

CONFORMALLY FLAT SEMI-PSEUDO SYMMETRIC MANIFOLDS

BY

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Abstract. In this paper we study a conformally flat semi pseudo symmetric manifold. Finally a special conformally flat semi pseudo symmetric manifold is studied.

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1. Introduction

In a paper [5], one of the authors, introduced a semi pseudo symmetric manifold. A non-flat Riemannian manifold (M^n, g) , $n > 3$, whose curvature tensor R satisfies the condition,

$$(1) \quad (\nabla_X R)(Y, Z)W = 2A(X)R(Y, Z)W + A(Y)R(X, Z)W \\ + A(Z)R(Y, X)W + A(W)R(Y, Z)X,$$

where A is a non-zero 1-form, with associated vector field ρ , that is

$$(2) \quad g(X, \rho) = A(X)$$

for every vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g , is called a semi-pseudo symmetric manifold and the 1-form A is called its associated 1-form. An n -dimensional semi-pseudo symmetric manifold is denoted by $(SPS)_n$.

In the year 1995, TARAFDAR [6] had studied conformally at pseudo-symmetric manifolds. Motivated by this, the authors in the present paper

dealt with a conformally flat $(SPS)_n$, $n > 3$. It is shown that, in such a manifold, the vector field ρ defined by (2) is a torse forming vector field [4]. Considering a conformally flat $(SPS)_n$, $n > 3$, of constant scalar curvature, it is shown that if in such a manifold, the associated 1-form A is not closed, then the vector field ρ is a proper torse forming vector field and if A is closed, then ρ is a proper concircular vector field. Further it is shown that if in a conformally flat $(SPS)_n$, $n > 3$, of constant scalar curvature, the associated 1-form is closed, then the manifold is a subprojective manifold in the sense of Kagan [1].

The notion of a special conformally flat manifold was introduced by CHEN and YANO [2]. The last section deals with a type of conformally flat $(SPS)_n$, $n > 3$, in which a certain scalar is constant. It is shown that such an $(SPS)_n$ is a special conformally flat manifold and by theorem [2], it is shown that such an $(SPS)_n$, $n > 3$, can be isometrically immersed in Euclidean space \mathbb{R}^{n+1} as a hypersurface.

2. Preliminaries

We now consider some formulae [3], [5] which will be required in the study of conformally flat $(SPS)_n$. Let S and r denote respectively the Ricci tensor of type $(0, 2)$ and the scalar curvature and L denote the symmetric endomorphism of the tangent space at each point of the manifold, corresponding to the Ricci tensor S i.e.,

$$(3) \quad S(X, Y) = g(LX, Y)$$

for any vector fields X, Y in M . Let B be the 1-form defined by,

$$(4) \quad B(X) = A(LX) = g(LX, \rho)$$

for any vector field X in M . From (1) we get,

$$(5) \quad \begin{aligned} (\nabla_X S)(Y, Z) &= 2A(X)S(Y, Z) + A(Y)S(X, Z) \\ &+ A(Z)S(Y, X) + A(R(X, Y)Z). \end{aligned}$$

Contracting (5) we have,

$$(6) \quad dr(X) = 2A(X)r + 3A(LX) = 2A(X)r + 3B(X).$$

Also,

$$(7) \quad \begin{aligned} (\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) &= A(X)S(Y, Z) - A(Z)S(X, Y) \\ &+ A(R(X, Z)Y). \end{aligned}$$

We shall use these formulae later on.

Let us recall:

Definition 1. A vector field ρ in a Riemannian manifold (M^n, g) is said to be torse forming [4] if it satisfies the equation $\nabla_X \rho = aX + \omega(X)\rho$, for every vector field X on M , where a is a non-zero scalar and ω is a 1-form.

Definition 2. A vector field ρ is said to be concircular if ω is a gradient vector field. If $d\omega = 0$, then ρ is said to be a proper concircular vector field [4].

3. Conformally flat $(SPS)_n, n > 3$

It is known that in a conformally flat $(SPS)_n, n > 3$, (on writing $Y = Z; Z = Y$ in the equation (5.5) of [5])

$$(8) \quad 0 = (n-1)A(X)S(Y, Z) - (n-1)A(Y)S(X, Z) - rA(X)g(Y, Z) + rA(Y)g(X, Z).$$

Putting $Z = \rho$ in (8) and using (3), (4) and (2), we get,

$$(9) \quad \begin{aligned} (n-1)A(X)B(Y) - (n-1)A(Y)B(X) &= 0, \\ \text{or, } A(X)B(Y) - A(Y)B(X) &= 0, \quad \text{as } n > 3. \end{aligned}$$

Hence,

$$(10) \quad B(X) = tA(X)$$

holds for any vector field X in M where t is a scalar. Using (2), (3), (4), (10) we find from (8), $(n-1)A(\rho)S(Y, Z) - (nt-t-r)A(Y)A(Z) - rA(\rho)g(Y, Z) = 0$, i.e.,

$$(11) \quad S(Y, Z) = \frac{r}{n-1} g(Y, Z) + \frac{nt-t-r}{(n-1)A(\rho)} A(Y)A(Z).$$

