Character of Locally Inequivalent Classes of States and Entropy of Entanglement

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In this letter we have established the physical character of pure bipartite states with the same amount of entanglement in the same Schmidt rank that either they are local unitarily connected or they are incomparable. There exist infinite number of deterministically locally inequivalent classes of pure bipartite states in the same Schmidt rank (starting from three) having same amount of entanglement. Further, if there exists incomparable states with same entanglement in higher Schmidt ranks (greater than three), then they should differ in at least three Schmidt coefficients.

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Quantum mechanics shows many counterintuitive properties in physical systems. However, these peculiar characteristics turn out as the fundamental features of different quantum systems. Entanglement is one of the most striking phenomena in recent times. Apart from its peculiar characteristics, entanglement is considered as an important physical resource in various quantum information processing tasks [1, 2, 3]. Thus the characterization and quantification of entanglement would always be one of the major issue to understand the behavior of composite quantum systems. It is almost convincing that the quantification of pure state entanglement in bipartite system through the von-Neumann entropy of reduced density matrices is complete. Popescu and Rohrlich showed that [4] von-Neumann entropy quantifies the unique measure of pure state entanglement in bipartite level. For pure bipartite states, both the entanglement of formation and distillable entanglement [5] are the same with the entropy of entanglement and every measures of entanglement that should satisfy some fundamental criteria, would necessarily collapse with the entropy of entanglement. Thus, it is assumed that the entropy of entanglement should necessarily reflect all the possible non-local features of pure entangled states in bipartite level. However, in this paper our findings are something beyond the entropy of entanglement. It generates with the connection between entanglement and LOCC. For pure bipartite states all states which are connected by local unitary operations, have the same amount of entanglement and have also the same set of Schmidt coefficients. But the general character of pure bipartite entangled states with the same amount of entanglement in the same Schmidt rank is not completely known to us. Also, if there exist such states with different Schmidt coefficients, then what is the physical nature of such states? In this letter, we would able to answer the above problems through the existence of incomparable states in pure bipartite level.

The notion of incomparable states in pure bipartite level is due to Nielsen [6] through the majorization criteria for deterministic conversion of pure entangled states under LOCC. Considerable amount of effort has been spent with the possibility and impossibility of manipulating pure entanglement under deterministic or stochastic LOCC [8, 9, 10]. However, the character of locally inequivalent, i.e., incomparable states is not clearly understood at least in pure bipartite level. It is peculiar that although there is no restriction from the amount of entanglement contained in a pure bipartite state, but it is impossible to convert this state to another lower entangled state, if they are incomparable to each other. To understand the basic nature of such states, recently it is found that there are some deeper relations between the existence of incomparable states and different no-go theorems and incomparability may be used as a detector of unphysical operations [11]. The important factor we would like to explore in this work is the existence of infinite number of pure bipartite entangled states with the same entanglement but all are incomparable to each other, i.e., there exists infinite number of pure entangled states having different Schmidt coefficients with the same entanglement in the same Schmidt rank. It starts from 3×3 systems. The nature of such states in higher dimensional systems are also quite surprising. They should differ in at least three Schmidt coefficients and up to that level equivalent to the case of incomparable states in 3×3 systems with the same amount of entanglement. Our proof is analytical, supported by numerical results and it would necessarily reflect the basic nature of entanglement as a non-local feature beyond the entropy of entanglement.

Let us first investigate the possible relations between entanglement of two pure bipartite states in 3×3 systems. Suppose, $\alpha_1,\alpha_2,\alpha_3$ and β_1,β_2,β_3 are the Schmidt coefficients of two pure bipartite state $|\Psi\rangle$ and $|\Phi\rangle$ of Schmidt rank three, i.e., $1>\alpha_i,\beta_i>0\ \ \forall\ i=1,2,3$ and $\alpha_1+\alpha_2+\alpha_3=1=\beta_1+\beta_2+\beta_3$. Then entanglement of the pure states are given by the von-Neumann entropy of the reduced density matrices as,

$$\begin{array}{l} E(|\Psi\rangle)=-(\alpha_1\log_2\alpha_1+\alpha_2\log_2\alpha_2+\alpha_3\log_2\alpha_3),\\ E(|\Phi\rangle)=-(\beta_1\log_2\beta_1+\beta_2\log_2\beta_2+\beta_3\log_2\beta_3)\end{array}\qquad(1)$$

