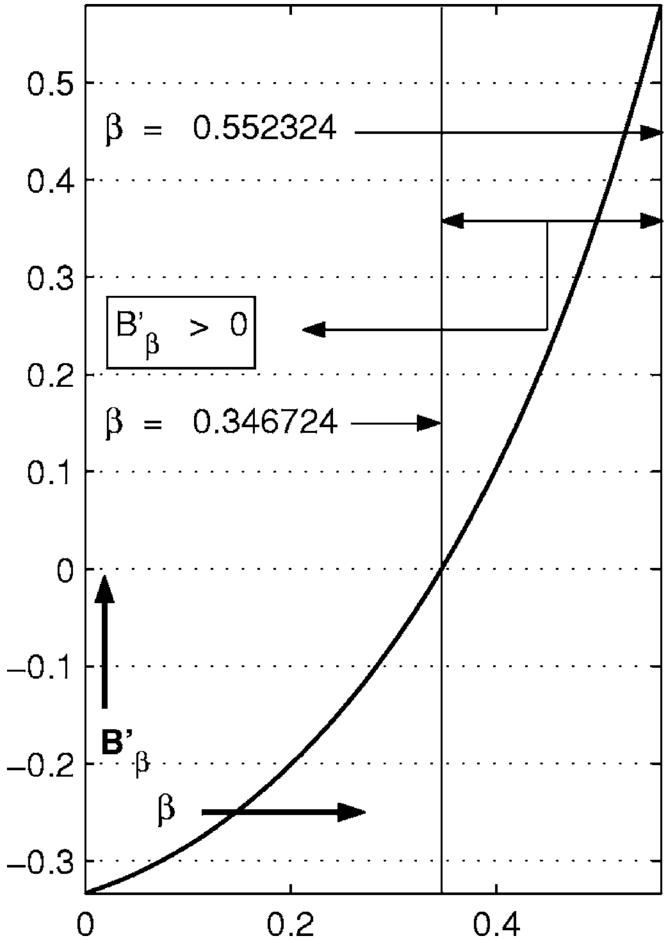

FIG. 2. B'_β is plotted against β.

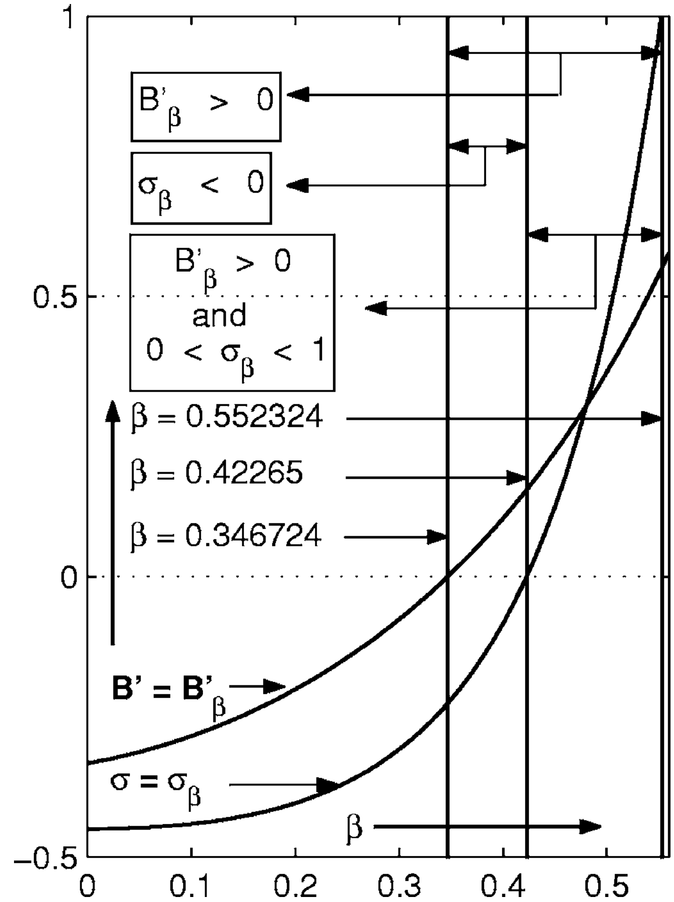

FIG. 3. σ_β and B'_β both are plotted against β.

