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CERTAIN CURVATURE CONDITIONS ON KENMOTSU MANIFOLDS ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. We study certain curvature properties of Kenmotsu manifolds with respect to the quarter-symmetric metric connection. First we consider Ricci semisymmetric Kenmotsu manifolds with respect to a quarter-symmetric metric connection. Next, we study ξ-conformally flat and ξ-concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Moreover, we study Kenmotsu manifolds satisfying the condition $\tilde{Z}(\xi, Y) \cdot \tilde{S} = 0$, where $\tilde{Z}$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection. Then, we prove the non-existence of ξ-projectively flat and pseudo-Ricci symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Finally, we construct an example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection for illustration.

1. Introduction

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M is said to be a quarter symmetric connection [**7**] if the torsion tensor T of the connection $\tilde{\nabla}$

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \tag{1.1}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. A linear connection $\tilde{\nabla}$ is called a metric connection of M if

$$(\tilde{\nabla}_X g)(Y,Z) = 0, \tag{1.3}$$

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where $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M. A linear connection $\tilde{\nabla}$ satisfying (1.2) and (1.3) is called a quarter-symmetric metric connection [**7**]. If we change ϕX by X, then the connection is called a semi-symmetric metric connection [**30**]. Thus the notion of quarter-symmetric connection generalizes the notion of the semi-symmetric connection. Semi-symmetric metric connections have been studied by several authors such as Özgür and Sular [**18, 19**], Ozen et al [**20, 21**], Prvanović [**23**], Smaranda and Andonie [**26**], Singh and Pandey [**27**] and many others.

A transformation in an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle of M, is said to be a concircular transformation [**31, 14**]. A concircular transformation is always a conformal transformation [**14**]. Here, we mean a geodesic circle by a curve in M whose first curvature is constant and second curvature is identically zero. Thus, the geometry of concircular transformations is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism [**2**]. An important invariant of a concircular transformation is the concircular curvature tensor Z, defined by [**31**]

$$Z(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}[g(Y,W)X - g(X,W)Y], \tag{1.4}$$

for all $X, Y, W \in \chi(M)$, where R is the Riemannain curvature tensor and r is the scalar curvature with respect to the Levi-Civita connection. The importance of concircular transformation and concircular curvature tensor is very well known in the differential geometry of certain F-structure such as complex, almost complex, Kähler, almost Kähler, contact and almost contact structure etc., [**3, 34, 33**].

Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

The Weyl conformal curvature tensor is defined by [**33**]

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{1.5}$$

where S is the Ricci tensor of type $(0,2)$ and Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y).$$

Let M be an n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geqslant 3$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. Here P is defined by [**28**]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.6}$$

for all $X, Y, Z \in \chi(M)$. In fact, M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

A non-flat n-dimensional Riemannian manifold (M, g), $n > 3$ is called pseudo Riccisymmetric [**5**] with respect to the quarter-symmetric metric connection if there exists a non-zero 1-form α on M such that

$$(\tilde{\nabla}_X \tilde{S})(Y, U) = 2\alpha(X)\tilde{S}(Y, U) + \alpha(Y)\tilde{S}(X, U) + \alpha(U)\tilde{S}(Y, X), \tag{1.7}$$

where $X, Y, U \in \chi(M)$.

Quarter-symmetric metric connection in a Riemannian manifold have been studied by several authors such as Mandal and De [**15**], Rastogi [**24, 25**], Yano and Imai [**32**], Mukhopadhyay, Roy and Barua [**16**], Han et al [**8, 9**], Biswas and De [**4**] and many others. Recently, Sular, Özgür and De [**29**] studied quarter-symmetric metric connection in a Kenmotsu manifold.

Motivated by the above studies in the present paper, we study quarter-symmetric metric connection in a Kenmotsu manifold. The paper is organized as follows: In section 2, we give a brief account of Kenmotsu manifolds. In section 3, we give the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the quarter-symmetric metric connection. Next, in section 4 we consider Ricci semisymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection is an Einstein manifold with respect to the Levi-Civita connection. Section 5 is devoted to study ξ-conformally flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that a ξ-conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an η-Einstein manifold with respect to the Levi-Civita connection. Section 6 deals with ξ-concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that if a Kenmotsu manifold is ξ-concircularly flat then the scalar curvature $r = -n(n-1)$. Next, we study Kenmotsu manifolds satisfying the condition $\tilde{Z}(\xi, Y) \cdot \tilde{S} = 0$, where $\tilde{Z}$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection and prove that in this case the manifold is η-Einstein with respect to the Levi-Civita connection. In section 8, we consider ξ-projectively flat and pseudo Riccisymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. We obtain the non-existence of these type manifolds. Finally, we construct an example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection to verify some theorems.

