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Effect of electron inertia on dispersive properties of Alfvén waves in cold plasmas

Sayanee Jana,[1,2] Samiran Ghosh,[3] and Nikhil Chakrabarti[1,2]
[1]Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata 700064, India
[2]Homi Bhaba National Institute, Training School Complex, Anushakti Nagar, Mumbai 400085, India
[3]Department of Applied Mathematics, University of Calcutta, 92, Acharya Prafulla Chandra Road, Kolkata 700 009, India

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The effect of electron inertia on Alfvén wave propagation is investigated in the framework of the two-fluid theory in a compressible magnetized plasma. The linear analysis of the governing equations manifests the dispersion relation of the circularly polarized Alfvén waves where the electron inertia is found to act as a source of dispersion. In the finite amplitude limit, the nonlinear Alfvén wave may be described by the Derivative Nonlinear Schrödinger equation (DNLSE) modified by third order dispersion arising due to finite electron inertia. The derived equation seems to be novel with respect to what exists in the literature of Alfvén wave dynamics. We have shown that this electron inertia modified DNLSE is completely integrable and an analytical solution is demonstrated with vanishing boundary conditions. The results are expected to be of special importance in the context of space and laboratory plasmas. *Published by AIP Publishing.*
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I. INTRODUCTION

Alfvén waves are transverse magnetohydrodynamic waves[1,2] travelling in the direction of the ambient magnetic field in a magnetized plasma. These waves have been extensively studied due to their applications in laboratory[3] and astrophysical and space plasmas.[4] The physical processes related to particle energization in magnetized plasmas,[5] self-modulation in strongly magnetized plasmas,[6] tokamak plasma heating,[7] and interplanetary shocks[8] are important areas where substantial research work can be done. The propagation of linearly polarized and circularly polarized Alfvén waves including their linear and nonlinear aspects is a subject of great interest in succeeding decades.[9–12] In a recent textbook,[13] the description of Alfvén waves has been highlighted using the two-fluid theory besides the usual single fluid description. Various interesting physics have come out especially related to the spatial scale in laboratory and space plasmas. Inspired by these formulations, we have shown here that how the electron inertia (normally undermines compared to ion) could be an important issue in Alfvén wave propagation. The electron inertia can introduce a dispersive effect which can profoundly alter the propagation characteristics of Alfvén waves.

Since Alfvén waves are long wavelength modes, they can transport energy from a distant part of the magnetosphere near to the Earth space and can play a significant role in solar corona and solar wind. Recently, various nonlinear phenomena of Alfvén waves ranging from the kinetic to inertial regime have been investigated by means of numerous laboratory observations and theoretical analysis.[14–20] Nonlinear effects of Alfvén waves in low beta plasmas have also drawn intense research interest to the plasma physicists.[21]

In plasmas, stable localized structures (viz., soliton) are observed when the nonlinearity is balanced by the wave dispersion. In the case of Alfvén waves, the dispersion arises from various plasma effects described by the generalized Ohms law, and this leads to the existence of Alfvénic solitons. Also, the coupling between the elliptically polarized magnetic field components introduces dispersive effects, and the dynamics of the finite amplitude nonlinear Alfvén wave (propagating parallel to the magnetic field) is governed by the well-known Derivative Nonlinear Schrödinger equation (DNLSE).[9,22] The DNLSE is valid in regions with low beta plasmas and magnetic fluctuations having lower order compared to the ambient magnetic field. So, near the Sun, the DNLSE describes the nonlinear evolution of finite-amplitude Alfvén waves very well and also describes Alfvénic soliton, Alfvén wave turbulence, etc.,[9,22,23] efficiently.

As mentioned before, several theoretical works have been reported on the propagation of nonlinear Alfvén waves[24–26] where the electron inertia response has been overlooked. Recently, a great deal of attention is devoted to the dispersive effects of Alfvén waves related to the electron inertial length (which ranges from the ionosphere to the magnetosphere).[27] Therefore, the main objective of the present work is to investigate how the electron inertia affects the dynamics of the finite amplitude nonlinear Alfvén wave besides the usual dispersive effects (due to coupling of the magnetic field components). For this purpose, we incorporate the electron inertia on Alfvén wave dynamics in the framework of the Lagrangian two-fluid model in a cold electron-ion plasma.[28–31] Interestingly, the electron inertia is shown to serve as a new dispersive effect, causing the amplitude decay of the perturbed magnetic field. In the quasi-linear limit, we have shown that the dynamics of the weakly nonlinear Alfvén wave is governed by a new type of modified DNLSE with the third order dispersion term arising due to the consideration of electron inertia effects. We have also investigated an important physical phenomenon known as the modulational instability which determines the conditions of the existence of solitons. This nonlinear evolution equation is found to be completely integrable.[32] An analytical solution of this novel equation is also presented.

This paper is organized as follows: in Sec. II, the basic equations in the Lagrangian coordinate are derived, and the linear analysis is presented in Sec. III. Section IV is devoted to the nonlinear analysis of weakly nonlinear Alfvén waves governed by the modified DNLS equation. In Sec. V, we have investigated the wave modulational characteristics of the nonlinear wave governed by the DNLS equation, and an analytical solution is derived in Sec. VI. Finally, in Sec. VII, we summarize the discussions of our work.