Now as $B \approx \bigcirc(\varepsilon^{1/2})$, M is large enough and consequently we can consider the limiting case where $\sigma \to \sigma_\beta \Leftrightarrow B \to 0 \Leftrightarrow M \to \infty$. For this limiting case, we get the following equation from Eq. (40):

$$\lim_{M\to\infty} \varphi_0 = \pm \sqrt{\frac{6U}{AB' \cos \delta}} \operatorname{sech} 2pZ \tag{50}$$

with

$$U = 4p^2 a_3. \tag{51}$$

Again we can write the Eq. (37) as

$$\varphi_0 = \pm \frac{\sqrt{6U}}{\sqrt{AB' \cos \delta}} \operatorname{sech} pZ \tag{52}$$

with

$$U = p^2 a_3. \tag{53}$$

Therefore Eqs. (50) and (51) are, respectively, the same as Eqs. (52) and (53) if we replace $2p$ by p in both Eqs. (50) and (51). Thus the alternative solitary wave solution (40) simply reduces to the solitary wave solution (37) of the MKdV-ZK Eq. (23) when $B \to 0$. This is expected because we use the same perturbation expansions of the dependent variables and the same stretching of coordinates and time to derive both MKdV-ZK and combined MKdV-KdV-ZK equations. The only exception is that we use the critical condition $B=0$ to derive the MKdV-ZK equation whereas we use the condition $B \approx \bigcirc(\varepsilon^{1/2})$ to derive the combined MKdV-KdV-ZK equation. On the other hand if we take the limit $B \to 0$ on both sides of the combined MKdV-KdV-ZK Eq. (26), this equation simply reduces to the MKdV-ZK Eq. (23) and consequently it is expected that the solitary wave solution of the combined MKdV-KdV-ZK Eq. (26) will converge to the solitary wave solution of the MKdV-ZK Eq. (23) for $B \to 0$. Therefore we can conclude that under certain condition [inequality (46)] the solitary wave solution (40) of the combined MKdV-KdV-ZK equation fills the gap between the solitary wave solution (sech^2 profile) of the KdV-ZK equation and the solitary wave solution (sech profile) of the MKdV-ZK equation. In the next section, we shall again see that the stability analysis of the alternative solitary wave solution (40) of the combined MKdV-KdV-ZK Eq. (26) is exactly the same as the stability analysis of the solitary wave solution (37) (sech profile) of the MKdV-ZK Eq. (23) when $B \to 0$. More specifically, we find that the first order growth rate of instability along with the instability condition of the alternative solitary wave solution (40) of the combined MKdV-KdV-ZK Eq. (26) are exactly the same as those of the solitary wave solution (37) (sech profile) of the MKdV-ZK Eq. (23) when $B \to 0$.

V. STABILITY OF ALTERNATIVE SOLITARY WAVE

To analyze the stability of an alternative solitary wave solution (40) of Eq. (30) by the multiple-scale perturbation expansion method of Allen and Rowlands,[28,29] we decompose $\varphi^{(1)}$ as

$$\varphi^{(1)}=\varphi_0(Z)+q(\xi,\eta,Z,\tau), \tag{54}$$

where $\varphi_0(Z)$ is the steady state alternative solitary wave solution (40) of Eq. (30) and $q\ (\xi,\eta,Z,\tau)$ is the perturbed part of $\varphi^{(1)}$.

Now for long-wavelength plane wave perturbation along a direction having direction cosines (l,m,n) we set

$$q=\bar{q}(Z)e^{i\{k(l\xi+m\eta+nZ)-\omega\tau\}}, \tag{55}$$

where k is small and $l^2+m^2+n^2=1$.

According to the multiple-scale perturbation expansion method of Allen and Rowlands,[28,29] we set

$$\bar{q}(Z)=\sum_{j=0}^{\infty}k^j q^{(j)}(Z,Z_1,Z_2,Z_3,\ldots), \tag{56}$$

$$\omega=\sum_{j=0}^{\infty}k^j\omega^{(j)} \quad \text{with } \omega^{(0)}=0, \tag{57}$$

where

$$Z_j=k^jZ,\quad Z_0=Z,\quad j=0,1,2,3,\ldots, \tag{58}$$

and each $q^{(j)}[=q^{(j)}(Z,Z_1,Z_2,Z_3,\ldots)]$ is a function of $Z,Z_1,Z_2,Z_3,\ldots$.

