JOURNAL OF MULTIVARIATE ANALYSIS 28, 1–8 (1989)

Characterizations of Some Bivariate Life-Distributions

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Communicated by C. G. Khatri

Bivariate exponential distribution of Marshall–Olkin form, bivariate Rayleigh distribution, and bivariate Weibull distribution have been characterized through some functional equations to be satisfied by the survival function and hazard function.

1. INTRODUCTION

While characterizing properties of the Bivariate Exponential Distribution of the Marshall–Olkin form (BVED–M & 0) have been studied by Marshall and Olkin [3] and by Samanta [5], little attempt has been made so far to characterize other bivariate life distributions. In the univariate case Roy and Mukherjee [4] have recently examined the characterization of Weibull distribution in terms of a functional relation to be satisfied by the hazard function.

In the present work we examine the role of bivariate hazard function in characterizing the Bivariate Rayleigh Distribution (BVRD) and the Bivariate Weibull Distribution (BVWD). Krishnaji [1] characterized the univariate exponential distribution through a property which is closely related to the lack of memory property. We examine a corresponding bivariate generalization to characterize BVED–M & O. We also derive a bivariate extension of a result of Marsaglia and Tubilla [2] for characterizing BVED–M & O.

Let $F(x, y)$ be the joint distribution function of the random variables X and Y denoting lives of two interdependent components. Let us denote by

Received January 20, 1986; revised February 14, 1987.
AMS 1980 subject classification: 62H05.
Key words and phrases: survival function, bivariate hazard function, loss of memory.

0047-259X/89 $3.00

$S(x, y)$ and $R(x, y)$ the corresponding survival function and hazard function and let $S(0,0)=1$. Obviously $R(x, y)=-\log S(x, y)$ and $S(x, y)=1-F_X(x)-F_Y(y)+F(x, y)$, where $F_X(\cdot)$ and $F_Y(\cdot)$ are the marginal cdf of X and Y.

2. BVED-Characterization

According to Marshall and Olkin, (X, Y) follows BVED–M&O if and only if

(i) $S(x+t, y+t)=S(t, t)\, S(x, y)\, \forall (x, y, t) \geqslant 0$ and

(ii) $F_X(\cdot)$ and $F_Y(\cdot)$ are univariate exponential cdf's.

This is the generalization of the famous loss of memory property. Our results regarding BVED–M & O are stated in the following theorems which basically provide conditions equivalent to (i).

Theorem 2.1. *(X, Y) follows* BVED–M&O *if and only if*

(i) $S(x+t, y+t)=S(t, t)\, S(x, y)\, \forall (x, y) \geqslant 0$ *for two non-negative points t_1 and t_2 of t such that t_1/t_2 is irrational,*

(ii) *$F_X(\cdot)$ and $F_Y(\cdot)$ are univariate exponential cdf*'s.

Proof. Let T be the set of all non-negative points of t satisfying (i). Then $t=0$ is in T and further T is closed under addition and ordered subtraction in a sense that if $r \in T$ and $s \in T$ because of

$$\text{(a)} \quad \begin{aligned} S(x+r+s, y+r+s) &= S(s, s)\, S(x+r, y+r) \\ &= S(s, s)\, S(r, r)\, S(x, y) \\ &= S(r+s, r+s)\, S(x, y), \end{aligned}$$

i.e., $r+s \in T$, and because of

$$\text{(b)} \quad \begin{aligned} S(x+r-s, y+r-s) &= S(x+r, y+r)/S(s, s) \\ &= S(x, y)\, S(r, r)/S(s, s) \\ &= S(x, y)\, S(r-s, r-s), \end{aligned}$$

i.e., $r-s \in T$ for $r>s$ and $S(s, s) \neq 0$.

Now, observing that T contains the points 0 and t_1 and t_2 where t_1/t_2 is irrational, we conclude that T is dense in $[0, \infty)$. The rest of the proof follows from that of Marshall and Olkin's result. Q.E.D.

THEOREM 2.2. *If X, Y, and Z are non-negative random variables and Z is independent of (X, Y), such that $P(X-Z>x_1,\ Y-Z>y_1)>0$ for some (x_1, y_1), then (X, Y) follows* BVED–M & O *if and only if*

(i) $P(X-Z>x+t, Y-Z>y+t) = P(X-Z>t, Y-Z>t)\,P(X>x, Y>y)\,\forall(x, y)>0$, *$t=0$, and two non-negative points t_1, t_2 of t such that t_1/t_2 is irrational, and*

(ii) *$F_X(\cdot)$ and $F_Y(\cdot)$ are univariate exponential cdf's.*

Proof. (Only if part) Writing $W(x, y)=P(X-Z>x, Y-Z>y)$ we observe from (i)

$$W(x+t, y+t)=W(t, t)\,S(x, y) \qquad \forall(x, y)\geqslant 0$$

for $t=0, t_1, t_2$. This implies that

$$W(x, y)=W(0, 0)\,S(x, y)$$

and hence

$$S(x+t, y+t)=S(t, t)\,S(x, y) \qquad \forall(x, y)\geqslant 0$$

and for $t=0, t_1, t_2$. This together with (ii) implies that (X, Y) follows BVED–M & O because of Theorem 2.1.