II. GOVERNING EQUATIONS

We consider a two-fluid model of a cold plasma with an uniform external magnetic field in the $\hat{e}_x$ direction ($B_0\hat{e}_x$), in which each distinct species of particle is specified by the index α, with mass m_α and charge q_α. Each collection of particles of a specific type is supposed to act as a fluid, with its own velocity V_α and number density n_α. We also assume a low-β_p plasma ($\beta_p = 8\pi nT/B_0^2 (\ll 1)$, where n is the plasma density, T is the plasma temperature, and B_0 is the strength of the magnetic field), i.e., the case where the magnetic pressure is considered to be large enough compared to the kinetic pressures of plasma species so that the cold plasma approximation is justified.[11] To investigate the dynamics of nonlinear Alfvén wave propagation parallel to the external magnetic field, the equation of motion of fluid corresponding to the species α is presented as

$$m_\alpha n_\alpha\left(\frac{\partial}{\partial t} + \mathbf{V}_\alpha\cdot\nabla\right)\mathbf{V}_\alpha = n_\alpha q_\alpha\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_\alpha\times\mathbf{B}\right). \quad (1)$$

Here, we consider only electron-ion plasma ($\alpha \equiv e, i$) in which $q_e \equiv -e$ and $q_i = e$, where e is the fundamental unit of electronic charge. The continuity equation for each fluid is

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot(n_\alpha\mathbf{V}_\alpha) = 0, \quad (2)$$

and the following Maxwell equations are

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \quad (3)$$

$$\nabla\times\mathbf{B} = \frac{4\pi\mathbf{J}}{c} = \frac{4\pi}{c}q_\alpha n_\alpha V_\alpha, \quad (4)$$

where summation convention is used. All symbols have their usual meaning. Since Alfvén waves are low-frequency modes ($\omega \ll \omega_{pe}$, electron plasma frequency), so we can neglect the displacement current compared to particle current in Eq. (4) for our study. This low frequency assumption is also consistent with the quasi-neutrality condition $n_i \approx n_e \equiv n$. To describe the Alfvén wave, we assume that all the dynamical variables are functions of x and t. On the basis of the above facts, from the continuity equations for both species, Eq. (2), we obtain

$$\frac{\partial}{\partial x}[n(V_{ix} - V_{ex})] = 0 \Rightarrow V_{ix} = V_{ex} = v\,(\text{say}), \quad (5)$$

where we assume that $V_{ix}(0,t) = V_{ex}(0,t) = 0$.

In Alfvén wave dynamics, the perturbed magnetic fields B_y and B_z arise from the spatial variation of polarization current and directed along the y-direction and z-direction, respectively. In component form, Eq. (4) can be written as

$$\frac{\partial B_z}{\partial x} = -\frac{4\pi en}{c}(V_{iy} - V_{ey}), \quad (6)$$

$$\frac{\partial B_y}{\partial x} = \frac{4\pi en}{c}(V_{iz} - V_{ez}). \quad (7)$$

In view of these continuity equations, we have

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)n = -n\frac{\partial v}{\partial x}. \quad (8)$$

There is convective operator in both Eqs. (1) and (8). The nonlinear operator can be simplified into a linear operator by introducing the Lagrangian variables (ξ, τ) through the following transformation relation:[29,33]

$$\xi = x - \int_0^\tau v(\xi, \tau')d\tau', \quad \tau = t. \quad (9)$$

With this transformation, the derivative operators are transformed accordingly similar to Ref. 31. Expressing momentum Eq. (1) in terms of these newly defined variables, we obtain the total momentum equation of the fluid

$$\frac{\partial}{\partial\tau}(m_e\mathbf{V}_e + m_i\mathbf{V}_i) = \frac{e}{c}(\mathbf{V}_i - \mathbf{V}_e)\times\mathbf{B}. \quad (10)$$

Expressing the total magnetic field as $\mathbf{B} = \hat{e}_x B_0 + \hat{e}_y B_y(x,t) + \hat{e}_z B_z(x,t)$, the x, y, and z components of Eq. (10) become

$$\frac{\partial v}{\partial\tau} = \frac{e}{(m_e + m_i)c}\left[B_z(V_{iy} - V_{ey}) - B_y(V_{iz} - V_{ez})\right], \quad (11)$$

$$\frac{\partial}{\partial\tau}(m_e V_{ey} + m_i V_{iy}) = \frac{eB_0}{c}(V_{iz} - V_{ez}), \quad (12)$$

$$\frac{\partial}{\partial\tau}(m_e V_{ez} + m_i V_{iz}) = -\frac{eB_0}{c}(V_{iy} - V_{ey}), \quad (13)$$

respectively. Here, the magnetic field associated with the wave has two components $\hat{e}_y B_y(x,t)$ and $\hat{e}_z B_z(x,t)$, which confirm the current propagation along the z and y directions, respectively. These facts can be further verified from Eqs. (6) and (7), which are given by

$$V_{iy} - V_{ey} = -\frac{c}{4\pi en(\xi,0)}\frac{\partial B_z}{\partial\xi}, \quad (14)$$

and

$$V_{iz} - V_{ez} = \frac{c}{4\pi en(\xi,0)}\frac{\partial B_y}{\partial\xi}, \quad (15)$$

respectively. Finally, combining these equations, we have in the Lagrangian variable space

$$(\mathbf{V}_i - \mathbf{V}_e)_\perp = \frac{c}{4\pi e n(\xi,0)}\left(\hat{e}_x \times \frac{\partial \mathbf{B}_\perp}{\partial \xi}\right). \quad (16)$$

