

A version of this article will appear in Journal of Mathematical Physics

$\mathcal{CP}\mathcal{T}$ -conserved effective mass Hamiltonians through first and higher order charge operator \mathcal{C} in a supersymmetric framework

B. Bagchi and A. Banerjee*

*Department of Applied Mathematics, University of Calcutta,
92 Acharya Prafulla Chandra Road, Kolkata – 700009, India
e-mail: bbagchi123@rediffmail.com, abhijit_banerjee@hotmail.com*

A. Ganguly

*Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, India
e-mail: gangulyasish@rediffmail.com, aganguly@maths.iitkgp.ernet.in*

(Dated: June 26, 2018)

This paper examines the features of a generalized position-dependent mass Hamiltonian H_m in a supersymmetric framework in which the constraints of pseudo-Hermiticity and $\mathcal{CP}\mathcal{T}$ are naturally embedded. Different representations of the charge operator are considered that lead to new mass-deformed superpotentials $\mathcal{W}_m(x)$ which are inherently \mathcal{PT} -symmetric. The qualitative spectral behavior of H_m is studied and several interesting consequences are noted.

PACS numbers: 02.30.Tb, 03.65.Ca, 03.65.Db, 03.65.Ge

Keywords: Position-dependent (Effective) mass, $\mathcal{CP}\mathcal{T}$ -conservation, supersymmetric quantum mechanics, \mathcal{PT} -symmetry, pseudo-Hermiticity

I. INTRODUCTION

Non-Hermitian systems admitting \mathcal{PT} -symmetry (i.e. invariance under a combined action of parity \mathcal{P} and time-reversal \mathcal{T}) have been a subject matter of intense interest [1, 2]. \mathcal{PT} -symmetry has an interesting implication that the whole class of Schrödinger Hamiltonians coming under its assignment namely, $H = p^2/2m + V(x)$ defined on the real line $x \in \mathbb{R}$, where the potential is typically $V(x) = V^*(-x)$, may possess real or conjugate pairs of energy eigenvalues under certain conditions related to \mathcal{PT} being unbroken (i.e. exact) or spontaneously broken. It has also been realized that the concept of \mathcal{PT} -symmetry has its roots in the theory of pseudo-Hermitian operators and that pseudo-Hermiticity serves as one of the plausible necessary and sufficient conditions for the reality of the spectrum [3].

In [4] a set of intertwining relations

$$H\zeta = \zeta H^\dagger \quad (1.1)$$

was studied in which a Hermitian operator ζ was proposed to be expressed as a product of the charge operator \mathcal{C} and parity operator \mathcal{P}

$$\zeta = \mathcal{C}\mathcal{P} \quad (\zeta = \zeta^\dagger). \quad (1.2)$$

It is straightforward to see that equations (1.1) and (1.2) together imply the $\mathcal{CP}\mathcal{T}$ conservation of the Hamiltonian H , \mathcal{T} being the time reversal operator

$$\mathcal{CP}\mathcal{T}H = H\mathcal{CP}\mathcal{T}. \quad (1.3)$$

Interestingly, it also follows from (1.1) that the operator ζ^{-1} , if it exists, also fulfills the intertwining relations

$$H^\dagger \zeta^{-1} = \zeta^{-1} H \quad (1.4)$$

implying that H is pseudo-Hermitian with respect to ζ^{-1} . This can be verified as follows:

$$\langle \psi, H\phi \rangle_{\zeta^{-1}} = \langle \psi, \zeta^{-1} H\phi \rangle = \langle \psi, H^\dagger \zeta^{-1} \phi \rangle = \langle H\psi, \zeta^{-1} \phi \rangle = \langle H\psi, \phi \rangle_{\zeta^{-1}}. \quad (1.5)$$

* Permanent address: Department of Mathematics, Krishnath College, Berhampore, Murshidabad 742101, India

Differential realizations for \mathcal{C} have been considered in the literature such as $\mathcal{C} = d/dx + \mathcal{W}(x)$ for the first-order [4] and $\mathcal{C} = d^2/dx^2 + \mathcal{W}(x)d/dx + \mathcal{U}_0(x)$ for the second-order case [5]. The aims of such models have been to search for closed-form solutions and to work out the solvability criterion of the embedded Hamiltonian.

In this article we intend to investigate these and related aspects of pseudo-Hermiticity and $\mathcal{CP}\mathcal{T}$ -conservation for extended versions of Schrödinger equation admitting SUSY in a position-dependent (effective) mass (PDM) framework. The 1-D effective mass Hamiltonian $H \rightarrow H_m$ obeys (in the atomic unit defined by $\hbar^2 = 2$) in real spatial coordinate [6]:

$$H_m(x)\psi_n(x) \equiv \left(-\partial \left[\frac{1}{m(x)} \partial \right] + \tilde{V}_m(x) \right) \psi_n(x) = E_n \psi_n(x), \quad \tilde{V}_m(x) = V_m(x) + \rho(m), \quad (1.6)$$