From (11) we get,

$$(12) \quad LY = \frac{r}{n-1} Y + \frac{nt-t-r}{n-1} \frac{A(Y)}{A(\rho)} \rho.$$

Again in a conformally flat space,

$$R(X, Y)Z = \frac{1}{n-2} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)LX - g(X, Z)LY] \\ + \frac{r}{(n-1)(n-2)} [g(X, Z)Y - g(Y, Z)X].$$

Again,

$$(13) \quad A(R(X, Y)\rho) = g(R(X, Y)\rho, \rho) = 0$$

and

$$(14) \quad A(R(X, \rho)Y) = \frac{t}{n-2} A(X)A(Y) - \frac{t}{n-2} g(X, Y)A(\rho).$$

Using (13) we get from (5) on using (3),(4),(10)

$$(15) \quad (\nabla_X S)(Y, \rho) = 2A(X)tA(Y) + tA(Y)A(X) + A(\rho)S(X, Y) \\ = A(\rho)S(X, Y) + 3tA(X)A(Y).$$

Again, $(\nabla_X B)(Y) = \nabla_X B(Y) - B(\nabla_X Y) = A(Y)\nabla_X t + t \nabla_X a(Y) - t g(\nabla_X Y, \rho)$, by (4) and (5) $= A(Y)(X \cdot t) + t \nabla_X g(Y, \rho) - t g(\nabla_X Y, \rho)$. Therefore,

$$(16) \quad (\nabla_X B)Y = (X \cdot t)A(Y) + t g(Y, \nabla_X \rho).$$

Again, $(\nabla_X S)(Y, \rho) = \nabla_X S(Y, \rho) - S(\nabla_X Y, \rho) - S(Y, \nabla_X \rho) = (\nabla_X B)(Y) - S(Y, \nabla_X \rho)$, by (3), (4). Therefore,

$$(17) \quad (\nabla_X S)(Y, \rho) = (X \cdot t)A(Y) + t g(Y, \nabla_X \rho) - S(Y, \nabla_X \rho), \text{ by (16).}$$

Hence from (15) and (17), it follows that,

$$(18) \quad A(\rho)S(X, Y) + 3tA(X)A(Y) = (X \cdot t)A(Y) + t g(Y, \nabla_X \rho) - S(Y, \nabla_X \rho),$$

that is,

$$(19) \quad A(\rho)LX + 3tA(X)\rho = (X \cdot t)\rho + t(\nabla_X \rho) - L(\nabla_X \rho).$$

Using (12), we get

$$\frac{r}{n-1} A(\rho)X + \left(\frac{nt-t-r}{n-1} \right) A(X)\rho \\ = (X \cdot t)\rho + \left(\frac{nt-t-r}{n-1} \right) \nabla_X \rho - \left(\frac{nt-t-r}{n-1} \right) \left(\frac{A(\nabla_X \rho)}{A(\rho)} \right) \rho.$$

We note that, $A(\nabla_X \rho) = g(\nabla_X \rho, \rho)$. As, $(\nabla_X g)(\rho, \rho) = 0$, we get $A(\nabla_X \rho) = \frac{1}{2} X \cdot g(\rho, \rho) = \frac{1}{2} X \cdot A(\rho)$.

Thus the above result reduces to

$$\begin{aligned} \nabla_X \rho &= \left(\frac{rA(\rho)}{nt-t-r} \right) X + \left(A(X) - \frac{(n-1)(X \cdot t)}{nt-t-r} + \frac{X \cdot A(\rho)}{2A(\rho)} \right) \rho \\ (20) \quad &= \sigma X + \omega(X)\rho, \end{aligned}$$

where

$$(21) \quad \sigma = \frac{r}{nt-t-r} A(\rho)$$

and

$$(22) \quad \omega(X) = A(X) - \frac{n-1}{nt-t-r} (X \cdot t) + \frac{X \cdot A(\rho)}{2A(\rho)}.$$

If $\sigma = 0$, then from (21) we get $r = 0$. Consequently from (6) we conclude that $B(X) = 0$. Using (10) this yields $tA(X) = 0$. Hence $A(X) = 0$, which is inadmissible by definition.