Substituting $V_{iy} - V_{ey}$ and $V_{iz} - V_{ez}$ in Eqs. (11), (12), and (13), we obtain

$$\frac{\partial v}{\partial \tau} = -\frac{1}{8\pi n(\xi,0)(m_e+m_i)}\frac{\partial |\mathbf{B}_\perp|^2}{\partial \xi}, \quad (17)$$

$$\frac{\partial}{\partial \tau}(m_e V_{ey} + m_i V_{iy}) = \frac{B_0}{4\pi n(\xi,0)}\frac{\partial B_y}{\partial \xi}, \quad (18)$$

and

$$\frac{\partial}{\partial \tau}(m_e V_{ez} + m_i V_{iz}) = \frac{B_0}{4\pi n(\xi,0)}\frac{\partial B_z}{\partial \xi}, \quad (19)$$

respectively. Further combining Eqs. (18) and (19), we have

$$\frac{\partial}{\partial \tau}(m_e \mathbf{V}_{e\perp} + m_i \mathbf{V}_{i\perp}) = \frac{B_0}{4\pi n(\xi,0)}\frac{\partial \mathbf{B}_\perp}{\partial \xi}. \quad (20)$$

The evolution equation for the magnetic field can further be expressed by taking curl in the electron momentum equation (1), where $\alpha \equiv e$, and using Eq. (3), we have

$$\frac{\partial \mathbf{B}_\perp}{\partial \tau} + \mathbf{B}_\perp \frac{n}{n(\xi,0)}\frac{\partial v}{\partial \xi} - B_0 \frac{n}{n(\xi,0)}\frac{\partial \mathbf{V}_{e\perp}}{\partial \xi} = \frac{m_e c}{e}\frac{n}{n(\xi,0)}\left[\hat{e}_x \times \frac{\partial}{\partial \xi}\left(\frac{\partial \mathbf{V}_{e\perp}}{\partial \tau}\right)\right]. \quad (21)$$

The continuity equation in terms of Lagrangian variables becomes

$$\frac{\partial}{\partial \tau}\left(\frac{1}{n}\right) = \frac{1}{n(\xi,0)}\frac{\partial v}{\partial \xi}. \quad (22)$$

Furthermore, combining Eqs. (17) and (22), we have

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{1}{n}\right) = -\frac{1}{8\pi n(\xi,0)(m_e+m_i)}\frac{\partial}{\partial \xi}\left(\frac{1}{n(\xi,0)}\frac{\partial |\mathbf{B}_\perp|^2}{\partial \xi}\right). \quad (23)$$

Next, replacing U_{iy} and U_{iz} in Eqs. (14) and (15), respectively, with the help of Eqs. (18) and (19), and further adding, we obtain

$$\frac{\partial \mathbf{V}_{e\perp}}{\partial \tau} = \frac{B_0}{4\pi n(\xi,0)(m_e+m_i)}\frac{\partial \mathbf{B}_\perp}{\partial \xi} - \frac{m_i c}{4\pi e n(\xi,0)(m_e+m_i)}\left(\hat{e}_x \times \frac{\partial^2 \mathbf{B}_\perp}{\partial \xi \partial \tau}\right). \quad (24)$$

Now, Eqs. (16), (21), (22), and (24) can be combined together to give the following equation in a more compact form as

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{\mathbf{B}_\perp}{n}\right) - \frac{B_0^2}{4\pi n(\xi,0)(m_e+m_i)}\frac{\partial}{\partial \xi}\left[\frac{1}{n(\xi,0)}\frac{\partial \mathbf{B}_\perp}{\partial \xi}\right] = -\frac{B_0 c(m_i-m_e)}{4\pi e n(\xi,0)(m_e+m_i)}\frac{\partial}{\partial \xi}\left[\frac{1}{n(\xi,0)}\left(\hat{e}_x \times \frac{\partial^2 \mathbf{B}_\perp}{\partial \tau \partial \xi}\right)\right] + \frac{c^2 m_e m_i}{4\pi e^2 n(\xi,0)(m_e+m_i)}\frac{\partial}{\partial \xi}\left[\frac{1}{n(\xi,0)}\frac{\partial^3 \mathbf{B}_\perp}{\partial \tau^2 \partial \xi}\right]. \quad (25)$$

Now, we are going to analyze the nonlinear system [Eqs. (23) and (25)] in a simplified form adopting the Lagrangian mass variable technique.[30,34] For this, let us define the following new Lagrangian mass variable ζ instead of ξ

$$\zeta = \int^{\xi} n(\xi',0)d\xi',$$

which yields the mathematical operator

$$\frac{\partial}{\partial \zeta} = \frac{1}{n(\xi,0)}\frac{\partial}{\partial \xi}.$$

Then, introducing this new mass variable ζ in Eqs. (23) and (25), we obtain the following simplified couple equations:

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{1}{n}\right) = -\frac{V_A^2}{2}\frac{\partial^2}{\partial \zeta^2}|\mathbf{B}_\perp|^2, \quad (26)$$