where $m(x)$ is a real valued mass function in the presence of a complex potential $\tilde{V}_m(x)$:

$$V_m(x) = V_m^R(x) + iV_m^I(x). \quad (1.7)$$

In equation (1.6), the mass-dependent function $\rho(m)$ has the form [7]

$$\rho(m) = \frac{1 + b m''(x)}{2} \frac{m'(x)}{m^2(x)} - c \frac{m'^2(x)}{m^3(x)}, \quad c = 1 + b + a(a + b + 1), \quad (1.8)$$

where a and b are the usual ambiguity parameters [8] typical to the effective-mass models. Position dependence in mass shows up in different areas of physics - semiconductors [9], quantum dots [10], ^3He clusters [11] and many more. A number of papers have been written on the issue of PDM in this rapidly expanding literature [12–34].

Note that the question of boundedness and invertibility of the operator ζ , assuming an explicit representation for it was addressed in [4] for the constant-mass case. In the PDM scenario, the problem is trickier and will be taken up elsewhere.

II. PSEUDO-HERMITICITY AND $\mathcal{CP}\mathcal{T}$ -SYMMETRY IN A SUPERSYMMETRIC FRAMEWORK

In the framework of supersymmetric (SUSY) quantum mechanics [35–38], an underlying anticommutator K of the supercharges Q and \bar{Q} can be explicitly constructed by specifying the following representation

$$K = \{Q, \bar{Q}\} = \begin{pmatrix} \zeta \zeta^* & 0 \\ 0 & \zeta^* \zeta \end{pmatrix}, \quad (2.1)$$

where Q and \bar{Q} are defined in terms of the operator ζ and its complex conjugate ζ^*

$$Q = \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & 0 \\ \zeta^* & 0 \end{pmatrix}. \quad (2.2)$$

Assuming polynomial expansions

$$\zeta \zeta^* = \sum_{k=0}^N l_k (H_m)^{N-k}, \quad \zeta^* \zeta = \sum_{k=0}^N l_k (H_m^*)^{N-k}, \quad [l_0 \equiv 1, H_m^0 \equiv I_2] \quad (2.3)$$

we get by post-multiplying the first relation and pre-multiplying the second relation above by ζ and subtracting

$$0 = \sum_{k=0}^{N-1} \left[l_k (H_m)^{N-k} \zeta \right] - \sum_{k=0}^{N-1} \left[l_k \zeta (H_m^*)^{N-k} \right]. \quad (2.4)$$

Similarly, by pre-multiplying the first relation and post-multiplying the second relation by ζ^* and subtracting

$$0 = \sum_{k=0}^{N-1} \left[l_k \zeta^* (H_m)^{N-k} \right] - \sum_{k=0}^{N-1} \left[l_k (H_m^*)^{N-k} \zeta^* \right]. \quad (2.5)$$

(2.4) and (2.5) lead to the intertwining relations

$$H_m \zeta = \zeta H_m^*, \quad H_m^* \zeta^* = \zeta^* H_m. \quad (2.6)$$

At play are also the following constraints

$$\text{Pseudo-hermiticity constraint: } \zeta^\dagger = \zeta \Rightarrow \mathcal{C}^\dagger[-x] = \mathcal{C}[x], \quad (2.7)$$

$$\mathcal{CPJ} \text{ constraint: } \mathcal{CPJ}H_m = H_m\mathcal{CPJ} \Rightarrow \mathcal{C}[x]H_m^*[-x] = H_m[x]\mathcal{C}[x], \quad (2.8)$$

$$\text{SUSY constraint: } \zeta\zeta^* = \sum_{k=0}^N l_k H_m^{N-k} \Rightarrow \mathcal{C}[x]\mathcal{C}^*[-x] = \sum_{k=0}^N l_k H_m^{N-k}. \quad (2.9)$$

Finally, in the context of the N -th order SUSY, a mass-deformed superpotential $\mathcal{W}_m(x)$ can be introduced which is given by the form [28]

$$\mathcal{W}_m(x) = \mathcal{W}(x) - \frac{N}{2} \left[\frac{1}{\sqrt{m^N(x)}} \right]'. \quad (2.10)$$

where $\mathcal{W}(x)$ corresponds to the superpotential of the constant mass case. A natural consequence of (2.10) is that unlike $\mathcal{W}(x)$ as in the constant-mass case $\mathcal{W}_m(x)$ turns out to be \mathcal{PJ} -symmetric from the pseudo-Hermiticity constraint (2.7) as will be revealed below.