Substituting Eq. (54) into Eq. (30) and then linearizing it with respect to q we get an equation for q, which is exactly similar to Eq. (31) of Das *et al.*[1] Due to the space-time dependence of q as given by Eq. (55), the equation of q reduces to an equation of $\bar{q}$, which is also exactly similar to the Eq. (33) of Das *et al.*.[1] Finally, substituting Eqs. (56) and (57) into the equation of $\bar{q}$ and then equating the coefficients of the different powers of k on both sides of the resulting equation, we get the following sequence of equations:

$$U\frac{\partial}{\partial Z}(L_Zq^{(j)})=Q^{(j)}, \tag{59}$$

where

$$L_Z=-1+6\frac{S}{\Psi}+6(M-1)\frac{S^2}{\Psi^2}+\frac{1}{4p^2}\frac{\partial^2}{\partial Z^2}$$

and $Q^{(j)}$ for $j=0,1,2$ are given by the following equations:

$$Q^{(0)}=0,$$

$$Q^{(1)}=i(u_1-u_2\varphi_0-u_3\varphi_0^2)q^{(0)}-iu_4q_{00}^{(0)}-2a_3q_{100}^{(0)} -U\frac{\partial}{\partial Z_1}(L_Zq^{(0)}),$$

$$Q^{(2)}=i\omega^{(2)}q^{(0)}+i(u_1-u_2\varphi_0-u_3\varphi_0^2)q^{(1)}-iu_4(q_{00}^{(1)}+2q_{10}^{(0)})+u_5q_0^{(0)}-a_3(2q_{100}^{(1)}+2q_{200}^{(0)}+3q_{110}^{(0)}) -U\frac{\partial}{\partial Z_2}(L_Zq^{(0)})-U\frac{\partial}{\partial Z_1}(L_Zq^{(1)}).$$

Here $u_1=\omega^{(1)}+nU$ and u_2, u_3, u_4, u_5 are given in Appendix B of Das *et al.*[1] and we have used the following notations to simplify the expression of $Q^{(j)}$ for $j=0$, 1, 2:

$$q_r^{(j)}=\frac{\partial q^{(j)}}{\partial Z_r},\quad q_{rs}^{(j)}=\frac{\partial^2 q^{(j)}}{\partial Z_r\,\partial Z_s},\quad q_{rst}^{(j)}=\frac{\partial^3 q^{(j)}}{\partial Z_r\,\partial Z_s\,\partial Z_t}.$$

The general solution of Eq. (59) can be put in the form

$$q^{(j)}=A_1^{(j)}f+A_2^{(j)}g+A_3^{(j)}h+\frac{4p^2}{U}\chi^{(j)}, \tag{60}$$

where

$$f=\frac{d\varphi_0}{dZ},\quad g=f\int\frac{1}{f^2}dZ,\quad h=f\int\frac{\varphi_0}{f^2}dZ,$$

and

$$\chi^{(j)}=f\int\frac{\int_{\infty}^{Z}f\int Q^{(j)}dZ}{f^2}dZ.$$

Here φ_0 is given by Eq. (40) and $A_1^{(j)}$, $A_2^{(j)}$, $A_3^{(j)}$ are all arbitrary functions of $Z_1,Z_2,Z_3,\ldots$.