2. Kenmotsu Manifolds

Let M be an n=$(2m+1)$-dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1.1) tensor field, ξ is the associated vector field, η is a 1-form and g is the Riemannian metric satisfying [**1**]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for any vector fields X, Y on M. If an almost contact metric manifold satisfies $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, where ∇ denotes the Levi-Civita connection of g, then M is called a Kenmotsu manifold [**12**]. From the above equations it follows

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y). \tag{2.5}$$

In addition to the above results in a Kenmotsu manifold the following conditions hold [**12, 29, 11**]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,$$

$$S(X, \xi) = -(n-1)\eta(X), \quad Q\xi = -(n-1)\xi, \quad R(\xi, X)\xi = X - \eta(X)\xi,$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.6}$$

$$S(\phi X, Y) = -S(X, \phi Y), \tag{2.7}$$

where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator. A Kenmotsu manifold is normal, that is, the Nijenhuis tensor of ϕ equals $-2d\eta \otimes \xi$ but not Sasakian. Moreover, Kenmotsu manifold is not compact since from the equation (2.4) we have $\text{div}\xi = n - 1$. In [**12**], Kenmotsu showed (1) that locally a Kenmotsu manifold is a warped product $I \times_f N$, where I is an interval, N is a Kähler manifold and f is a warping function defined by $f(t) = se^t$, s is a nonzero constant; (2) that a Kenmotsu manifold of constant ϕ-sectional curvature is a space of constant curvature -1, hence it is hyperbolic space. Let M be a Kenmotsu manifold. M is said to be an η-Einstein manifold if there exists the real valued functions a and b such that $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$. For $b = 0$, the manifold M is an Einstein manifold. Now we state the following:

Lemma 2.1. [**12**] *Let M be an η-Einstein Kenmotsu manifold of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$. Then $a + b = -(n-1)$. If $b =$ constant (or, $a =$ constant), then M is an Einstein one.*

Kenmotsu manifolds have been studied by several authors such as Pitis [**22**], De and Pathak [**6**], Jun, De and Pathak [**11**], Özgür and De [**17**], Kirichenko [**13**], Hong et al [**10**] and many others.

3. Curvature Tensor

The quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ on a Kenmotsu manifold are related by [**29**]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y, \tag{3.1}$$

for all vector fields X, Y on M. Let $\tilde{R}$ and R be the Riemannian curvature tensor with respect to the quarter-symmetric metric connection and Levi-Civita connection respectively of a Kenmotsu manifold. Then the relation between $\tilde{R}$ and R is given by [**29**]

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \eta(X)g(\phi Y,Z)\xi - \eta(Y)g(\phi X,Z)\xi - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X.$$

Also in a Kenmotsu manifold with respect to the quarter-symmetric metric connection the following relations hold [**29**]

$$\tilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X - \eta(X)\phi Y + \eta(Y)\phi X, \tag{3.2}$$

$$\tilde{R}(X,\xi)Y = g(X,Y)\xi - \eta(Y)X - g(\phi X,Y)\xi + \eta(Y)\phi X, \tag{3.3}$$

$$\tilde{R}(\xi,X)\xi = X - \eta(X)\xi - \phi X, \tag{3.4}$$

$$\tilde{S}(X,Y) = S(X,Y) + g(\phi X,Y), \tag{3.5}$$

$$\tilde{S}(X,\xi) = S(X,\xi) = -(n-1)\eta(X), \tag{3.6}$$

$$\tilde{r} = r, \tag{3.7}$$

where $\tilde{S}$ and $\tilde{r}$ are the Ricci tensor and the scalar curvature respectively with respect to the quarter-symmetric metric connection. Moreover, it is noted that the Ricci tensor $\tilde{S}$ with respect to the quarter-symmetric metric connection is not symmetric. Using expressions (3.2) and (3.3), the following are easily obtained from (1.4)

$$\tilde{Z}(\xi,Y)U = \Big[1 + \frac{\tilde{r}}{n(n-1)}\Big][\eta(U)Y - g(Y,U)\xi] + g(\phi Y,U)\xi - \eta(U)\phi Y \tag{3.8}$$

$$\tilde{Z}(\xi,Y)\xi = \Big[1 + \frac{\tilde{r}}{n(n-1)}\Big][Y - \eta(Y)\xi] - \phi Y. \tag{3.9}$$

4. Ricci Semisymmetric Kenmotsu Manifolds with Respect to the Quarter-symmetric Metric Connection

In this section we consider Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$.