III. FIRST ORDER CHARGE OPERATOR

The first order representation of the charge operator \mathcal{C} in a PDM scheme is given by

$$\mathcal{C} = \frac{1}{\sqrt{m(x)}} \frac{d}{dx} + \mathcal{W}(x). \quad (3.1)$$

From (2.3) we have for $N = 1$ the projections

$$\zeta\zeta^* = H_m + l_1, \quad \zeta^*\zeta = H_m^* + l_1. \quad (3.2)$$

Imposing the pseudo-hermiticity restriction (2.7), we have the solutions:

$$m(x) = m(-x), \quad \mathcal{W}(x) = \mathcal{W}^*(-x) - \frac{1}{2} \frac{m'(x)}{m^{3/2}(x)}. \quad (3.3)$$

It is evident from (2.10) and (3.3) that

$$\mathcal{W}_m(x) = \mathcal{PJ}\mathcal{W}_m(x). \quad (3.4)$$

implying $\mathcal{W}_m(x)$ to be \mathcal{PJ} -symmetric and the mass function to be parity-invariant. As remarked earlier, $\mathcal{W}(x)$ ceases to be \mathcal{PJ} -symmetric. A \mathcal{PJ} -symmetric $\mathcal{W}_m(x)$ can be implemented by choosing for $\mathcal{W}(x)$ the form say, $\mathcal{W}(x) = \exp(i\alpha x) + h(x)$ where $\alpha \in \mathbb{R}$ and a non- \mathcal{PJ} $h(x)$ can be confronted with a suitable parity-invariant mass function leaving $\mathcal{W}_m(x)$ to be \mathcal{PJ} -symmetric. The following concrete example is one we have in mind

$$\mathcal{W}(x) = \exp(i\alpha x) - \sin(x), \quad m(x) = \frac{1}{4} \sec^2(x) \Rightarrow \mathcal{W}_m(x) = \exp(i\alpha x) \quad 0 < x < \frac{\pi}{2} \quad (3.5)$$

where $\mathcal{W}_m(x)$ is a periodic potential.

Turning to the \mathcal{CPJ} -constraint (2.8) and using (3.1) we get two relations. While comparison of the ∂ -term yields the difference

$$\tilde{V}_m(x) - \mathcal{PJ}\tilde{V}_m(x) = 2 \frac{\mathcal{W}'_m(x)}{\sqrt{m(x)}}, \quad (3.6)$$

the remaining part results in a first-order differential equation which can be readily integrated to provide for $\tilde{V}_m(x)$ the expression

$$\tilde{V}_m(x) = \mathcal{W}_m^2(x) + \frac{\mathcal{W}'_m(x)}{\sqrt{m(x)}} + \frac{1}{4} \frac{m''}{m^2} - \frac{7}{16} \frac{m'^2}{m^3} + \Lambda, \quad (3.7)$$

where Λ is an arbitrary constant of integration.

A non-trivial form for $\tilde{V}_m(x)$ is also obtained on employing the SUSY constraint (2.9) for $N = 1$ namely

$$\mathcal{C}[x]\mathcal{C}^*[-x] = H_m + l_1. \quad (3.8)$$

A comparison between (3.7) and (3.8) fixes $\Lambda = -l_1$. (3.8) is our final form of $\tilde{V}_m(x)$ for the $N = 1$ case. Note that the underlying $\mathcal{CP}\mathcal{T}$ -invariance has the implication

$$H_m \mathcal{CP}\mathcal{T}\psi(x) = \mathcal{CP}\mathcal{T}H_m\psi(x) = \mathcal{CP}\mathcal{T}E\psi(x) = E^* \mathcal{CP}\mathcal{T}\psi(x). \quad (3.9)$$

Thus if (ψ, E) is an eigenpair of a $\mathcal{CP}\mathcal{T}$ -invariant PDM Hamiltonian H_m , then $(\mathcal{CP}\mathcal{T}\psi, E^*)$ must form another eigenpair provided $\mathcal{CP}\mathcal{T}\psi \neq 0$. Thus $\mathcal{CP}\mathcal{T}$ -invariance of ψ leads to the corresponding Hamiltonian having real eigenvalues.

From the first relation of (2.3) and (1.6), the ground state ψ_0 in the $N = 1$ case has to obey

$$\zeta^* \psi_0(x) = 0 \Rightarrow \left[\frac{1}{\sqrt{m(x)}} \partial + \mathcal{W}^*(x) \right] \psi_0(-x) = 0, \quad (3.10)$$

with the lowest eigenvalue $-l_1$. From (3.10) we find

$$\psi_0(x) = \mathcal{N}_0 m^{1/4}(x) \exp \left[\int^x \sqrt{m(y)} \mathcal{W}_m(y) dy \right], \quad (3.11)$$

\mathcal{N}_0 being the normalization constant. Note that $\psi_0(x)$ is non- \mathcal{PT} -symmetric.

IV. SECOND ORDER CHARGE OPERATOR

We now look at the following mass-dependent second order representation of the charge operator \mathcal{C}

$$\mathcal{C} = \frac{1}{m(x)} \frac{d^2}{dx^2} + \mathcal{W}(x) \frac{d}{dx} + \mathcal{U}_0(x), \quad (4.1)$$

accompanied by the $N = 2$ SUSY representations

$$\zeta \zeta^* = H_m^2 + l_1 H_m + l_2 I_2. \quad (4.2)$$

as follows from (2.3). For the literature on $N = 2$ SUSY in the constant-mass case we refer to the readers [39–44].