Theorem-1: *Let $|\Psi\rangle, |\Phi\rangle$ be two pure bipartite states with Schmidt rank three, having same amount of entanglement. If one pair of Schmidt coefficient for the two pure states are equal ($\alpha_i = \beta_i$ for some $i = 1, 2, 3$), then so also for the other two Schmidt coefficients, i.e., $\alpha_i = \beta_i, \ \forall \ i = 1, 2, 3$.*

Proof. Let $\alpha_1 = \beta_1$, i.e., the largest Schmidt coefficients of the pure states $|\Psi\rangle, |\Phi\rangle$ be equal. Also, we assume that $E(|\Psi\rangle) = E(|\Phi\rangle)$. Then, $\alpha_2 + \alpha_3 = 1 - \alpha_1 = 1 - \beta_1 = \beta_2 + \beta_3$, Thus we may construct two pure bipartite states of Schmidt rank two, as

$$|\Psi'\rangle \equiv (\alpha, 1-\alpha), \ |\Phi'\rangle \equiv (\beta, 1-\beta) \tag{2}$$

where $\alpha = \frac{\alpha_2}{\alpha_1}, \ \beta = \frac{\beta_2}{\beta_1}$. Then using $E(|\Psi\rangle) = E(|\Phi\rangle)$ we observe that $E(|\Psi'\rangle) = E(|\Phi'\rangle)$, which would necessarily imply that $|\Psi'\rangle$ and $|\Phi'\rangle$ have the same set of Schmidt coefficients, i.e., $\alpha = \beta$. Thus, $\alpha_2 = \beta_2$ and $\alpha_3 = \beta_3$. So, the pure states $|\Psi\rangle, |\Phi\rangle$ must necessarily have the same set of Schmidt coefficients.

In a similar manner, if either of $\alpha_2 = \beta_2$ or $\alpha_3 = \beta_3$, then it is also be the case that the pure bipartite states $|\Psi\rangle, |\Phi\rangle$ must have the same set of Schmidt coefficients.

Before going to describe the next result, we recall the notion of incomparable states in pure bipartite level [6]. Two pure bipartite states $|\Psi\rangle$ and $|\Phi\rangle$ of $m \times n$ system with $\min\{m, n\} \leq d$ are said to be comparable to each other if and only if the Schmidt coefficients $\alpha_1, \alpha_2, \cdots, \alpha_d$, and $\beta_1, \beta_2, \cdots, \beta_d$ corresponding to the states $|\Psi\rangle$ and $|\Phi\rangle$ should satisfy the following relations,

$$\textstyle\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i, \ \ \forall \ \ k = 1, 2, \cdots, d \tag{3}$$

where $\alpha_i \geq \alpha_{i+1} \geq 0$ and $\beta_i \geq \beta_{i+1} \geq 0$, for $i = 1, 2, \cdots, d-1$, and $\sum_{i=1}^{d} \alpha_i = 1 = \sum_{i=1}^{d} \beta_i$. It is known as majorization [12] criteria of two vectors formed by the Schmidt coefficients of the states and it provides us the necessary and sufficient condition for converting $|\Psi\rangle$ to $|\Phi\rangle$ under deterministic LOCC. As a consequence of non-increase of entanglement by LOCC, if $|\Psi\rangle \rightarrow |\Phi\rangle$ is possible under LOCC with certainty, then $E(|\Psi\rangle) \geq E(|\Phi\rangle)$ where,

$$\begin{array}{l} E(|\Psi\rangle) = -\sum_{i=1}^{d} \alpha_i \log_2 \alpha_i, \\ E(|\Phi\rangle) = -\sum_{i=1}^{d} \beta_i \log_2 \beta_i. \end{array} \tag{4}$$