Hence, $\sigma \neq 0$ and we can state:

Theorem 3.1. *In a conformally flat (SPS) $_n$, $n > 3$, the vector field ρ given by (2) is a torse forming vector field.*

4. Conformally flat (SPS) $_n$, $n > 3$, with constant scalar curvature

If r is a non-zero constant, then from (6) we get $2A(X)r + 3B(X) = 0$, that is,

$$(23) \quad B(X) = -\frac{2}{3} rA(X).$$

In this case, taking $t = -\frac{2}{3}r$, from (21) and (22) we get $\sigma = -\frac{3}{2n+1} A(\rho)$ and $\omega(X) = A(X) + \frac{X \cdot A(\rho)}{2A(\rho)}$. Thus,

$$(24) \quad \nabla_X \rho = \sigma X + \omega(X)\rho,$$

where

$$(25) \quad \sigma = -\frac{3}{2n+1} A(\rho)$$

and

$$(26) \quad \omega(X) = A(X) + \frac{X \cdot A(\rho)}{A(\rho)}.$$

From (25), it follows that σ cannot be zero. Again from (26), it follows that ω is closed if A is closed and ω is not closed if A is not so. Hence we can state:

Theorem 4.1. *In a conformally flat $(SPS)_n$ ($n > 3$) with constant scalar curvature, the vector field ρ defined by (2) is a proper torse forming vector field if the associated 1-form is not closed and ρ is a proper concircular vector field if A is closed.*

It is known [1] that if a conformally flat manifold admits a proper concircular vector field, then, it is a subprojective manifold in the sense of Kagan.

Thus we can state

Theorem 4.2. *If in a conformally flat $(SPS)_n$, $n > 3$, with constant scalar curvature, the associated 1-form is closed, the manifold is a subprojective manifold in the sense of Kagan.*

5. Conformally flat $(SPS)_n$, $n > 3$, with constant scalar t

It is to be noted that if in a conformally flat $(SPS)_n$, $n > 3$, the scalar curvature r is constant, then the scalar t defined by (10) is equal to $-\frac{2}{3}r$ and is thus a constant. On the other hand, if the scalar t is a constant, then the scalar curvature r is not necessarily a constant. In this section, we consider a conformally flat $(SPS)_n$, $n > 3$, with constant scalar t .

From (11) we have, $S(Y, Z) = \frac{r}{n-1} g(Y, Z) + \frac{nt-t-r}{n-1} T(Y)T(Z)$, where

$$(27) \quad T(X) = \frac{A(X)}{\sqrt{A(\rho)}}.$$

Let,

$$(28) \quad H(X, Y) = -\frac{1}{n-2} S(X, Y) + \frac{r}{2(n-1)(n-2)} g(X, Y).$$

On using (26), we find from above,

$$(29) \quad H(X, Y) = -\frac{r}{2(n-1)(n-2)} g(X, Y) - \frac{nt-t-r}{(n-1)(n-2)} T(X)T(Y).$$

We now put

$$(30) \quad \alpha^2 = \frac{r}{(n-1)(n-2)}.$$

Then $2\alpha(X \cdot \alpha) = \frac{X \cdot r}{(n-1)(n-2)}$. Again from (6) we have $X \cdot r = dr(X) = 2A(X)r + 3B(X) = A(X)(2r + 3t)$. Thus from above

$$(31) \quad \alpha(X \cdot \alpha) = \frac{A(X)(2r + 3t)}{2(n-1)(n-2)}.$$

Hence $\alpha(X \cdot \alpha)\alpha(Y \cdot \alpha) = \frac{A(X)A(Y)(2r+3t)^2}{4(n-1)^2(n-2)^2}$ or, $(X \cdot \alpha)(Y \cdot \alpha) \frac{r}{(n-1)(n-2)} = \frac{A(X)A(Y)}{4(n-1)^2(n-2)^2} (2r + 3t)^2$ by (29). Thus

$$(32) \quad (X \cdot \alpha)(Y \cdot \alpha) = \frac{A(\rho)(2r + 3t)^2}{4r(n-1)(n-2)} T(X)T(Y).$$

Thus (28) takes the form,

$$(33) \quad H(X, Y) = -\frac{\alpha^2}{2} g(X, Y) + \beta(X \cdot \alpha)(Y \cdot \alpha)$$

where

$$(34) \quad \beta = \frac{4r(r + t - nt)}{A(\rho)(2r + 3t)^2}.$$

If $\alpha=0$, then $r=0$ and consequently from (6), $A(X) = 0$, which is inadmissible. Thus α cannot be zero and hence we may take α as positive in (32).

According to Chen and Yano [2], if in a conformally flat manifold, a (0,2) tensor H defined by

$$(35) \quad H(X, Y) = -\frac{1}{n-2} S(X, Y) + \frac{r}{2(n-1)(n-2)} g(X, Y)$$

is expressible in the form

$$(36) \quad H(X, Y) = -\frac{\alpha^2}{2} g(X, Y) + \beta(X \cdot \alpha)(Y \cdot \alpha)$$

where α, β are two scalars such that α is positive, then the manifold is said to be a special conformally flat manifold. Hence from (33), it follows that

$(SPS)_n$ under consideration is a special conformally flat manifold. It is known from a theorem [2] that every simply connected special conformally flat manifold can be isometrically immersed in a Euclidean space \mathbb{R}^{n+1} as a hypersurface. Thus we state,

Theorem 5.1. *Every simply connected conformally flat $(SPS)_n$, ($n > 3$) with constant scalar t can be isometrically immersed in a Euclidean space \mathbb{R}^{n+1} as a hypersurface.*

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