Employing the pseudo-Hermiticity requirement (2.7) gives the following solutions

$$m(-x) = m(x), \quad \mathcal{W}_m(x) = \mathcal{PT} \mathcal{W}_m(x) \quad (4.3)$$

which are similar to the $N = 1$ case i.e. $\mathcal{W}_m(x)$ is \mathcal{PT} -symmetric and the mass function $m(x)$ is parity-invariant. Note that according to (2.10), $\mathcal{W}_m(x)$ is related to the constant-mass superpotential $\mathcal{W}(x)$ by

$$\mathcal{W}_m(x) = \mathcal{W}(x) - \left(\frac{1}{m} \right)'. \quad (4.4)$$

As an illustrative example we can take this time

$$\mathcal{W}(x) = \exp(i\alpha x) - \sin(x), \quad m(x) = \sec(x) \quad \alpha \in \mathbb{R}, \quad 0 < x < \frac{\pi}{2} \quad (4.5)$$

leading again to a periodic \mathcal{PT} -symmetric $\mathcal{W}_m(x) = \exp(i\alpha x)$.

Apart from (4.3), the pseudo-Hermiticity condition also furnishes another relation namely

$$\Delta \mathcal{U}_0(x) \equiv \mathcal{U}_0(x) - \mathcal{PT} \mathcal{U}_0(x) = \mathcal{W}'_m(x), \quad (4.6)$$

which reflects the non- \mathcal{PT} -symmetric character of the function $\mathcal{U}_0(x)$ present in (4.1).

Next, consideration of the $\mathcal{CP}\mathcal{T}$ requirement (2.8) furnishes

$$\Delta \tilde{V}_m(x) \equiv \tilde{V}_m(x) - \mathcal{PT} \tilde{V}_m(x) = 2\mathcal{W}'_m(x) + \frac{m'}{m} \mathcal{W}_m(x) \quad (4.7)$$

which is slightly different in form from the $N = 1$ result (3.7). In (4.7) $\tilde{V}_m(x)$ is restricted by

$$\tilde{V}_m(x) = \Delta \tilde{V}_m(x) + f(x) - \mathcal{U}_0(x) + \Lambda \quad (4.8)$$

where

$$f(x) = \frac{1}{2} \left[m \mathcal{W}_m^2(x) - \frac{m'}{m} \mathcal{W}_m(x) - \mathcal{W}_m'(x) \right] \quad (4.9)$$

on making use of (4.3) and (4.6). The constant Λ appears in (4.8) through the process of integration and is left arbitrary at this stage.

In addition to (4.7) and (4.8), the non- \mathcal{PT} -function $\mathcal{U}_0(x)$ has to satisfy the differential equation

$$\left[\frac{\mathcal{U}_0'(x)}{m(x)} \right]' - \mathcal{U}_0(x) \Delta \tilde{V}_m(x) + \mathcal{W}_m(x) \left\{ \mathcal{PT} \tilde{V}_m(x) \right\}' + \left[\frac{\left\{ \mathcal{PT} \tilde{V}_m(x) \right\}'}{m(x)} \right]' = 0. \quad (4.10)$$

Substitution of (4.8) into (4.10) converts it to the form

$$\mathcal{U}_0'(x) \mathcal{W}_m(x) + \mathcal{U}_0(x) \left[2 \mathcal{W}_m'(x) + \frac{m'}{m} \mathcal{W}_m(x) \right] = \frac{f''(x)}{m} + f'(x) \left[\mathcal{W}_m(x) - \frac{m'}{m^2} \right], \quad (4.11)$$

which may be solved to arrive at

$$\mathcal{U}_0(x) = \frac{f'(x)}{m(x) \mathcal{W}_m(x)} + \frac{f^2(x)}{m(x) \mathcal{W}_m^2(x)} + \frac{\Theta}{m \mathcal{W}_m^2(x)}. \quad (4.12)$$

Θ being an arbitrary constant of integration.

We now attend to the SUSY constraint (2.9). Here we need to compare the five coefficients of ∂^ℓ , $\ell = 0, 1, 2, 3, 4$. While the first two produce solutions similar to (4.3), the last three respectively yields the following three relations:

$$2 \frac{1}{m(x)} \tilde{V}_m(x) + \frac{1}{m(x)} [\mathcal{U}_0(x) + \mathcal{PT} \mathcal{U}_0(x)] - 2 \frac{1}{m(x)} \mathcal{W}_m'(x) = \mathcal{W}_m^2(x) - \left(\frac{1}{m(x)} \right)' \mathcal{W}_m(x) - l_1 \frac{1}{m(x)}, \quad (4.13)$$

$$\frac{1}{m(x)} [\Delta \mathcal{U}_0(x)]' + \mathcal{W}_m(x) \Delta \mathcal{U}_0(x) = \frac{1}{m(x)} \mathcal{W}_m''(x) + [\mathcal{W}_m^2(x)/2]', \quad (4.14)$$

$$\left[\frac{1}{m(x)} \left\{ \mathcal{PT} \mathcal{U}_0(x) \right\}' \right]' + \mathcal{W}_m(x) \left\{ \mathcal{PT} \mathcal{U}_0(x) \right\}' + \mathcal{U}_0(x) \mathcal{PT} \mathcal{U}_0(x) = \tilde{V}_m^2(x) + l_1 \tilde{V}_m(x) + l_2 - \left[\frac{1}{m(x)} \tilde{V}_m'(x) \right]' \quad (4.15)$$