If the above criterion [eqn. (3)] does not hold, then we usually denote it by $|\Psi\rangle \not\rightarrow |\Phi\rangle$ and if both $|\Psi\rangle \not\rightarrow |\Phi\rangle$ and $|\Phi\rangle \not\rightarrow |\Psi\rangle$ occur, then we denote it as $|\Psi\rangle \not\leftrightarrow |\Phi\rangle$ and call $(|\Psi\rangle, |\Phi\rangle)$ as a pair of incomparable states [6, 9]. For 3×3 states $|\Psi\rangle, |\Phi\rangle$ with Schmidt coefficients $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ in decreasing order, the condition for incomparability can be written in the simplified form

$$\begin{array}{ll} \texttt{either,} & \alpha_1 > \beta_1 \ \texttt{and} \ \alpha_3 > \beta_3 \\ \texttt{or,} & \alpha_1 < \beta_1 \ \texttt{and} \ \alpha_3 < \beta_3. \end{array} \tag{5}$$

So, if there exists two incomparable states in 3×3 system with the same amount of entanglement, then all the Schmidt coefficients must be different.

Theorem-2: *The amount of entanglement of any two comparable, pure, bipartite states of $d \times d, \ d \geq 3$ systems with different Schmidt coefficients must necessarily be different.*

In other words, for any two Schmidt rank d (≥ 3) states $|\Psi\rangle$ and $|\Phi\rangle$ with different Schmidt coefficients,

$$|\Psi\rangle \rightarrow |\Phi\rangle \Rightarrow E(|\Psi\rangle) > E(|\Phi\rangle) \tag{6}$$

To prove the theorem, we use the concept of Schur Convexity and its connection with the majorization of vectors.

Schur Convex Function:[7, 12, 14] A function $F : I^n \rightarrow \Re$ is called Schur Convex if,

$$x \prec y \Longrightarrow F(x) \leq F(y), \quad \forall \ x, y \in I^n \tag{7}$$

where $I \subset \Re$, $\Re$ is the set of all real numbers and $x \prec y$ means x is majorized by y.

The function $F(x) \equiv F(x_1, x_2, \cdots, x_n), x_i \in I, \forall i = 1, 2, \ldots, n$, is called Strictly Schur Convex, if and only if, the above inequality is strict for all $x \in I^n$. All Schur convex functions are symmetric in nature, i.e., invariant under any permutation, but the converse is not true [7, 12, 14].

Also, a function $F : I^n \rightarrow \Re$ where $I \subset \Re$ is called Schur Concave(strictly) if and only if the function $F' = -F$ is Schur Convex(strictly).

Lemma[7, 12, 14]: Suppose a function $F : I^n \rightarrow \Re$ where $I \subset \Re$, is symmetric and have continuous partial derivatives on I^n. Then $F(\cdot)$ is Schur Convex, if and only if,

$$(x_i - x_j)\left(\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial x_j}\right) \geq 0, \ \forall \ x_i, x_j \in I; i, j = 1, 2, \cdots, n \tag{8}$$

It is strictly Schur convex if and only if the above inequality is strict for all $x_i \neq x_j$.

The well known example of strict Schur concave function is the Shannon entropy of a probability distribution, i.e., $H(p) = -\sum_{i=1}^{n} p_i \log_2 p_i, \ 0 \leq p_i \leq 1, \sum_{i=1}^{n} p_i = 1$ [7].