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{\mathbf{B}_\perp}{n}\right) - V_A^2 \frac{\partial^2 \mathbf{B}_\perp}{\partial \zeta^2} = -V_A \lambda \frac{\partial^2}{\partial \zeta^2}\left(\hat{e}_x \times \frac{\partial \mathbf{B}_\perp}{\partial \tau}\right) + \delta^2 \frac{\partial^2}{\partial \zeta^2}\left(\frac{\partial^2 \mathbf{B}_\perp}{\partial \tau^2}\right), \quad (27)$$

where $\mathbf{B}_\perp \equiv \mathbf{B}_\perp / B_0$, $n \equiv n/n_0$, $V_A = B_0/\sqrt{4\pi n_0(m_e+m_i)}$ is the Alfvén velocity, $\lambda = (c^2(m_i-m_e)^2/[4\pi n_0 e^2(m_i+m_e)])^{1/2}$ is the ion inertial length, and $\delta = (c^2 m_e m_i/[4\pi n_0 e^2(m_e+m_i)])^{1/2}$ is the skin depth arising due to electron's finite mass. These two couples of partial differential Eqs. (26) and (27) are the governing equations of the nonlinear, dispersive circularly polarized Alfvén wave in electron-ion plasmas. These equations are very complicated to solve exactly with their full nonlinearity. Therefore, in Sec. IV, we will investigate the finite amplitude nonlinear solutions keeping up to the third order nonlinear term.

III. LINEAR ANALYSIS

Before going to detailed nonlinear analysis, we implement the perturbative scheme to identify the basic linear modes involved in this study. So, we linearize Eq. (27) by considering the perturbation entities with a value much smaller than unity such as $n = 1 + \tilde{n}$ and $\mathbf{B}_\perp = \tilde{\mathbf{B}}_\perp$ and obtain the following linear equation:

$$\left(\frac{\partial^2}{\partial t^2} - V_A^2 \frac{\partial^2}{\partial x^2}\right)\tilde{\mathbf{B}}_\perp = -V_A \lambda \hat{e}_x \times \frac{\partial^3 \tilde{\mathbf{B}}_\perp}{\partial t \partial x^2} + \delta^2 \frac{\partial^4 \tilde{\mathbf{B}}_\perp}{\partial t^2 \partial x^2}. \quad (28)$$

Then, assuming the solution in the form of Fourier mode $\tilde{f} \sim f_k \exp[-i(\omega t - kx)]$ (where ω and k are the oscillation frequency and wave number), we obtain the following dispersion relation in dimensionless form:

$$(1+\delta^2 k^2)^2 \omega^4 - (2 + 2\delta^2 k^2 + \lambda^2 k^2)k^2 \omega^2 + k^4 = 0, \quad (29)$$

which describes the left-hand (ω_-) and right-hand (ω_+) circularly polarized waves

$$\omega_{\pm}^2 = k^2 \frac{1+(\lambda^2+2\delta^2)\frac{k^2}{2}}{(1+\delta^2k^2)^2} \times \left[1 \pm \sqrt{1-\left(\frac{(1+\delta^2k^2)}{1+(\lambda^2+2\delta^2)\frac{k^2}{2}}\right)^2}\right], \tag{30}$$

where ω, k, λ, and δ are made dimensionless using $\omega \to \omega L/V_A$, $k \to kL$, $\lambda \to \lambda/L$, and $\delta \to \delta/L$ respectively. Here, L is the typical system length and V_A is the Alfvén velocity. Here, the wave dynamics is regulated by magnetic pressure, ion inertia, and electron inertia. The dispersion relation clearly shows that the waves are dispersive in nature. Here, we will focus to study the dynamics of the right-hand polarized wave. Figure 1 presents dispersion relations of the right-hand polarized wave in the absence and presence of electron inertia. It shows that in the absence of electron inertia, the dispersive curve asymptotically increases, whereas the dispersive effect of electron inertia arrests the wave propagation and saturation occurs.

IV. WEAK AMPLITUDE NONLINEAR DYNAMICS

Having complete perception of the linear modes involved in our study, we continue to study the nonlinear regime of nonlinear Alfvén waves under the weak amplitude limit. For dependent variables, proposing

$$n = 1+\tilde{n} \text{ and } \mathbf{B}_\perp = \tilde{\mathbf{B}}_\perp \text{ with } \tilde{n},\ |\mathbf{B}_\perp| < 1,$$

keeping up to the second order term and substituting, from Eq. (27), we obtain

$$\left(\frac{\partial^2}{\partial\tau^2} - V_A^2\frac{\partial^2}{\partial\zeta^2}\right)\tilde{\mathbf{B}}_\perp = -V_A\lambda\left(\hat{e_x}\times\frac{\partial^3\tilde{\mathbf{B}}_\perp}{\partial\tau\partial\zeta^2}\right) + \delta^2\frac{\partial^4\tilde{\mathbf{B}}_\perp}{\partial\tau^2\partial\zeta^2} + \frac{\partial^2}{\partial\tau^2}(\tilde{\mathbf{B}}_\perp\tilde{n}). \tag{31}$$

The above equation represents both left-hand and right-hand polarized Alfvén waves modified by the dispersions (arising due to both ion inertia and electron inertia effects) and nonlinearity.