The converse is immediate from the fact that

$$\begin{aligned} W(x+t, y+t) &= \int_0^\infty S(x+t+z, y+t+z)\,dF_Z(z) \\ &= S(x, y)\int_0^\infty S(t+z, t+z)\,dF_Z(z) \\ &= S(x, y)\,W(t, t) \end{aligned}$$

because of independence of Z. Q.E.D.

It may be noted from above that independence of Z is redundant for proving the only if part of the theorem. But for the if part of the proof this condition of independence cannot be relaxed as can be observed from the following counterexample.

EXAMPLE. Let $S(x, y)=\exp(-x-y)$, a special case of BVED–M & O, and $Z=X/2$. Then $W(x, y)$ works out as $2/3\exp(-3x-y)$ for which $W(x+t, y+1)\neq W(t, t)\,S(x, y)$.

PROPERTY A. A non-negative random variable Z is said to follow Property A if for each interval $I\subset[0, \infty)$ with positive Lebesgue measure, $P(Z\in I)>0$.

THEOREM 2.3. *Assume that*

(i) (X, Y) *is a non-negative bivariate random variable with* $S(x, y)$ *continuous on* $0 \leqslant x < \infty$, $0 \leqslant y < \infty$ *and either* (X, Y) *is Bivariate New Better than Used* (NBU) *or Bivariate New Worse than Used* (NWU) *in the weak sense,*

(ii) (U, V) *and* W *are independent random variables distributed independently of* (X, Y) *with each of* U, V, *and* W *satisfying Property* A, *and*

(iii) X *and* Y *have marginal exponential cdf's.*

Then, (X, Y) *follows* BVED–M & O *if and only if*

$$P(X > U + W, Y > V + W) = P(X > U, Y > V)\, P(X > W, Y > W). \tag{2.1}$$

Proof. (If part) Let (2.1) be true. Then writing

$$D(x, y, t) = S(x + t, y + t) - S(t, t)\, S(x, y) \tag{2.2}$$

we have from the property of NBU for (X, Y), $D(x, y, t) \geqslant 0$ for all $(x, y, t) \geqslant 0$. Let $D(x_0, y_0, t_0) > 0$ for some (x_0, y_0, t_0). Because of continuity of $D(\cdot)$ there exists a neighbourhood N of (x_0, y_0, t_0) such that sides are of positive Lebesgue measure and $D(\cdot) > 0$ in N.

Writing

$$P(X > U + W, Y > V + W) - P(X > U, Y > V)\, P(X > W, Y > W)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \{S(u + w, v + w) - S(u, v)\, S(w, w)\}\, dG(u, v)\, dH(w)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty D(u, v, w)\, dG(u, v)\, dH(w), \tag{2.3}$$

where $G(u, v)$ is the cdf of (U, V) and $H(w)$ is the cdf of W, we get from Property A that (2.3) is greater than zero, which is in contradiction with (2.1). Hence $D(x, y, t) = 0$ for all $(x, y, t) \geqslant 0$, i.e., (X, Y) follows BVED–M & O. The proof for the case (X, Y) is NWU will be similar.

(Only if part) The result follows from the fact that (2.3) equated to zero results in (2.1). Q.E.D.

THEOREM 2.4. *Under the setup as in Theorem* 2.3, (X, Y) *follows* BVED–M & O *if and only if*

$$P(X > U + W + x + t, Y > V + W + y + t)$$
$$= P(X > W + t, Y > W + t)\, P(X > U + x, Y > V + y)$$

for all $(x, y, t) \geqslant 0$.

Proof is similar to that of Theorem 2.3.

3. BVRD-CHARACTERIZATION

A vector variable (X, Y) is said to follow BVRD $(\delta_1, \delta_2, \delta_3)$ if the survival function is given by

$$S(x, y) = \exp(-\delta_1 x^2 - \delta_2 y^2 - \delta_3 \max(x^2, y^2)) \tag{3.1}$$

for $(\delta_1, \delta_2, \delta_3) > 0$.