Now, consider a pure bipartite state $|\Psi\rangle$ of Schmidt rank d with Schmidt vector $\lambda_{|\Psi\rangle} = (\lambda_1, \lambda_2, \cdots, \lambda_d)$. Then, $E(|\Psi\rangle) = -\sum_{i=1}^{d} \lambda_i \log_2 \lambda_i$, where $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{d} \lambda_i = 1$. It is easy to check that,

$$\begin{array}{l} (\lambda_i - \lambda_j)\left(\frac{\partial E}{\partial \lambda_i} - \frac{\partial E}{\partial \lambda_j}\right) \\ = (\lambda_i - \lambda_j)\{(-\log_2 \lambda_i - \log_2 e) - (-\log_2 \lambda_j - \log_2 e)\} \\ = (\lambda_i - \lambda_j)\{\log_2(\frac{\lambda_j}{\lambda_i})\} < 0, \quad \forall \ \lambda_i \neq \lambda_j \end{array} \tag{9}$$

So from the lemma, we conclude that $E : I^d \rightarrow \Re$ where $I = [0, 1]$ is a strictly Schur Concave function.

Proof of Theorem-2: Suppose, $|\Psi\rangle$, $|\Phi\rangle$ are any two pure bipartite states of Schmidt rank d, $d \geq 3$. Then $|\Psi\rangle \longrightarrow |\Phi\rangle$ under deterministic LOCC if and only if $\lambda_{|\Psi\rangle} \prec \lambda_{|\Phi\rangle}$, where $\lambda_{|\Psi\rangle}, \lambda_{|\Phi\rangle}$ are Schmidt vectors of $|\Psi\rangle$ and $|\Phi\rangle$ [6]. Now, for different Schmidt vectors of $|\Psi\rangle$ and

$|\Phi\rangle$, the strict Schur concavity of the function E implies,

$$\begin{aligned} & -E(\lambda_{|\Psi\rangle}) < -E(\lambda_{|\Phi\rangle}) \\ \Rightarrow \quad & E(\lambda_{|\Psi\rangle}) > E(\lambda_{|\Phi\rangle}). \end{aligned} \tag{10}$$

One may also check the results by algebraic method by considering the different possible cases of majorization relation. We have mentioned a case explicitly in the appendix. The above result is also true for $d = 2$.

Thus, we may conclude that the amount of entanglement of any two comparable, $d \times d$ pure bipartite states with different Schmidt coefficients, must necessarily be different. This result is quite compatible with our natural intuition. However, it would not imply immediately that all the pure bipartite states with same entanglement are locally unitarily connected, or at least locally connected. In contrary, there are infinite number of states with the same entanglement even in the lowest possible dimension, i.e., in 3×3 systems, but incomparable in nature. The theorems 1 and 2 readily imply that if there exist pure entangled states with the same entanglement but different Schmidt coefficients in $d \times d$, $(d \geq 3)$ system, then they must be incomparable to each other. The following example is a way how one could find such states numerically.

Example: Firstly, consider the pure bipartite state $|\Psi\rangle$, with Schmidt coefficients .45,.39,.16. The entropy of entanglement of the state is $E(|\Psi\rangle) \approx 1.471215431$. Next, consider the following pair of states represented by their Schmidt vectors,

$$\begin{aligned} |\Phi_1\rangle &\equiv (.49, .33676028, .17323972), \\ |\Phi_2\rangle &\equiv (.49, .33676030, .17323970) \end{aligned} \tag{11}$$

Both the states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are incomparable with $|\Psi\rangle$ and have the entanglement $E(|\Phi_1\rangle) \approx 1.471215442$, $E(|\Phi_2\rangle) \approx 1.471215423$. Then, upto the 8 significant digits, $E(|\Psi\rangle) = E(|\Phi_1\rangle) = E(|\Phi_2\rangle) = 1.4712154$ and precisely, $E(|\Phi_1\rangle) > E(|\Psi\rangle) > E(|\Phi_2\rangle)$. From the continuity of the von-Neumann entropy function on Schmidt coefficients of the states, we may conclude that there exist a pure 3×3 state $|\Phi\rangle \equiv (.49, .33676028 + \delta, .17323972 - \delta)$ where $0 < \delta < .00000002$ between the states $|\Phi_1\rangle$ and $|\Phi_2\rangle$, such that $E(|\Psi\rangle) = E(|\Phi\rangle)$ exactly. By construction the states $|\Psi\rangle$ and $|\Phi\rangle$ are incomparable with each other. Now, further, if we look at the largest Schmidt coefficients .45 and .49 of the states $|\Psi\rangle$ and $|\Phi\rangle$ respectively, we observe that they are widely separated. The states have certain distance with respect to Schmidt coefficients and if we consider the largest Schmidt coefficient any value between .45 and .49, then we could always find a state $|\Phi'\rangle$ that has the same amount of entanglement with $|\Psi\rangle$ and incomparable with it. Thus, there are infinite number of pure bipartite states which have same amount of entanglement, but incomparable to each other.