Evaluating f, g, and h with the help of MATHEMATICA,[35] we can simplify the expression of $q^{(j)}$ as

$$\begin{aligned}q^{(j)}=&A_1^{(j)}\frac{d\varphi_0}{dZ}-\frac{M^{3/2}A_2^{(j)}}{8a\lambda p^2}\frac{S^{-1}}{\Psi^2}-\frac{M(4A_2^{(j)}+aA_3^{(j)})}{4ap^2}\frac{1}{\Psi^2}\\&+\left\{\frac{A_2^{(j)}(2+12M+3M^2)}{8a\lambda p^2\sqrt{M}}+\frac{2aA_3^{(j)}(1+3M)}{8a\lambda p^2\sqrt{M}}\right\}\frac{S}{\Psi^2}\\&+\frac{4A_2^{(j)}(1+2M)+aA_3^{(j)}(3+2M)}{4ap^2}\frac{S^2}{\Psi^2}\\&+\frac{3\{A_2^{(j)}(4+M)+2aA_3^{(j)}\}}{8a^2p^2}Z\frac{d\varphi_0}{dZ}\\&+\frac{4p^2}{U}f\int\frac{\int_{\infty}^{Z}f\int Q^{(j)}dZ}{f^2}.\end{aligned} \tag{61}$$

Zeroth order equation:

As $Q^{(0)}=0$, the solution of Eq. (59) for $j=0$ can be written as

$$\begin{aligned}q^{(0)}=&A_1^{(0)}\frac{d\varphi_0}{dZ}-\frac{M^{3/2}A_2^{(0)}}{8a\lambda p^2}\frac{S^{-1}}{\Psi^2}-\frac{M(4A_2^{(0)}+aA_3^{(0)})}{4ap^2}\frac{1}{\Psi^2}\\&+\left\{\frac{A_2^{(0)}(2+12M+3M^2)}{8a\lambda p^2\sqrt{M}}+\frac{2aA_3^{(0)}(1+3M)}{8a\lambda p^2\sqrt{M}}\right\}\frac{S}{\Psi^2}\\&+\frac{4A_2^{(0)}(1+2M)+aA_3^{(0)}(3+2M)}{4ap^2}\frac{S^2}{\Psi^2}\\&+\frac{3\{A_2^{(0)}(4+M)+2aA_3^{(0)}\}}{8a^2p^2}Z\frac{d\varphi_0}{dZ}.\end{aligned} \tag{62}$$

Now we note that $\Psi = S + \lambda\sqrt{M} \to \lambda\sqrt{M}$ as $|Z| \to \infty$, therefore to make $q^{(0)}$ bounded we must have

$$-\frac{M^{3/2}A_2^{(0)}}{8a\lambda p^2} = 0. \tag{63}$$

Again to make $q^{(0)}$ consistent with the boundary condition, i.e. $q^{(0)} \to 0$ as $|Z| \to \infty$, we must have

$$-\frac{M(4A_2^{(0)} + aA_3^{(0)})}{4ap^2} = 0. \tag{64}$$

Solving Eqs. (63) and (64), we get

$$A_2^{(0)} = A_3^{(0)} = 0. \tag{65}$$

Therefore $q^{(0)}$ assumes the following form:

$$q^{(0)} = A_1^{(0)}\frac{d\varphi_0}{dZ}. \tag{66}$$

With the help of this equation one can easily verify that

$$L_Z q^{(0)} = 0. \tag{67}$$

First order equation:

Substituting Eqs. (66) and (67) in the expression of $Q^{(1)}$ and then performing the integrals appearing in the expression of $\chi^{(1)}$ with the help of MATHEMATICA,[35] the expression of $\chi^{(1)}$ can be simplified as

$$\chi^{(1)} = \frac{U}{4p^2}\left\{ iA_0^{(1)}\left(s_1\frac{S^2}{\Psi^2} + s_2\frac{S}{\Psi^2}\right) - \left(\frac{\partial A_0^{(1)}}{\partial Z_1} - is_3A_0^{(1)}\right)Z\frac{d\varphi_0}{dZ}\right\}, \tag{68}$$

where

$$s_1 = \frac{a}{6U}(6u_1 - au_2),$$

$$s_2 = \frac{a}{12U\lambda\sqrt{M}}\{6(1+M)u_1 - a(2u_2 + au_3)\},$$

$$s_3 = \frac{1}{2U}(u_1 - 4p^2u_4).$$

Now using the expression of $\chi^{(1)}$ as given by Eq. (68), we obtain the complete solution for $q^{(1)}$ from Eq. (61) for $j=1$. Using the same argument as given in the lowest order solution $q^{(0)}$, $q^{(1)}$ can be made bounded and consistent if and only if

$$A_2^{(1)} = A_3^{(1)} = 0, \tag{69}$$

and consequently $q^{(1)}$ assumes the following form:

$$q^{(1)} = A_1^{(1)}\frac{d\varphi_0}{dZ} + iA_0^{(1)}\left(s_1\frac{S^2}{\Psi^2} + s_2\frac{S}{\Psi^2}\right) - \left(\frac{\partial A_0^{(1)}}{\partial Z_1} - is_3A_0^{(1)}\right)Z\frac{d\varphi_0}{dZ}. \tag{70}$$