It is easy to examine that marginally X follows Rayleigh distribution with scale parameter $(\delta_1 + \delta_3)$ and Y follows Rayleigh distribution with scale parameter $(\delta_2 + \delta_3)$. We shall state a characterizing property of the univariate Rayleigh distribution which, in fact, follows as a special case of a subsequent characterization of BVRD.

THEOREM 3.1. *For a nondegenerate non-negative random variable X with hazard function $R(x)$,*

(i) $R(x+c) = R(x) + R(c) + 2R(\sqrt{cx})\ \forall (x, c) \geq 0$ *and*

(ii) $\lim_{x \to 0} x^{-2} R(x) = \delta$ *if and only if X follows univariate Rayleigh distribution with scale parameter δ.*

Proof. Notice that for any finite x,

$$\lim_{c \to 0} \frac{R(x+c) - R(c)}{c} = \lim_{c \to 0} c \frac{R(c)}{c^2} + 2 \lim_{c \to 0} \frac{R(\sqrt{cx})}{c}$$

$$= 2x \lim_{y \to 0} \frac{R(y)}{y^2} = 2x\,\delta.$$

Hence $R'(x) = 2x\delta$ and this gives $R(x) = \delta x^2$ because $R(0) = 0$. Q.E.D.

It may be noted that condition (ii) of the above theorem can be replaced by continuity of $R(x)$ to arrive at a similar characterizing property.

THEOREM 3.2. *For non-degenerate non-negative random variable (X, Y) with hazard function $R(x, y)$,*

(i) $R(x+c, y+c) = R(x, y) + R(c, c) + 2R(\sqrt{cx}, \sqrt{cy})$ *for all* $(x, y, c) > 0$,

(ii) $\lim_{x \to 0} x^{-2} R(x, x) = \delta$, *and*

(iii) *marginally X and Y follow Rayleigh distributions if and only if (X, Y) follows* BVRD.

Proof. (Only if part) Let

$$R(x+c, y+c) = R(x, y) + R(c, c) + 2R(\sqrt{cx}, \sqrt{cy}) \tag{3.2}$$

for all $(x, y, c) > 0$.

Notice that if $y=x$ in (3.2) and $f(x)=R(x, x)$, then

$$f(x+c)=f(x)+f(c)+2f(\sqrt{cx})$$

and with condition (ii), Theorem 3.1 gives

$$R(x, x)=x^2\delta. \tag{3.3}$$

Using (3.3) in (3.2) we get

$$R(x+c, y+c)=R(x, y)+\delta c^2+2R(\sqrt{cx}, \sqrt{cy}). \tag{3.4}$$

Because of (iii) let $R(x, 0)=ax^2$ and $R(0, y)=by^2$. Hence for a choice of $c=v$, $y=0$, $x+c=u$, such that $u>v$, in (3.4) we have

$$R(u, v)=au^2+(\delta-a)\,v^2, \qquad u>v. \tag{3.5}$$

Similarly for a choice of $c=u$, $x=0$, $y+c=v$, such that $v>u$, in (3.4) we have

$$R(u, v)=(\delta-b)\,u^2+bv^2, \qquad v>u. \tag{3.6}$$

Thus combining (3.5) and (3.6)

$$R(u, v)=(\delta-b)\,u^2+(\delta-a)\,v^2+(a+b-\delta)\max(u^2, v^2).$$

Observing that $\delta=R(1, 1)$, $a=R(1, 0)$, and $b=R(0, 1)$ it is easy to verify that $\delta-a>0$, $\delta-b>0$. Further consideration of the distribution of $\text{Max}(X, Y)$ implies that $a+b-\delta>0$. Hence (X, Y) follows BVRD as in (3.1) with $\delta_1=\delta-b$, $\delta_2=\delta-a$, and $\delta_3=a+b-\delta$.

(If part) Let (X, Y) follows BVRD given by $S(x, y)$ as given (3.1). Then (iii) is immediate and (i) follows after some algebraic manipulation. Also (ii) is true for $\delta=\delta_1+\delta_2+\delta_3$. Q.E.D.

4. BVWD-Characterization

A vector variable (X, Y) will be said to follow BVW distribution with parameters $\boldsymbol{\delta}$ and α (abbreviated as BVWD $(\boldsymbol{\delta}, \alpha)$, $\boldsymbol{\delta}=(\delta_1, \delta_2, \delta_3)'$) if the survival function is of the form

$$S(x, y)=\exp(-\delta_1 x^\alpha-\delta_2 y^\alpha-\delta_3\max(x^\alpha, y^\alpha)). \tag{4.1}$$

In particular if $\alpha=2$ we get the corresponding BVRD$(\delta_1, \delta_2, \delta_3)$ and if $\alpha=1$ we get the corresponding BVED $(\delta_1, \delta_2, \delta_3)$ of Marshall–Olkin type.