Moreover, the small amplitude nonlinear wave equations are derived by assuming that the equilibrium density is homogeneous, i.e., $n(\xi,0)=1$; therefore, $\zeta = \xi\,[\equiv x - \int v(\xi,\tau')d\tau']$. Also, in this weak amplitude limit, $\xi(\zeta)\equiv x$ and $\tau \equiv t$ (actually in this case, $\xi(\zeta)$ and τ are no longer remain Lagrangian variables but become equivalent to x and t). Therefore, we rewrite the above Eq. (31) in the following form:

$$\left(\frac{\partial}{\partial t} - V_A\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + V_A\frac{\partial}{\partial x}\right)\tilde{\mathbf{B}}_\perp - \frac{\partial^2}{\partial x^2}(\tilde{\mathbf{B}}_\perp\tilde{n}) - V_A\lambda\left(\hat{e_x}\times\frac{\partial^3\tilde{\mathbf{B}}_\perp}{\partial t\partial x^2}\right) + \delta^2\frac{\partial^4\tilde{\mathbf{B}}_\perp}{\partial t^2\partial x^2}. \tag{32}$$

For the Alfvén wave propagating only in the positive x direction, from the above relation, we get

$$\frac{\partial}{\partial t} = -V_A\frac{\partial}{\partial x},$$

and this approximation yields [from Eq. (26)]

$$\tilde{n} \approx \frac{|\tilde{\mathbf{B}}_\perp|^2}{2}.$$

Then, substituting all these in Eq. (32) and integrating the transformed equation once with the boundary condition at $x\to\infty$, $|\tilde{\mathbf{B}}_\perp|\to 0$, we obtain

$$\left(\frac{\partial}{\partial t} + V_A\frac{\partial}{\partial x}\right)\tilde{\mathbf{B}}_\perp + \frac{V_A}{4}\frac{\partial}{\partial x}(\tilde{\mathbf{B}}_\perp|\tilde{\mathbf{B}}_\perp|^2) = -\frac{V_A\lambda}{2}\left(\hat{e_x}\times\frac{\partial^2\tilde{\mathbf{B}}_\perp}{\partial x^2}\right) - \frac{\delta^2}{2}\frac{\partial^3\tilde{\mathbf{B}}_\perp}{\partial x^3}. \tag{33}$$

Finally, a further transformation of coordinates

$$\hat{x} = x - V_A t \text{ and } \hat{t} = t,$$

renders the following equation:

$$\frac{\partial\tilde{\mathbf{B}}_\perp}{\partial\hat{t}} + \frac{V_A}{4}\frac{\partial}{\partial\hat{x}}(\tilde{\mathbf{B}}_\perp|\tilde{\mathbf{B}}_\perp|^2) = -\frac{V_A\lambda}{2}\left(\hat{e_x}\times\frac{\partial^2\tilde{\mathbf{B}}_\perp}{\partial\hat{x}^2}\right) - \frac{\delta^2}{2}\frac{\partial^3\tilde{\mathbf{B}}_\perp}{\partial\hat{x}^3}. \tag{34}$$

Then, normalizing Eq. (34) by $\bar{t}\to\hat{t}/t_0, \bar{x}\to\hat{x}/L$ and $\tilde{\mathbf{B}}_\perp \to \alpha\mathbf{b}_\perp$, we arrive at the Vector Derivative Nonlinear Schrödinger Equation (VDNLS) with third order dispersion

$$\frac{\partial\mathbf{b}_\perp}{\partial\bar{t}} + \frac{\partial}{\partial\bar{x}}(\mathbf{b}_\perp|\mathbf{b}_\perp|^2) + \left(\hat{e_x}\times\frac{\partial^2\mathbf{b}_\perp}{\partial\bar{x}^2}\right) + \frac{\partial^3\mathbf{b}_\perp}{\partial\bar{x}^3} = 0, \tag{35}$$

where

$$t_0 = \frac{2c(m_em_i)^2}{eB_0(m_i-m_e)^3},\ L = \frac{cm_em_i}{(m_i-m_e)\sqrt{4\pi n_0e^2(m_e+m_i)}},$$

and $\alpha = (m_i - m_e)\sqrt{\frac{2}{m_em_i}}$.

Finally, we can write Eq. (35) as the complex Derivative Nonlinear Schrödinger Equation (DNLSE) for the right-hand polarized Alfvén wave

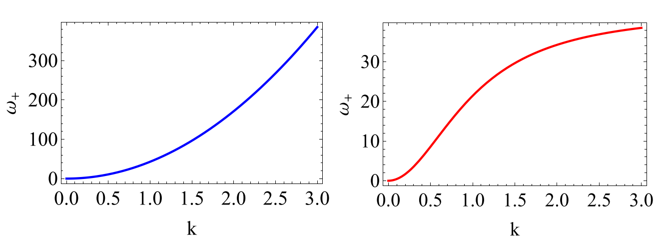

FIG. 1. Comparison of the dispersion curve for the right-hand polarized wave in the absence and presence of electron inertia with $\lambda = 42.8$. The left figure (blue solid curve) is drawn in the absence of electron inertia ($\delta = 0$), and the right figure (red solid curve) is drawn in the presence of inertia ($\delta = 1$).