Definition 4.1. A manifold M is said to be Ricci semisymmetric manifold if it satisfies $R(X,Y)\cdot S = 0$, where R is $(1,3)$-type curvature tensor and S is the $(0,2)$-type Ricci tensor.

Let us consider an n-dimensional Kenmotsu manifold which is Ricci semisymmetric with respect to the quarter-symmetric metric connection, that is,

$$\tilde{R}(X,Y)\cdot\tilde{S} = 0,$$

where $\tilde{R}(X,Y)$ denotes the derivation of the tensor algebra at each point of the manifold. This implies

$$\tilde{S}(\tilde{R}(X,Y)U,V) + \tilde{S}(U,\tilde{R}(X,Y)V) = 0. \tag{4.1}$$

Substituting $X = \xi$ in (4.1) and using (3.5) we get

$$S(\tilde{R}(\xi,Y)U,V) + S(U,\tilde{R}(\xi,Y)V) - g(\tilde{R}(\xi,Y)U,\phi V) + g(\phi U,\tilde{R}(\xi,Y)V) = 0.$$

Putting $U = \xi$ in the above equation and in view of (3.3) and (3.4) yields

$$S(Y,V) - S(\phi Y,V) + (n-2)g(\phi V,Y) + ng(Y,V) - \eta(Y)\eta(V) = 0. \tag{4.2}$$

Interchanging Y and V in the above equation we have

$$S(V,Y)-S(\phi V,Y)+(n-2)g(\phi Y,V)+ng(V,Y)-\eta(V)\eta(Y)=0. \tag{4.3}$$

Adding (4.2) and (4.3) and using the facts (2.3) and (2.7) we obtain

$$S(Y,V)=-ng(Y,V)+\eta(Y)\eta(V). \tag{4.4}$$

Hence by Lemma 2.1, (4.4) implies that the manifold is an η-Einstein manifold. Now, in this position we can state the following:

THEOREM 4.1. *Let M be a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

Now, using Lemma 2.1 and Theorem 4.1 we can state the following:

THEOREM 4.2. *Let M be a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

Since Ricci symmetric manifold ($\tilde{\nabla}\tilde{S}=0$) with respect to the quarter-symmetric metric connection implies $\tilde{R}\cdot\tilde{S}=0$, therefore we obtain the following:

COROLLARY 4.1. *If a Kenmotsu manifold is Ricci symmetric with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

5. ξ-conformally Flat Kenmotsu Manifolds with Respect to the Quarter-symmetric Metric Connection

ξ-conformally flat K-contact manifolds have been studied by Zhen et al [**35**]. Since at each point $p\in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n)=\phi(T_p(M^n))\oplus\{\xi_p\}$, where $\{\xi_p\}$ is the one-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, the conformal curvature tensor C is a map

$$C: T_p(M^n)\times T_p(M^n)\times T_p(M^n)\to\phi(T_p(M^n))\oplus\{\xi_p\}.$$

DEFINITION 5.1. [**35**] An almost contact metric manifold M^n is called ξ-conform-ally flat if the projection of the image of C onto $\{\xi_p\}$ is zero, that is, $C(X,Y)\xi=0$, where C is the conformal curvature tensor defined in (1.5).

This section deals with ξ-conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$, that is, $\tilde{C}(X,Y)\xi=0$. From this and (1.5) we have

$$\tilde{R}(X,Y)\xi-\frac{1}{n-2}[\tilde{S}(Y,\xi)X-\tilde{S}(X,\xi)Y+\eta(Y)\tilde{Q}X-\eta(X)\tilde{Q}Y] + \frac{\tilde{r}}{(n-1)(n-2)}[\eta(Y)X-\eta(X)Y]=0. \tag{5.1}$$

With the help of (3.2), (3.5) and (3.6) we have from (5.1)

$$\eta(Y)QX-\eta(X)QY=\Big[1+\frac{\tilde{r}}{n-1}\Big][\eta(Y)X-\eta(X)Y]$$

$$+ (n-3)[\eta(Y)\phi X - \eta(X)\phi Y].$$

Substituting $Y = \xi$ in the above equation and using (3.7) we have

$$QX = \Big(1 + \frac{r}{n-1}\Big)X - \Big(n + \frac{r}{n-1}\Big)\eta(Y)\xi + (n-3)\phi X. \tag{5.2}$$