To tackle the set of equations (4.13)–(4.15), we observe that the second equation here can be integrated out entirely to have

$$\Delta \mathcal{U}_0(x) = \mathcal{W}_m'(x) + C \exp \left[- \int^x m(y) \mathcal{W}_m(y) dy \right], \quad (4.16)$$

where C is a constant of integration. But C has to be set equal to zero to be consistent with (4.6). So we are left with (4.6) only. Incorporating it along with (4.7), (4.9) and (4.13) $\tilde{V}_m(x)$ reads

$$\tilde{V}_m(x) = \Delta \tilde{V}_m(x) + f(x) - \mathcal{U}_0(x) - l_1/2, \quad (4.17)$$

Then looking at (4.8) prompts us to identify $\Lambda = -\frac{l_1}{2}$ and recast $\tilde{V}_m(x)$ as

$$\tilde{V}_m(x) = \frac{3}{2} \mathcal{W}_m'(x) + \frac{m'}{2m} \mathcal{W}_m(x) + \frac{m}{2} \mathcal{W}_m^2(x) - \mathcal{U}_0(x) - \frac{l_1}{2}. \quad (4.18)$$

We now focus on the remaining SUSY constraint (4.15). This can be converted to a second-order differential equation

$$\left[\frac{1}{m(x)} \left\{ \mathcal{U}_0(x) + \tilde{V}_m^*(-x) \right\}' \right]' - \mathcal{U}_0'(x) \mathcal{W}_m(x) + [\mathcal{U}_0(x) - \mathcal{W}_m'(x)] \mathcal{U}_0(x) = \left[\tilde{V}_m^*(-x) + \frac{l_1}{2} \right]^2 + \left(l_2 - \frac{l_1^2}{4} \right), \quad (4.19)$$

by applying the \mathcal{PJ} -operator on both sides and rearranging. Note that the action of \mathcal{PJ} on any function $g(x)$ is to be understood in the usual sense: $\mathcal{PJ} g(x) = g^*(-x)$, $\mathcal{PJ} g'(x) = -g'^*(-x)$ and so on. The nonlinear term \mathcal{U}_0^2 in (4.19) is redundant and can be eliminated in the following way. Using the relation $\tilde{V}_m^*(-x) \equiv \mathcal{PJ} \tilde{V}_m(x) = \tilde{V}_m(x) - \Delta \tilde{V}_m(x)$, (4.18) results in

$$\tilde{V}_m^*(-x) + \mathcal{U}_0(x) = f(x) - \frac{l_1}{2}. \quad (4.20)$$

Employing (4.20), (4.19) can be reduced to the first order form

$$\mathcal{W}_m(x)\mathcal{U}'_0(x) + [\mathcal{W}'_m(x) - 2f(x)]\mathcal{U}_0(x) = \left[\frac{1}{m(x)}(x)f'(x) \right]' - f^2(x) + \left[\frac{l_1^2}{4} - l_2 \right]. \quad (4.21)$$

Equation (4.21) which essentially results from the SUSY constraint (4.15) is consistent with the \mathcal{CPJ} equation (4.11) for $\mathcal{U}_0(x)$ given by (4.12) should we identify $\Theta = \left[l_2 - \frac{l_1^2}{4} \right]$. In terms of $\delta = +\sqrt{l_1^2 - 4l_2}$ we express $\mathcal{U}_0(x)$ as

$$\begin{aligned} \mathcal{U}_0(x) = & \frac{m(x)\mathcal{W}_m^2(x)}{4} + \frac{\mathcal{W}'_m(x)}{2} - \frac{\mathcal{W}_m''(x)}{2m(x)\mathcal{W}_m(x)} + \frac{1}{m(x)} \left(\frac{\mathcal{W}'_m(x)}{2\mathcal{W}_m(x)} \right)^2 \\ & + \frac{3}{4} \frac{m''(x)}{m^3(x)} - \frac{m''(x)}{2m^2(x)} - \frac{1}{m(x)} \left(\frac{\delta}{2\mathcal{W}_m(x)} \right)^2. \end{aligned} \quad (4.22)$$

As a specific example we can go for the choice (4.5) which would give

$$\begin{aligned} \mathcal{U}_0(x) = & \frac{1}{4} \sec(x) \exp(2i\alpha x) - \frac{\delta^2}{4} \cos(x) \exp(-2i\alpha x) + \frac{i\alpha}{2} \exp(i\alpha x) \\ & + \frac{\alpha^2}{4} \cos(x) + \frac{1}{4} \sin^2(x) \sec(x) - \frac{1}{2} \sec(x). \end{aligned} \quad (4.23)$$

Evidently $\mathcal{U}_0(x)$ is non- \mathcal{PJ} -symmetric.