Again we can omit the first term of the right-hand side of Eq. (70), as this type of term has already been included in the lowest order term $q^{(0)}$. Again according to Allen and Rowlands,[28,29] the last term of the right-hand side of Eq. (70) is a ghost secular term and can be removed by setting

$$\frac{\partial A_1^{(0)}}{\partial Z_1} = iA_1^{(0)}s_3. \tag{71}$$

Therefore $q^{(1)}$ assumes the following form:

$$q^{(1)} = iA_0^{(1)}\left(s_1\frac{S^2}{\Psi^2} + s_2\frac{S}{\Psi^2}\right). \tag{72}$$

Now for the existence of the solution of Eq. (59), its right-hand side must be perpendicular to the kernel of the operator adjoint to the operator $U(\partial/\partial Z)L_Z$; this kernel, which must tend to zero as $|Z| \to \infty$, is φ_0. Thus we get the following consistency condition for the existence of the solution of the Eq. (59)

$$\int_{-\infty}^{\infty} \varphi_0 Q^{(j)} dZ = 0. \tag{73}$$

It can be easily verified that the condition (73) holds good for $j=0$ and also for $j=1$. But for the existence of the solution of Eq. (59) for $j=2$, the consistency condition (73) reduces to the following quadratic equation for $u_1(=\omega^{(1)} + nU)$:

$$u_1^2 + G_1u_1 + H_1 + \pi_\lambda(G_2u_1 + H_2) = 0, \tag{74}$$

where

$$G_1 = \frac{1}{3(1-N)}\{N(3p_2 - 2p_3) - 4p_3 + 24p_4\},$$

$$G_2 = -\frac{2\sqrt{N}}{(1-N)^{3/2}}\{p_2 - 2p_3 + 8p_4\},$$

$$H_1 = \frac{1}{72(1-N)^2}\{2N^2(4p_2 - 3p_3)(2p_2 - p_3) + N(32p_2^2 - 130p_2p_3 + 83p_3^2 + 432p_2p_4 - 352p_3p_4 - 192p_4^2 + 192p_5) + 16(p_3^2 - 8p_3p_4 - 24p_4^2 + 24p_5)\},$$

$$H_2 = -\frac{\sqrt{N}}{12(1-N)^{5/2}}\{4N(4p_2^2 - 10p_2p_3 + 5p_3^2 + 24p_2p_4 - 16p_3p_4) + 3(p_3 - 8p_4)(5p_3 + 8p_4) - 10p_2p_3 + 48p_2p_4 + 192p_5\},$$

with

$$N = \frac{1}{M}, \quad \pi_\lambda = \arctan\left(\frac{\lambda - \sqrt{N}}{\sqrt{1-N}}\right),$$

$$p_2 = au_2, \quad p_3 = \frac{a^2u_3}{M-1},$$

$$p_4 = p^2u_4, \quad p_5 = Up^2u_5$$

and it can be easily checked that p_2, p_3, p_4, p_5 are all independent of $N(=1/M)$. Actually, p_2, p_3, p_4, p_5 are all functions of A, D, and δ.