Taking the inner product of (5.2) with Y, we have

$$S(X,Y) = \Big(1 + \frac{r}{n-1}\Big)g(X,Y) - \Big(n + \frac{r}{n-1}\Big)\eta(X)\eta(Y) + (n-3)g(\phi X, Y), \tag{5.3}$$

where $S(X,Y) = g(QX,Y)$. Interchanging X and Y in the above equation we obtain

$$S(Y,X) = \Big(1 + \frac{r}{n-1}\Big)g(Y,X) - \Big(n + \frac{r}{n-1}\Big)\eta(Y)\eta(X) + (n-3)g(\phi Y, X). \tag{5.4}$$

Adding (5.3) and (5.4) and using fact (2.3) yields

$$S(X,Y) = \Big(1 + \frac{r}{n-1}\Big)g(X,Y) - \Big(n + \frac{r}{n-1}\Big)\eta(X)\eta(Y). \tag{5.5}$$

Therefore by Lemma 2.1, (5.5) shows that the manifold under consideration is an η-Einstein manifold. This leads to the following:

THEOREM 5.1. *A ξ-conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an η-Einstein manifold.*

Since the conformally flatness implies ξ-conformally flat, therefore from the above theorem we state the following:

COROLLARY 5.1. *A conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an η-Einstein manifold.*

6. ξ-concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogously to the definition of ξ-conformally flat almost contact metric manifold, we define ξ-concircularly flat Kenmotsu manifolds.

DEFINITION 6.1. A Kenmotsu manifold M is said to be ξ-concircularly flat with respect to the quarter-symmetric metric connection if it satisfies

$$\tilde{Z}(X,Y)\xi = 0, \tag{6.1}$$

for any vector fields $X, Y \in \chi(M)$ and $\tilde{Z}$ is the concircular curvature tensor defined by (1.4) with respect to the quarter-symmetric metric connection.

In this section we study ξ-concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then from (1.4) and (6.1), we have

$$\tilde{R}(X,Y)\xi - \frac{\tilde{r}}{n(n-1)}[\eta(Y)X - \eta(X)Y] = 0. \tag{6.2}$$

Using (3.2) and (3.7) we obtain from (6.2)

$$\Big[1 + \frac{r}{n(n-1)}\Big][\eta(X)Y - \eta(Y)X] - \eta(X)\phi Y + \eta(Y)\phi X = 0. \tag{6.3}$$

Taking the inner product of (6.3) with U we obtain

$$\Big[1+\frac{r}{n(n-1)}\Big][\eta(X)g(Y,U)-\eta(Y)g(X,U)] -\eta(X)g(\phi Y,U)+\eta(Y)g(\phi X,U)=0. \tag{6.4}$$

Now putting $Y=U=e_i$ in (6.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i=1,2,\dots,n$ we get $\big[1+\frac{r}{n(n-1)}\big](n-1)\eta(X)=0$. Hence for $n\geqslant 3$, the scalar curvature $r=-n(n-1)$. Therefore we conclude the following:

THEOREM 6.1. *If an n-dimensional $(n\geqslant 3)$ Kenmotsu manifold is ξ-concircularly flat with respect to the quarter-symmetric metric connection, then the scalar curvature $r=-n(n-1)$.*

7. Kenmotsu Manifolds Satisfying the Condition $\tilde{Z}(\xi,Y)\cdot\tilde{S}=0$

This section is devoted to study Kenmotsu manifold satisfying the condition

$$\tilde{Z}(\xi,Y)\cdot\tilde{S}=0, \tag{7.1}$$

where $\tilde{Z}$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection. Equation (7.1) implies

$$\tilde{S}(\tilde{Z}(\xi,Y)U,V)+\tilde{S}(U,\tilde{Z}(\xi,Y)V)=0. \tag{7.2}$$

In view of (3.5) and (7.2) we have

$$S(\tilde{Z}(\xi,Y)U,V)-g(\tilde{Z}(\xi,Y)U,\phi V) + S(U,\tilde{Z}(\xi,Y)V)+g(\phi U,\tilde{Z}(\xi,Y)V)=0. \tag{7.3}$$

Putting $U=\xi$ in (7.3), we get

$$S(\tilde{Z}(\xi,Y)\xi,V)-g(\tilde{Z}(\xi,Y)\xi,\phi V)+S(\xi,\tilde{Z}(\xi,Y)V)=0. \tag{7.4}$$