Let us now analyze the solution of the zero-mode equation

$$\zeta^* \psi(x) = 0 \Rightarrow \left[\frac{1}{m} \partial^2 + \mathcal{W}^*(x) \partial + \mathcal{U}_0^*(x) \right] \psi(-x) = 0. \quad (4.24)$$

Two linearly independent solutions of zero-mode equation (4.24) may be expressed in the following compact form (see for details [28]):

$$\psi_j(x) = \mathcal{N}_j \sqrt{m(x)\mathcal{W}_m(x)} \exp \left[\int^x F_j(y) dy \right], \quad (4.25)$$

where

$$F_j(x) = \frac{m(x)\mathcal{W}_m^2(x) + (-1)^j \delta}{2\mathcal{W}_m(x)}, \quad j = 1, 2. \quad (4.26)$$

These solutions will correspond the ground and first excited states of H_m , a feature known in the quadratic SUSY algebra.

Now it follows from the quadratic SUSY algebra (4.2) that the lowest eigenvalues of H_m are roots of the following quadratic equation

$$E^2 + l_1 E + l_2 = 0 \Rightarrow E_0 = -\frac{l_1 + \delta}{2}, E_1 = -\frac{l_1 - \delta}{2}. \quad (4.27)$$

It is clear that the lowest two eigenvalues $E_{0,1}$ will be purely real if and only if the SUSY constants l_1, l_2 satisfy following inequality

$$l_1^2 \geq 4l_2. \quad (4.28)$$

It may be pointed out that the condition $l_1^2 \geq 4l_2$ was identified with the reducibility of the second-order SUSY construction [40] in the context of Hermitian QM. In non-Hermitian QM, we have shown that the same condition is related with the reality of the spectra.

V. N -TH ORDER CHARGE OPERATOR

The charge conjugate operator \mathcal{C} may be represented as N -th order differential operator with N coefficient functions

$$\mathcal{C} = \frac{1}{\sqrt{m^N}} \partial^N + \mathcal{W}(x) \partial^{N-1} + \sum_{j=0}^{N-2} \mathcal{U}_j(x) \partial^j, \quad N = 1, 2, \dots \quad (5.1)$$

Some of the previous results are possible to generalize. Firstly, the pseudo-Hermiticity constraint (2.7) need to be compared order by order from both sides for N -th order representation (5.1) of \mathcal{C} . To do this, we note that the contributions from the adjoint operation on the term $g(x) \partial^\ell$ may be computed using the Libneitz rule as follows

$$[g(x) \partial^\ell]^\dagger = (-1)^\ell \partial^\ell [g(x)] = (-1)^\ell \sum_{r=0}^{\ell} [\ell C_r \partial^r \{g(x)\} \partial^{\ell-r}]. \quad (5.2)$$

Then order by order comparison gives the following restrictions on the coefficient functions in the charge operator \mathcal{C} given by (5.1)

$$\left. \begin{aligned} \ell = N : \quad & \mathcal{P}[m(x)] = m(x), \quad (\text{mass is parity-invariant}) \\ \ell = N - 1 : \quad & \mathcal{P}\mathcal{J}[\mathcal{W}_m(x)] = \mathcal{W}_m(x). \quad (\text{superpotential is } \mathcal{P}\mathcal{J}\text{-invariant}) \end{aligned} \right\}, \quad N = 1, 2, 3, \dots \quad (5.3)$$

One may compare the general result derived above with the corresponding results for $N = 1$ [see (3.3) and (3.4)] and for $N = 2$ [see (4.3)]. Note that the second condition means as usual that the $\text{Re } \mathcal{W}_m(x)$ is an even function while its imaginary part $\text{Im } \mathcal{W}_m(x)$ is an odd function.

In contrast to the superpotential \mathcal{W}_m , the functions $\mathcal{U}_j, j = 0$ to $N - 2$, are not $\mathcal{P}\mathcal{J}$ -symmetric. For instance, for $\ell = N - 2$ and $\ell = N - 3$ we have

$$\mathcal{U}_{N-2}(x) - \mathcal{P}\mathcal{J}[\mathcal{U}_{N-2}(x)] = (N - 1) \mathcal{W}'_m(x), \quad N = 2, 3, \dots \quad (5.4)$$

$$\mathcal{U}_{N-3}(x) - \mathcal{P}\mathcal{J}[\mathcal{U}_{N-3}(x)] = (N - 2) \left[-\frac{N-1}{2} \left\{ \frac{N}{6} \left(\frac{1}{\sqrt{m^N}} \right)''' + \mathcal{W}''_m \right\} + \mathcal{U}'_{N-2}(x) \right], \quad N = 3, 4, \dots, \quad (5.5)$$

and so on. More generally,

$$\Delta \mathcal{U}_{N-s}(x) = \mathcal{U}_{N-s}(x) - \mathcal{P}\mathcal{J} \mathcal{U}_{N-s}(x) = {}^N C_s \partial^s \left(\frac{1}{\sqrt{m^N}} \right) + {}^{N-1} C_{s-1} \partial^{s-1} [\mathcal{P}\mathcal{J} \mathcal{W}(x)] + \sum_{j=1}^{s-2} {}^{N-s+j} C_j \partial^j [\mathcal{P}\mathcal{J} \{ \mathcal{U}_{N-s+j}(x) \}], \quad (5.6)$$