$$\frac{\partial \phi}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\left(|\phi|^2\phi\right) - i\frac{\partial^2 \phi}{\partial \bar{x}^2} + \frac{\partial^3 \phi}{\partial \bar{x}^3} = 0, \tag{36}$$

where $\phi = b_y - ib_z$. It is very important to note that the Nonlinear Schrödinger Equation governs only nonlinear modulation of the complex amplitude of the carrier wave; in contrast to this, the DNLSE describes the whole nonlinear modulation of the complex field. In our study, the modified DNLSE, with the third order dispersion term arising due to the electron inertia effect, describes the nonlinear modulation of the Alfvén wave. For the sake of a more generalized equation, a parameter ϵ can be set to Eq. (36) as

$$\frac{\partial \phi}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\left(|\phi|^2\phi\right) - i\frac{\partial^2 \phi}{\partial \bar{x}^2} + \epsilon\frac{\partial^3 \phi}{\partial \bar{x}^3} = 0, \tag{37}$$

where ϵ can either be 1 or 0. If we neglect the second order dispersive term, the modified DNLSE (36) is reduced to the following well-known Complex modified Korteweg-de Vries (CMKdV) equation[35]

$$\frac{\partial \phi}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\left(|\phi|^2\phi\right) + \epsilon\frac{\partial^3 \phi}{\partial \bar{x}^3} = 0. \tag{38}$$

On the other hand, if we put $\epsilon = 0$, we recover the following well-known DNLSE

$$\frac{\partial \phi}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\left(|\phi|^2\phi\right) - i\frac{\partial^2 \phi}{\partial \bar{x}^2} = 0. \tag{39}$$

Here, we must emphasize that the DNLSE was derived in a fluid system (water waves) and its solution, soliton, is applicable in nearly all branches of physics. Thus, the derived modified DNLSE (36) should be applicable mostly to all physical systems and is a very generalized equation from any point of view.

V. MODULATIONAL INSTABILITY

In this section, following the way used in Refs. 12,16,17, and 36–39 we analyze the modulational characteristics of the right-hand polarized Alfvén wave. For this purpose, we perform a linear stability analysis of the plane wave solution for Eq. (36). For the simplification of notation removing the bar signs, Eq. (36) for the right-hand polarized wave becomes

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\left(|\phi|^2\phi\right) - i\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^3 \phi}{\partial x^3} = 0. \tag{40}$$

It is easy to find out that Eq. (40) possesses the plane-wave solution with constant amplitude as

$$\phi = \phi_0 \exp\left[-i(ax + \gamma t)\right], \tag{41}$$

where ϕ_0 and a are real parameters, and $\gamma = a(a + a^2 - \phi_0^2)$. Therefore, for stability analysis, we consider the perturbation about this stable solution in the following standard procedure:

$$\phi = \left[\phi_0 + \tilde{\phi}(x,t)\right]\exp\left[-i(ax + \gamma t)\right], \tag{42}$$

where $\tilde{\phi}(x,t)(|\tilde{\phi}| \ll \phi_0)$ is the perturbed amplitude of the modulated wave. Then, the substitution of Eq. (42) into Eq. (40) yields the following linearized two coupled equations:

$$\frac{\partial \tilde{\phi}_R}{\partial t} + \frac{\partial^3 \tilde{\phi}_R}{\partial x^3} + \left(3\phi_0^2 - 3a^2 - 2a\right)\frac{\partial \tilde{\phi}_R}{\partial x} + (3a+1)\frac{\partial^2 \tilde{\phi}_I}{\partial x^2} = 0$$

$$\text{and}\quad \frac{\partial \tilde{\phi}_I}{\partial t} + \frac{\partial^3 \tilde{\phi}_I}{\partial x^3} + \left(\phi_0^2 - 3a^2 - 2a\right)\frac{\partial \tilde{\phi}_I}{\partial x} - (3a+1)\frac{\partial^2 \tilde{\phi}_R}{\partial x^2} - 2a\phi_0^2\tilde{\phi}_R = 0, \tag{43}$$

where $\tilde{\phi} = \tilde{\phi}_R + i\tilde{\phi}_I$, $\phi_{R(I)}$ is the real (imaginary) part of ϕ.

Finally, the space-time dependence of the perturbation of the form $\tilde{\phi} \sim \exp(i\vartheta)$, where $\vartheta(= \Lambda x - \Omega t)$ is the modulated phase with $(\Lambda \ll a)$ and $\tilde{\Omega}(\ll \gamma)$ being the wave number and modulation frequency, respectively, yields the following dispersion relation:

$$\Omega = \Lambda(2\phi_0^2 - 3a^2 - 2a - \Lambda^2) \pm \Lambda\sqrt{(3a+1)^2\Lambda^2 - 2a(3a+1)\phi_0^2 + \phi_0^4}, \tag{44}$$

from which we can say that the dispersion relation depends on the values of the plane wave amplitude ϕ_0 together with the wave number Λ. When $(3a+1)^2\Lambda^2 - 2a(3a+1)\phi_0^2 + \phi_0^4 < 0$, the frequency becomes complex at any value of the wave number Λ and the disturbance will grow(decay) into bright(dark) solitons depending on the positive(negative) complex part of the frequency.