With the help of (3.7), (3.8) and (3.9) we have from (7.4)

$$\begin{aligned}\Big[1+\frac{r}{n(n-1)}\Big]&[S(Y,V)+(n-1)\eta(Y)\eta(V)]\\ &- S(\phi Y,V)-\Big\{\Big[1+\frac{r}{n(n-1)}\Big]g(Y,\phi V)-g(\phi Y,\phi V)\Big\}\\ &-(n-1)\Big\{\Big[1+\frac{r}{n(n-1)}\Big][\eta(V)\eta(Y)-g(Y,V)]+g(\phi Y,V)\Big\}=0.\end{aligned} \tag{7.5}$$

Interchanging Y and V in (7.5) yields

$$\begin{aligned}\Big[1+\frac{r}{n(n-1)}\Big]&[S(V,Y)+(n-1)\eta(V)\eta(Y)]\\ &- S(\phi V,Y)-\Big\{\Big[1+\frac{r}{n(n-1)}\Big]g(V,\phi Y)-g(\phi V,\phi Y)\Big\}\\ &-(n-1)\Big\{\Big[1+\frac{r}{n(n-1)}\Big][\eta(Y)\eta(V)-g(V,Y)]+g(\phi V,Y)\Big\}=0.\end{aligned} \tag{7.6}$$

Subtracting (7.5) from (7.6), we get

$$S(\phi Y,V)-S(\phi V,Y)-\Big[1+\frac{r}{n(n-1)}\Big][g(V,\phi Y) - g(Y,\phi V)]-(n-1)[g(\phi V,Y)-g(\phi Y,V)]=0. \tag{7.7}$$

In view of (2.7), (2.3) and (7.7) we have

$$S(\phi Y,V)=\Big[2-n+\frac{r}{n(n-1)}\Big]g(\phi Y,V). \tag{7.8}$$

Substituting $V=\phi V$ in (7.8) and using (2.6) and (2.2) we obtain

$$S(Y,V)=\Big[2-n+\frac{r}{n(n-1)}\Big]g(Y,V)-\Big[1+\frac{r}{n(n-1)}\Big]\eta(Y)\eta(V). \tag{7.9}$$

Hence by Lemma 2.1, (7.9) shows the manifold under consideration is an η-Einstein manifold.

From the above discussions we have the following:

THEOREM 7.1. *If a Kenmotsu manifold satisfies the condition $\tilde{Z}(\xi,Y)\cdot\tilde{S}=0$ with respect to the quarter-symmetric metric connection, then the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

8. Non-existence of Certain Kinds of Kenmotsu Manifolds with Respect to the Quarter-symmetric Metric Connection

THEOREM 8.1. *There is no ξ-projectively flat Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$.*

PROOF. Let us suppose that there exists a ξ-projectively flat Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then we have $\tilde{P}(X,Y)\xi=0$, for any vector fields X and Y on M. From this and (1.6) we have

$$\tilde{R}(X,Y)\xi-\frac{1}{n-1}[\tilde{S}(Y,\xi)X-\tilde{S}(X,\xi)Y]=0. \tag{8.1}$$

With the help of (3.2) and (3.6) we obtain from (8.1)

$$\eta(Y)\phi X-\eta(X)\phi Y=0. \tag{8.2}$$

Taking $Y=\xi$ in (8.2) and using the facts $\phi\xi=0$ and $\eta(\xi)=1$, we get $\phi X=0$, which is a contradiction. Therefore the statement of this theorem follows. □

THEOREM 8.2. *There is no pseudo Riccisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$.*

PROOF. Suppose that there exists a pseudo Riccisymmetric Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then from (1.7) we have

$$(\tilde{\nabla}_X\tilde{S})(Y,U)=2\alpha(X)\tilde{S}(Y,U)+\alpha(Y)\tilde{S}(X,U)+\alpha(U)\tilde{S}(Y,X). \tag{8.3}$$

Taking $U=\xi$ in (8.3) and using (3.6) we obtain

$$(\tilde{\nabla}_X\tilde{S})(Y,\xi)=-2(n-1)\alpha(X)\eta(Y)-(n-1)\alpha(Y)\eta(X)+\alpha(\xi)\tilde{S}(Y,X). \tag{8.4}$$

On the other hand, by the covariant differentiation of the Ricci tensor $\tilde{S}$ with respect to the quarter-symmetric metric connection $\tilde{\nabla}$, we have

$$(\tilde{\nabla}_X \tilde{S})(Y,U) = \tilde{\nabla}_X \tilde{S}(Y,U) - \tilde{S}(\tilde{\nabla}_X Y, U) - \tilde{S}(Y, \tilde{\nabla}_X U). \tag{8.5}$$