From the results, it is clear that the pseudo-Hermiticity constraints measure the amount of $\mathcal{P}\mathcal{J}$ -asymmetry in the coefficient functions. In particular, the measure is zero for first coefficient $m(x)$ and mass-deformed superpotential $\mathcal{W}_m(x)$.

Next comparing the coefficients of each derivative ∂^ℓ for $\ell = 0, 1, \dots, N + 2$ from both sides of the $\mathcal{C}\mathcal{P}\mathcal{J}$ -constraint (2.8), a straightforward calculation shows

$$\Delta \tilde{V}_m(x) \equiv \tilde{V}_m(x) - \mathcal{P}\mathcal{J}[\tilde{V}_m(x)] = \frac{\sqrt{m^N}}{m^2} [2m \mathcal{W}'_m(x) + (N - 1) m' \mathcal{W}_m(x)], \quad N = 1, 2, 3, \dots \quad (5.7)$$

Comparison for $\ell = N - 1$ gives a closed expression for the potential due to the integrability of the equation

$$\begin{aligned} N \tilde{V}_m(x) = & \frac{\sqrt{m^N}}{m^2} \left[{}^N C_2 m' \mathcal{W}_m(x) + m \left\{ \sqrt{m^N} \mathcal{W}_m^2(x) + (2N - 1) \mathcal{W}'_m(x) - 2 \mathcal{U}_{N-2}(x) \right\} \right] \\ & + \frac{N(N-2)}{48} \left[4(2N+1) \left(\frac{1}{m} \right)'' + 3N(N-2)m \left(\frac{1}{m} \right)'^2 \right] + \Lambda, \quad N = 1, 2, 3, \dots, \end{aligned} \quad (5.8)$$

where we set a convention that ${}^N C_j \equiv 0, \mathcal{U}_{N-j} \equiv 0$ for $N < j$. Continuing this comparison up to the term ∂^0 , we find that for all order N , only two coefficient functions in the representation of charge operator \mathcal{C} remain independent,

which are the mass function $m(x)$ and the superpotential \mathcal{W}_m . As for instance, comparing ∂^{N-2} from both sides of (2.8), one obtains for $N \geq 3$

$$\begin{aligned} & \left[\frac{\mathcal{U}'_{N-2}(x)}{m} \right]' - \left[\Delta \tilde{V}_m(x) + {}^{N-1}C_2 \left(\frac{1}{m} \right)'' \right] \mathcal{U}_{N-2}(x) + \frac{2}{m} \mathcal{U}'_{N-3}(x) - (N-3) \left(\frac{1}{m} \right)' \mathcal{U}_{N-3}(x) \\ &= {}^{N+1}C_4 \frac{1}{\sqrt{m^N}} \left(\frac{1}{m} \right)^{IV} - {}^N C_2 \frac{(\tilde{V}_m^*)''(-x)}{\sqrt{m^N}} + \mathcal{W}(x) \left[{}^N C_3 \left(\frac{1}{m} \right)''' - (N-1)(\tilde{V}_m^*)'(-x) \right], \quad N = 2, 3, \dots, \end{aligned} \quad (5.9)$$

Similar to the first and second order cases, a general nonlinear SUSY algebra can be set up. The energy in such an algebra are zeros of the same N -th degree polynomial

$$E^N + l_1 E^{N-1} + l_2 E^{N-2} + \dots + l_{N-2} E^2 + l_{N-1} E + l_N = 0 \quad (5.10)$$

from which we conclude that for an odd-order charge operator, the Hamiltonian H_m possesses at least one real energy eigenvalue.

VI. CONCLUSION

In this article we have studied a generalized PDM Schrödinger equation in a non-Hermitian framework. We have proposed new differential realization for the charge operator and sought for the solvability of the model. Several interesting consequences due to PDM and non-Hermiticity of the Hamiltonian are derived. It should be noted that not all the results of the constant-mass non-Hermitian system are carried over to the PDM case. In constant-mass case, we showed that the superpotential $\mathcal{W}(x)$ had to be \mathcal{PT} -symmetric to preserve $\mathcal{CP}\mathcal{T}$ -symmetry and pseudo-hermiticity. In contrast, in the present work we have shown that the superpotential $\mathcal{W}(x)$ loses its \mathcal{PT} -symmetric property. Instead a new mass-deformed superpotential $\mathcal{W}_m(x)$ can be defined which turns out to be \mathcal{PT} -symmetric. Our work uncovers a new class of potentials $\tilde{V}_m(x)$ admitting $\mathcal{CP}\mathcal{T}$ -symmetry in PDM non-Hermitian systems. We have also obtained extension of some of our results to a general N -th order charge operator wherein the mass function remains even and the mass-deformed superpotential \mathcal{PT} -symmetric.