The character of any pair of pure bipartite states in $d \times d$ system with equal entanglement have a nice relation with the lower dimensional incomparable states and ultimately we find that any pair incomparable states should differ in at least three Schmidt coefficients. Consider a pair of pure bipartite states in $d \times d$ system having the Schmidt vectors,

$$\begin{aligned} |\Psi\rangle &\equiv (\alpha_1, \alpha_2, \cdots \alpha_d), \\ |\Phi\rangle &\equiv (\beta_1, \beta_2, \cdots \beta_d) \end{aligned} \tag{12}$$

If they have the same entanglement, then either $\alpha_i = \beta_i$, $\forall\ i = 1, 2, \cdots, d$, or, $\alpha_i \neq \beta_i$ for at least 3 values of $i \in \{1, 2, \cdots, d\}$, i.e., they are either locally unitarily connected or they are incomparable with at least three different Schmidt coefficients. To be precise, if there is exactly $k \leq (d-3)$ number of values of $i \in \{1, 2, \cdots, d\}$ for which $\alpha_i = \beta_i$, then there exists a pair of incomparable pure bipartite states with Schmidt rank $d - k$, having same amount of entanglement for which all the other $d - k$ Schmidt coefficients are different. We would now show it for $k = 1$. Suppose, the j^{th} Schmidt coefficients of $|\Psi\rangle$ and $|\Phi\rangle$ are equal, i.e., $\alpha_j = \beta_j = \kappa$(say) and $\alpha_i \neq \beta_i$ for all other i. Then we may construct two pure bipartite states with Schmidt rank $d - 1$ as follows,

$$\begin{aligned} |\Upsilon\rangle &= (\chi_1, \chi_2, \cdots \chi_{d-1}), \\ |\Omega\rangle &= (\eta_1, \eta_2, \cdots \eta_{d-1}) \end{aligned} \tag{13}$$

with $\chi_i = \frac{\alpha_i}{1-\kappa}$ and $\eta_i = \frac{\beta_i}{1-\kappa}$ for $1 \leq i < j$ and $\chi_i = \frac{\alpha_{i+1}}{1-\kappa}$ and $\eta_i = \frac{\beta_{i+1}}{1-\kappa}$ for $j \leq i \leq d-1$. Now,