Therefore there is instability if

$$G^2 - 4H < 0, \tag{75}$$

where

$$G = G_1 + \pi_\lambda G_2, \quad H = H_1 + \pi_\lambda H_2,$$

and if this condition is satisfied, then the growth rate of instability, $\gamma_1[=\mathrm{Im}(\omega^{(1)})]$ is given by the following equation:

$$\gamma_1 = \frac{1}{2}\sqrt{4H - G^2}. \tag{76}$$

VI. GROWTH RATE OF INSTABILITY FOR LARGE VALUES OF *M*

Now as $B \approx \bigcirc(\varepsilon^{1/2})$, M is large enough and consequently we can consider the limiting case where $\sigma \to \sigma_\beta \Leftrightarrow B \to 0 \Leftrightarrow M \to \infty \Leftrightarrow N \to 0$. For this limiting case, the values of G_1, G_2, H_1, H_2 are, respectively, given by the following equations:

$$\lim_{M\to\infty} G_1 = -\frac{4}{3}(p_3 - 6p_4),$$

$$\lim_{M\to\infty} G_2 = 0,$$

$$\lim_{M\to\infty} H_1 = \frac{2}{9}(p_3^2 - 8p_3p_4 - 24p_4^2 + 24p_5),$$

$$\lim_{M\to\infty} H_2 = 0,$$

and the instability condition (75) and the growth rate of instability, γ_1, as given by Eq. (76) are, respectively, simplified as follows:

$$\Gamma > 0, \tag{77}$$

$$\gamma_1^2 = \frac{4DU^2}{3(\cos^2\delta + D\sin^2\delta)^2}\Gamma, \tag{78}$$

where

$$\Gamma = l^2(1 - 3D\tan^2\delta) + m^2(\cos^2\delta + D\sin^2\delta).$$

Equation (78) is the same as Eq. (77) of Bandyopadhyay and Das,[11] where Eq. (77) of Bandyopadhyay and Das[11] gives the first order growth rate of instability of the solitary wave solution (37) of the MKdV-ZK Eq. (23). Therefore the steady state solitary wave solution (40) of the combined MKdV-KdV-ZK Eq. (26) and its first order growth rate of instability along with the instability condition are exactly the same as those of the solitary wave solution (37) (sech-profile) of the MKdV-ZK Eq. (23).

VII. CONCLUSIONS

The KdV-ZK equation describes the nonlinear behavior of the long wavelength, weakly nonlinear ion-acoustic wave propagating obliquely to a uniform static magnetic field in a nonthermal plasma.[7,8] But if the values of the parameters β and σ lie in the small neighborhood of the curve $\sigma = \sigma_\beta$ in the $\beta\sigma$-parametric plane on which the coefficient of the nonlinear term of the KdV-ZK equation vanishes, the same ion-acoustic wave is described by a combined MKdV-KdV-ZK equation, which has been derived by Bandyopadhyay and Das.[8] In that paper they have shown that this equation admits an alternative solitary wave solution of fixed width, whose profile is different from sech^2 or sech profile. The present paper investigates the existence and stability of the alternative solitary wave solution of variable width (instead of fixed width) of the same combined MKdV-KdV-ZK equation of Bandyopadhyay and Das.[8] The condition for existence of the alternative solitary wave solution of variable width of this combined MKdV-KdV-ZK equation and its first order growth rate of instability along with the instability condition are obtained correctly. For a limiting case, where $B \to 0$, i.e., when the coefficient of the nonlinear term of the KdV-ZK equation approaches zero, the steady state solitary wave solution (40) of the combined MKdV-KdV-ZK Eq. (26) and its first order growth rate of instability along with the instability condition are exactly the same as the solitary wave solution (37) (sech-profile) and its first order growth rate of instability along with the instability condition of the MKdV-ZK Eq. (23). In other wards, the alternative solitary wave solution as given by Eq. (40) of the present paper fills the gap between the sech^2-profile and sech-profile. This equivalence, on the other hand proves the correctness of derivation of the combined MKdV-KdV-ZK equation as well as the steady state solution of this equation because $N \to 0 \Leftrightarrow M \to \infty \Leftrightarrow B \to 0$ and consequently the combined MKdV-KdV-ZK Eq. (26) reduces to the MKdV-ZK Eq. (23). Numerical study on the existence and stability of the alternative solitary wave solution (40) of the combined MKdV-KdV-ZK Eq. (26) for any arbitrary value of p and for any values of the other parameters involved in the system can be done following the same procedure of Das *et al.*[1] and exactly the same type of observations are found as given in Sec. 7 of Das *et al.*[1]

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