VI. ANALYTICAL SOLUTION

In this section, we try to solve Eq. (36) analytically. As is well known, the DNLSE is completely integrable and preserves the infinite number of conserved quantities.[40,41] Like the DNLSE, the newly developed modified DNLSE is also found to be completely integrable and preserves the infinite number of conserved quantities. According to the conservation laws, the first three of those conserved quantities are

$$\text{Energy}(\mathcal{E}) = \int_{-\infty}^{+\infty} |\phi|^2 dx, \tag{45}$$

$$\text{Momentum}(\mathcal{M}) = \int_{-\infty}^{+\infty}\left[i\left(\phi\phi_x^* - \phi^*\phi_x\right) - |\phi|^4\right]dx, \tag{46}$$

$$\text{Hamiltonian}(\mathcal{H}) = \int_{-\infty}^{+\infty}\left[|\phi_x|^2 + \frac{i}{4}|\phi|^2\left(\phi^*\phi_x - \phi\phi_x^*\right)\right]dx. \tag{47}$$

So, it is possible to find the analytical solution of the derived modified DNLSE (40).

Therefore, we solve Eq. (40) using moving frame analysis. To find out nonlinear solution, we transform Eq. (40) into the moving frame $\xi = x - ut$, where u is the phase velocity of the wave. Then, the first integral of the transform equation leads to the following equation:

$$\frac{d^2\phi}{d\xi^2} + (|\phi|^2 - u)\phi - i\frac{d\phi}{d\xi} = 0, \tag{48}$$

subject to the boundary conditions $\phi \to 0$, and all derivatives $\to 0$, as $\xi \to \pm\infty$.

Assuming a stationary solution of the form

$$\phi(\xi) = \sqrt{\psi}(\xi)e^{i\theta(\xi)}, \tag{49}$$

with real functions ψ and θ, and substituting in Eq. (48), we obtain a pair of coupled equations for ψ and θ

$$\frac{d^2\psi}{d\xi^2} - \frac{1}{2\psi}\left(\frac{d\psi}{d\xi}\right)^2 - 2\psi\left(\frac{d\theta}{d\xi}\right)^2 + 2\psi\frac{d\theta}{d\xi} + 2(\psi - u)\psi = 0, \tag{50}$$

$$2\frac{d\psi}{d\xi}\frac{d\theta}{d\xi} + 2\psi\frac{d^2\theta}{d\xi^2} - \frac{d\psi}{d\xi} = 0 \Rightarrow \frac{d}{d\xi}\left[2\psi\frac{d\theta}{d\xi} - \psi\right] = 0. \tag{51}$$

Integrating once using the boundary condition $\phi \to 0$ as $\xi \to \pm\infty$, we have

$$\theta(\xi) = \theta_0 + \frac{1}{2}\xi. \tag{52}$$

With Eq. (52), we can rewrite Eq. (50) as the second order differential equation in ψ

$$\frac{d^2\psi}{d\xi^2} - \frac{1}{2\psi}\left(\frac{d\psi}{d\xi}\right)^2 + \left(\frac{1}{2} - 2u\right)\psi + 2\psi^2 = 0. \tag{53}$$

Multiplying Eq. (53) by $\frac{2}{\psi}\frac{d\psi}{d\xi}$, we have

$$\frac{d}{d\xi}\left[\frac{1}{\psi}\left(\frac{d\psi}{d\xi}\right)^2\right] = \frac{d}{d\xi}\left[(4u-1)\psi - 2\psi^2\right]. \tag{54}$$

Integrating Eq. (54) using the boundary conditions $\phi \to 0$ as $\xi \to \pm\infty$, we can arrive at

$$\int \frac{d\psi}{\psi\sqrt{a^2 - 2\psi}} = \xi + c_1, \tag{55}$$

where we have defined $a^2 = 4u - 1$ and considered $c_1 = 0$, which is consistent with the boundary conditions.

Finally, integrating Eq. (55), we get

$$\psi = \frac{a^2}{2}\operatorname{sech}^2\left(\frac{a\xi}{2}\right). \tag{56}$$

Therefore, the final solution is given by

$$\phi = \sqrt{\left(2u - \frac{1}{2}\right)}\operatorname{sech}\left(\sqrt{u - \frac{1}{4}}(x - ut)\right) \times \exp\left[i\left(\theta_0 + \frac{1}{2}(x - ut)\right)\right]. \tag{57}$$

The soliton solution is shown graphically in Fig. 2 for wave phase velocity $u = 1$. It is found that the dispersive effect caused by the electron inertia together with ion inertia can balance the nonlinear steepening of waves, leading to the formation of a soliton.

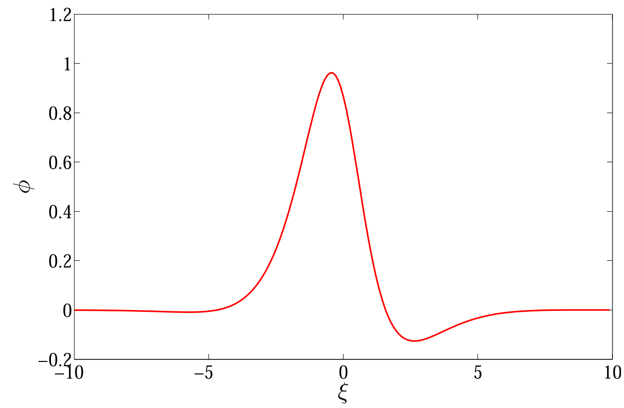

FIG. 2. Formation of single soliton in the moving frame ξ with $u = 1$.