$$\begin{aligned} & E(|\Psi\rangle) = E(|\Phi\rangle) \\ & \Rightarrow -\sum_{i=1}^{d} \alpha_i \log_2 \alpha_i = -\sum_{i=1}^{d} \beta_i \log_2 \beta_i \\ & \Rightarrow \sum_{i=1}^{j-1} \alpha_i \log_2 \alpha_i + \sum_{i=j+1}^{d} \alpha_i \log_2 \alpha_i \\ & \quad = \sum_{i=1}^{j-1} \beta_i \log_2 \beta_i + \sum_{i=j+1}^{d} \beta_i \log_2 \beta_i \\ & \Rightarrow \sum_{i=1}^{j-1} \tfrac{\alpha_i}{1-\kappa} \log_2 \tfrac{\alpha_i}{1-\kappa} + \sum_{i=j+1}^{d} \tfrac{\alpha_i}{1-\kappa} \log_2 \tfrac{\alpha_i}{1-\kappa} \\ & \quad = \sum_{i=1}^{j-1} \tfrac{\beta_i}{1-\kappa} \log_2 \tfrac{\beta_i}{1-\kappa} + \sum_{i=j+1}^{d} \tfrac{\beta_i}{1-\kappa} \log_2 \tfrac{\beta_i}{1-\kappa} \\ & \Rightarrow \sum_{i=1}^{j-1} \tfrac{\alpha_i}{1-\kappa} \log_2 \tfrac{\alpha_i}{1-\kappa} + \sum_{i=j}^{d-1} \tfrac{\alpha_{i+1}}{1-\kappa} \log_2 \tfrac{\alpha_{i+1}}{1-\kappa} \\ & \quad = \sum_{i=1}^{j-1} \tfrac{\beta_i}{1-\kappa} \log_2 \tfrac{\beta_i}{1-\kappa} + \sum_{i=j}^{d-1} \tfrac{\beta_{i+1}}{1-\kappa} \log_2 \tfrac{\beta_{i+1}}{1-\kappa} \\ & \Rightarrow -\{\sum_{i=1}^{d-1} \chi_i \log_2 \chi_i\} = -\{\sum_{i=1}^{d-1} \eta_i \log_2 \eta_i\} \\ & \Rightarrow E(|\Upsilon\rangle) = E(|\Omega\rangle). \end{aligned}$$

Also $(|\Psi\rangle, |\Phi\rangle)$ are incomparable imply, either $\alpha_1 \leq \beta_1$ and $\sum_{i=1}^{m} \alpha_i > \sum_{i=1}^{m} \beta_i$ for some $m \in \{2, 3, \cdots, d-1\}$ or $\beta_1 \leq \alpha_1$ and $\sum_{i=1}^{n} \beta_i > \sum_{i=1}^{n} \alpha_i$ for some $n \in \{2, 3, \cdots, d-1\}$. Then, either $\chi_1 \leq \eta_1$ and $\sum_{i=1}^{m} \chi_i > \sum_{i=1}^{m} \eta_i$ for some $m \in \{2, 3, \cdots, d-2\}$ or $\eta_1 \leq \chi_1$ and $\sum_{i=1}^{n} \eta_i > \sum_{i=1}^{n} \chi_i$ for some $n \in \{2, 3, \cdots, d-2\}$, i.e., $(|\Upsilon\rangle, |\Omega\rangle)$ are incomparable.

Proceeding in the same way, it is always possible to construct a lower dimensional incomparable pair of states with same entanglement from an upper dimensional one and they should differ in at least three Schmidt coefficients.

The above results have some immediate consequences in quantum information theory. If someone is restricted to use non-maximally pure bipartite entangled states as teleportation channel, then the optimal fidelity [13] for sending qudits are different for a pair of incomparable states with equal entanglement. Thus, the capacity as channel is not always equal, however, same resource in the sense of amount of entanglement, are used for the task. Again, if we mix such incomparable states with simply identity (garbage state), then sometimes they are PPT states and sometimes they are NPT states with the same mixing probabilities but differ only with the incomparable states considered. Also, the behavior of these locally inequivalent classes of states are different, if we consider different measures of correlations [15]. In particular, it easy to check the non-monotonicity of concurrence with entanglement of formation [16].

In conclusion we have found the physical character of pure bipartite states with the same amount of entanglement in the same Schmidt rank. The number of such incomparable states are infinite and the higher Schmidt rank incomparable states with equal entanglement must differ with at least three Schmidt coefficients. Thus it is observed that the entropy of entanglement is not always able to characterize the non-local properties of pure bipartite states. The relations between locally inequivalent pure bipartite states and different entanglement measures will be the important future issues to understand the proper behavior of entanglement. We hope our result would have far reaching consequences in entanglement dynamics.