First we observe that

THEOREM 4.1. *(X, Y) follows* BVWD $(\boldsymbol{\delta}, \alpha)$ *if and only if* (X^α, Y^α) *follows* BVED $(\boldsymbol{\delta})$ *of Marshall–Olkin type.*

Next we generalize a univariate characterizing property of the Weibull distribution given by

$$R(cx)\, R(1) = R(x)\, R(c) \qquad \forall x > 0$$

and two non-negative points c_1 and c_2 of c, such that $(\log c_1)/(\log c_2)$ is irrational as in Roy and Mukherjee [4].

THEOREM 4.2. *For non-degenerate non-negative random variable (X, Y),*

(i) $R(cx, cy)\, R(1, 1) = R(x, y)\, R(c, c)$ $\forall (x, y) > 0$ *for two non-negative points c_1 and c_2 of c such that* $(\log c_1)/(\log c_2)$ *is irrational,*

(ii) $R(x, 1) + R(1, 0) = R(x, 0) + R(1, 1)$ $\forall x \geqslant 1$, *and*

(iii) $R(1, y) + R(0, 1) = R(0, y) + R(1, 1)$ $\forall y \geqslant 1$ *if and only if (X, Y) follows* BVWD.

Proof. (Only if part) Let the conditions (i), (ii), and (iii) be true. Writing C as the set of points satisfying (i) we observe that C is closed under multiplication and division in a sense that if r and s belong to C then rs belongs to C and r/s belongs to C. This implies that if (i) is true for two points c_1 and c_2 such that $(\log c_1)/(\log c_2)$ is irrational it is true for all c as C is dense in $[0, \infty)$. Further (i) implies that for $y = x$,

$$R(cx, cx)\, R(1, 1) = R(x, x)\, R(c, c) \qquad \forall (x, c)$$

and hence $R(x, x) = x^\alpha R(1, 1)$, where $\alpha > 0$ from the fact that $R(x, x)$ is an increasing function in x.

Rewriting (i) we get

$$R(cx, cy) = c^\alpha R(x, y) \qquad \forall (x, y, c) \tag{4.2}$$

Putting $c = v$, $y = 1$, and $x = u/v$ in (4.2) we have

$$R(u, v) = v^\alpha R(u/v, 1) \qquad \forall (u, v). \tag{4.3}$$

Also

$$R(cx, 0) = c^\alpha R(x, 0) \qquad \forall (x, c). \tag{4.4}$$

Now from (ii) we have for $x \geqslant y$, i.e., $x/y \geqslant 1$,

$$R(x/y, 1) + R(1, 0) = R(x/y, 0) + R(1, 1)$$

or

$$R(x, y) + y^\alpha R(1, 0) = y^\alpha R(1, 1) + x^\alpha R(1, 0)$$

by an application of (4.3) and (4.4). Hence

$$R(x, y) = R(1, 0)\, x^\alpha + (R(1, 1) - R(1, 0))\, y^\alpha \qquad \text{for} \qquad x \geqslant y.$$

Similarly from (iii), proceeding as above,

$$R(x, y) = (R(1, 1) - R(0, 1))\, x^\alpha + R(0, 1)\, y^\alpha \qquad \text{for} \qquad x \leqslant y.$$

Combining these two results we express

$$\begin{aligned} R(x, y) &= (R(1, 1) - R(0, 1))\, x^\alpha + (R(1, 1) - R(1, 0))\, y^\alpha \\ &\quad + (R(0, 1) + R(1, 0) - R(1, 1)) \max(x^\alpha, y^\alpha) \\ &= \delta_1 x^\alpha + \delta_2 y^\alpha + \delta_3 \max(x^\alpha, y^\alpha). \end{aligned}$$

Thus (X, Y) follows BVWD $(\boldsymbol{\delta}, \alpha)$, where it is easy to verify that $\boldsymbol{\delta} > \mathbf{0}$.

(If part) Observing that for (X, Y) following BVWD $(\boldsymbol{\delta}, \alpha)$

$$R(x, y) = \delta_1 x^\alpha + \delta_2 y^\alpha + \delta_3 \max(x^\alpha, y^\alpha),$$

conditions (i), (ii), and (iii) follow from some straightforward algebraic calculation. Q.E.D.

Acknowledgment

The authors are grateful to the referee for his or her valuable comments.

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