VII. DISCUSSION

In this paper, we have investigated the effect of electron inertia on the Alfvén wave propagation in the frame work of the Lagrangian two-fluid model in a cold electron-ion plasma. The complete linear analysis indicates the saturation of the right-hand polarized wave in the presence of the dispersive effect of electron inertia. We have also shown that the dynamics of the weakly nonlinear Alfvén wave is governed by modified DNLSE with the third order dispersion term. The equation reflects that the third order dispersion arises solely due to the consideration of finite electron inertia. This nonlinear equation is analyzed by means of analytical calculation, and soliton type solutions are obtained. These results could be useful in interpreting solitary Alfvén wave propagation, reported by satellites in different parts of the magnetosphere and near earth space.

[1]H. Alfvén, Nature **150**, 405 (1942).
[2]H. L. Pecseli, *Waves and Oscillations in Plasmas* (CRC Press, 2012).
[3]B. Buti, V. L. Galinski, V. I. Shevchenko, G. Lakhina, B. T. Tsurutani, B. E. Goldstein, P. Diamond, and M. V. Medvedev, Astrophys. J. **523**, 849 (1999).
[4]N. F. Cramer, *The Physics of Alfvén Waves* (Wiley-VCH, Germany, 2001).
[5]D. J. Wu and L. Yang, Astron. Astrophys. **452**, L7 (2006).
[6]R. A. López, F. A. Asenjo, V. Muñoz, C.-L. C. Abraham, and J. A. Valdivia, Phys. Rev. E **88**, 023105 (2013).
[7]M. N. Rosenbluth and P. H. Rutherford, Phys. Rev. Lett. **34**, 1428 (1975).
[8]L. Stenflo, A. B. Shvartsburg, and J. Weiland, Phys. Lett. A **225**, 113 (1997).
[9]C. F. Kennel, B. Buti, T. Hada, and R. Pellat, Phys. Fluids **31**, 1949 (1988).
[10]J. Zhao, Phys. Plasmas **22**, 042115 (2015).
[11]A. Sarkar, N. Chakrabarti, and H. Schamel, Phys. Plasmas **22**, 072307 (2015).
[12]S. Jana, S. Ghosh, and N. Chakrabarti, Phys. Plasmas **23**, 072304 (2016).
[13]P. M. Bellan, *Fundamental of Plasma Physics* (The Cambridge University Press, Cambridge, 2006).
[14]T. A. Carter, B. Brugman, P. Pribyl, and W. Lybarger, Phys. Rev. Lett. **96**, 155001 (2006).
[15]S. Dorfman and T. A. Carter, Phys. Rev. Lett. **110**, 195001 (2013).
[16]A. Panwar, H. Rizvi, and C. M. Ryu, Phys. Plasmas **20**, 052103 (2013).
[17]A. Panwar, H. Rizvi, and C. M. Ryu, Phys. Plasmas **20**, 082101 (2013).
[18]Y. Lin, J. R. Johnson, and X. Wang, Phys. Rev. Lett. **109**, 125003 (2012).
[19]J. S. Zhao, Y. Voitenko, D. J. Wu, and J. D. Keyser, Astrophys. J. **785**, 139 (2014).
[20]J. S. Zhao, Y. Voitenko, J. D. Keyser, and D. J. Wu, Astrophys. J. **799**, 222 (2015).

[21]M. L. Rinawa, N. Gaur, and R. P. Sharma, Phys. Plasmas **22**, 022310 (2015).
[22]E. Mjølhus, J. Plasma Phys. **19**, 437 (1978).
[23]C. Kennel and R. Sagdeev, J. Geophys. Res. **72**, 3327, doi:10.1029/JZ072i013p03327 (1967).
[24]K. Mio, T. Ogino, K. Minami, and S. Takeda, J. Phys. Soc. Jpn. **41**, 265 (1976).
[25]E. Mjølhus and J. Wyller, Phys. Scr. **33**, 442 (1986).
[26]E. Mjølhus, Phys. Scr. **40**, 227 (1989).
[27]K. Stasiewicz, J. Geophys. Res.: Space Phys. **110**, A03220, doi:10.1029/2004JA010852 (2005).
[28]J. M. Dawson, Phys. Rev. **113**, 383 (1959).
[29]R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic Press, New York, 1972).
[30]H. Schamel, Phys. Rep. **392**, 279 (2004).
[31]N. Chakrabarti, C. Maity, and H. Schamel, Phys. Rev. E **88**, 023102 (2013).
[32]M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, 1991), Vol. 149.
[33]R. C. Davidson and P. P. Schram, Nucl. Fusion **8**, 183 (1968).
[34]R. Courant, K. Friedrichs, and H. Lewy, IBM J. Res. Dev. **11**, 215 (1967).
[35]C. F. Karney, A. Sen, and F. Y. Chu, Phys. Fluids **22**, 940 (1979).
[36]M. J. Ablowitz and H. Suger, *Solitons and the Inverse Scattering Transform* (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1981).
[37]O. Bang, W. Krolikowski, J. Wyller, and J. J. Rasmussen, Phys. Rev. E **66**, 046619 (2002).
[38]W. Krolikowski, O. Bang, J. J. Rasmussen, and J. Wyller, Phys. Rev. E **64**, 016612 (2001).
[39]S. Ghosh, S. Sarkar, M. Khan, and M. R. Gupta, Phys. Rev. E **84**, 066401 (2011).
[40]D. J. Kaup and A. C. Newell, J. Math. Phys. **19**, 798 (1978).
[41]J. Borhanian and A. Rezaei, Phys. Plasmas **24**, 022302 (2017).