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Appendix: Consider two pure bipartite states $|\Psi\rangle, |\Phi\rangle$ of Schmidt rank d having Schmidt vectors, $(\alpha_1, \alpha_2, \cdots \alpha_d)$, and $(\beta_1, \beta_2, \cdots \beta_d)$ respectively, where $1 > \alpha_i \geq \alpha_{i+1} > 0$ and $1 > \beta_i \geq \beta_{i+1} > 0$, $\forall \ i = 1, 2, \ldots, d-1$, $\sum_{i=1}^d \alpha_i = 1 = \sum_{i=1}^d \beta_i$. Suppose it is possible to convert $|\Psi\rangle$ to $|\Phi\rangle$ under deterministic LOCC, i.e., $|\Psi\rangle \to |\Phi\rangle$. Then, from Nielsen's criteria we have,

$$\sum_{i=1}^k \alpha_i \leq \sum_{i=1}^k \beta_i \ \ ; \quad \forall \ \ k = 1, 2, \ldots, d-1 \tag{14}$$

The above relation could be restated as,

$$\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i - \epsilon_k; \ \epsilon_k \geq 0, \ \forall \ k = 1, 2, \ldots, d-1 \tag{15}$$

Now, for the case $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_{d-1} > 0$, we have,

$$\begin{aligned}
& E(|\Psi\rangle) - E(|\Phi\rangle) \\
& = \{-\textstyle\sum_{i=1}^d \alpha_i \log_2 \alpha_i\} - \{-\textstyle\sum_{i=1}^d \beta_i \log_2 \beta_i\} \\
& = \beta_1 \log_2 \beta_1 - (\beta_1 - \epsilon_1) \log_2(\beta_1 - \epsilon_1) \\
& \quad + \textstyle\sum_{i=2}^{d-1} \{\beta_i \log_2 \beta_i - (\beta_i + \epsilon_{i-1} - \epsilon_i) \log_2(\beta_i + \epsilon_{i-1} - \epsilon_i)\} \\
& \quad + \beta_d \log_2 \beta_d - (\beta_d + \epsilon_{d-1}) \log_2(\beta_d + \epsilon_{d-1}) \\
& = -\beta_1 \log_2(1 - \tfrac{\epsilon_1}{\beta_1}) - \textstyle\sum_{i=2}^{d-1} \beta_i \log_2(1 + \tfrac{\epsilon_{i-1} - \epsilon_i}{\beta_i}) \\
& \quad - \beta_d \log_2(1 + \tfrac{\epsilon_{d-1}}{\beta_d}) + \epsilon_1 \log_2 \alpha_1 + \textstyle\sum_{i=2}^{d-1} (\epsilon_i - \epsilon_{i-1}) \log_2 \alpha_i \\
& \quad - \epsilon_{d-1} \log_2 \alpha_d \\
& \geq \log_e(2)(\beta_1 \textstyle\sum_{j=1}^{\infty} \frac{(\frac{\epsilon_1}{\beta_1})^j}{j} + \textstyle\sum_{i=2}^{d-1} \beta_i (\tfrac{\epsilon_{i-1} - \epsilon_i}{\beta_i}) \\
& \quad + \beta_d (\tfrac{\epsilon_{d-1}}{\beta_d})) + \textstyle\sum_{i=1}^{d-1} \epsilon_i (\log_2 \alpha_i - \log_2 \alpha_{i+1}) \\
& = \log_e(2)(\epsilon_1 + \beta_1 \textstyle\sum_{j=2}^{\infty} \frac{(\frac{\epsilon_1}{\beta_1})^j}{j} - \textstyle\sum_{i=2}^{d-1} (\epsilon_{i-1} - \epsilon_i) + \epsilon_{d-1}) \\
& \quad + \textstyle\sum_{i=1}^{d-1} \epsilon_i \log_2(\tfrac{\alpha_i}{\alpha_{i+1}}) \\
& = \log_e(2)(\beta_1 \textstyle\sum_{j=2}^{\infty} \frac{(\frac{\epsilon_1}{\beta_1})^j}{j}) + \textstyle\sum_{i=1}^{d-1} \epsilon_i \log_2(\tfrac{\alpha_i}{\alpha_{i-1}}) > 0.
\end{aligned}$$

The proof of all other alternatives are similar.