-
- [1] Bender C M and Boettcher S 1998 Phys. Rev. Lett. **80** 4243
[2] Bender C M 2007 Reports on Progress in Physics **70** 947
[3] Mostafazadeh A 2010 Int. J. Geom. Meth. Mod. Phys. **7** 1191
[4] Caliceti E, Cannata F, Znojil M and Ventura A 2005 Phys. Lett. A **335** 26
[5] Bagchi B, Banerjee A, Caliceti E, Cannata F, Geyer H B, Quesne C and Znojil M 2005 Int. J. Mod. Phys. A **20** 7107
[6] Bagchi B, Gorain P, Quesne C and Roychoudhury R 2004 Mod. Phys. Lett. A **19** 2765
[7] von Roos O 1983 Phys. Rev. B **27** 7547
[8] Mustafa O and Mazharimousavi S H 2006 Phys. Lett. A **358** 259
[9] Bastard G 1988 *Wave Mechanics Applied to Semiconductor Heterostructures* (Les Editions de Physique, Les Ulis, France)
[10] Serra L and Lipparini E 1997 Europhys. Lett. **40** 667
[11] Barranco M, Pi M, Gatica S M, Hernandez E S and Navarro J 1997 Phys. Rev. B **56** 8997
[12] Jonas R. F. Lima, Vieira M, Furtado C, Moraes F and Filgueiras C 2012 J. Math. Phys. **53** 072101
[13] Bagchi B, Banerjee A, Quesne C and Tkachuk V M 2005 J. Phys. A: Math.Gen. **38** 2929
[14] Quesne C 2009 SIGMA **5** 046
[15] Çapak M and Gönül B 2011 J. Math. Phys. **52** 122103
[16] Koç R and Koca M 2003 J. Phys. A: Math. Gen. **36** 8105
[17] Alhaidari A D 2003 Int. J. Theor. Phys. **42** 2999
[18] O Mustafa 2011 J. Phys. A: Math. Theor. **44** 355303
[19] Quesne C and Tkachuk V M 2004 J. Phys. A: Math. Gen. **37** 4267
[20] Ou Y C, Cao Z and Shen Q 2004 J. Phys. A: Math. Gen. **37** 4283
[21] Yu J, Dong S and Sun G 2004 Phys. Lett. A **322** 290
[22] Znojil M and Levai G 2012 Phys Lett A **376** 3000
[23] Levai G, Siegl P and Znojil M 2009 J. Phys. A: Math. Theor. **42** 295201
[24] Bagchi B, Gorain P, Quesne C and Roychoudhury R 2005 Eur. Phys. Lett. **72** 155
[25] Ganguly A, Ioffe V and Nieto L M 2006 J. Phys. A **39** 14659
[26] Quesne C 2006 Ann. Phys. **321** 1221
[27] Ganguly A, Kuru S, Negro J and Nieto L M 2006 Phys. Lett. A **360** 228

- [28] Ganguly A and Nieto L M 2007 J. Phys. A **40** 7265
- [29] Midhya B, Roy B, and Roychoudhury R 2010 J. Math. Phys. **51** 022109
- [30] Yesiltas, Ö 2010 J. Pys. A **43** 095305
- [31] Dutra A de Souza and Almeida C A S 2000 Phys. Lett. A **275** 25
- [32] Roy B and Roy P 2002 J. Phys. A : Math. Gen. **35** 2961
- [33] Tanaka T 2006 J. Phys. A : Math. Gen. **39** 219
- [34] Bagchi B and Tanaka T 2008 Phys. Lett. A **372** 5390
- [35] Cooper F, Khare A and Sukhatme U 2001 “*Supersymmetry in Quantum Mechanics* ” (World Scientific, Singapore)
- [36] Junker G 1996 “*Supersymmetric methods in Quantum and statistical Mechanics* ” (Springer: Berlin)
- [37] Bagchi B 2000 “*Supersymmetry in Quantum and Classical Mechanics* ” (Chapman Hall/CRC: Boca Raton, Florida)
- [38] Levai G and Özer O 2010 J. Math. Phys. **51** 092103
- [39] Andrianov A A, Ioffe M V and Spiridonov V P 1993 Phys. Lett. A **174** 273
- [40] Andrianov A A, Ioffe M V, Cannata F and Dedonder J -P 1995 Int. J. Mod. Phys. A **10** 2683
- [41] Bagchi B, Mallik S and Quesne C 2002 Int. J. Mod. Phys. A **17** 51
- [42] Andrianov A A and Ioffe M V 2012 J. Phys. A **45** 503001.
- [43] Correa F, Jakubsky V, Nieto L M and Plyushchay M S 2008 Phys. Rev. Lett. **101** 030403
- [44] Fernández D J and Fernández - Garcia N 2005 A.I.P Conf. Proc. **744** 236