So putting $U = \xi$ in (8.5) and using (3.5), (3.1) and (2.4) we get

$$(\tilde{\nabla}_X \tilde{S})(Y,\xi) = -(n-1)g(X,Y) - S(X,Y) - g(X,\phi Y). \tag{8.6}$$

Then comparing the right hand sides of equations (8.4) and (8.6), we obtain

$$\begin{aligned} -2(n-1)\alpha(X)\eta(Y) - (n-1)\alpha(Y)\eta(X) + \alpha(\xi)\tilde{S}(Y,X) \\ = -(n-1)g(X,Y) - S(X,Y) - g(X,\phi Y). \end{aligned}$$

Substituting X and Y with ξ in the above equation we find (since $n > 3$)

$$\alpha(\xi) = 0. \tag{8.7}$$

Now we show that $\alpha = 0$ holds for any vector field on M. Taking $Y = \xi$ in (8.4) and using (8.7) we have $(\tilde{\nabla}_X \tilde{S})(\xi,\xi) = -2(n-1)\alpha(X)$. By the use of (8.6) we find $\alpha(X) = 0$ for every vector field X on M, which implies that $\alpha = 0$ on M. This contradicts to the definition of pseudo Riccisymmetry. □

9. Example of a 5-dimensional Kenmotsu Manifold Admitting a Quarter-symmetric Metric Connection

We consider the 5-dimensional manifold $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$e_1 = e^{-v}\frac{\partial}{\partial x}, \quad e_2 = e^{-v}\frac{\partial}{\partial y}, \quad e_3 = e^{-v}\frac{\partial}{\partial z}, \quad e_4 = e^{-v}\frac{\partial}{\partial u}, \quad e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$\begin{gathered} g(e_i, e_j) = 0, \qquad i \neq j, \quad i,j = 1,2,3,4,5 \\ g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = g(e_5,e_5) = 1. \end{gathered}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_5)$, for any $Z \in \chi(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = e_3, \quad \phi e_2 = e_4, \quad \phi e_3 = -e_1, \quad \phi e_4 = -e_2, \quad \phi e_5 = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_5) = 1, \quad \phi^2(Z) = -Z + \eta(Z)e_5, \quad g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U),$$

for any $U, Z \in \chi(M)$. Thus, for $e_5 = \xi$, $M(\phi,\xi,\eta,g)$ defines an almost contact metric manifold. The 1-form η is closed. We have

$$\Omega\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\Big) = g\Big(\frac{\partial}{\partial x}, \phi\frac{\partial}{\partial z}\Big) = g\Big(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\Big) = -e^{2v}.$$

Hence, we obtain $\Omega = -e^{2v}dx \wedge dz$. Thus, $d\Omega = -2e^{2v}dv \wedge dx \wedge dz = 2\eta \wedge \Omega$. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold. It can be seen that $M(\phi, \xi, \eta, g)$ is normal. So, it is a Kenmotsu manifold. Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = 0, [e_1, e_5] = e_1,$$
$$[e_4, e_5] = e_4,\ [e_2, e_4] = [e_3, e_4] = 0,\ [e_2, e_5] = e_2,\ [e_3, e_5] = e_3.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Taking $e_5 = \xi$ and using Koszul's formula we get the following

$$\begin{array}{lllll} \nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -e_5, & \nabla_{e_4} e_5 = e_4, \\ \nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0. \end{array}$$

Using the above relations in (3.1), we obtain

$$\begin{array}{lllll} \tilde{\nabla}_{e_1} e_1 = -e_5, & \tilde{\nabla}_{e_1} e_2 = 0, & \tilde{\nabla}_{e_1} e_3 = 0, & \tilde{\nabla}_{e_1} e_4 = 0, & \tilde{\nabla}_{e_1} e_5 = e_1, \\ \tilde{\nabla}_{e_2} e_1 = 0, & \tilde{\nabla}_{e_2} e_2 = -e_5, & \tilde{\nabla}_{e_2} e_3 = 0, & \tilde{\nabla}_{e_2} e_4 = 0, & \tilde{\nabla}_{e_2} e_5 = e_2, \\ \tilde{\nabla}_{e_3} e_1 = 0, & \tilde{\nabla}_{e_3} e_2 = 0, & \tilde{\nabla}_{e_3} e_3 = -e_5, & \tilde{\nabla}_{e_3} e_4 = 0, & \tilde{\nabla}_{e_3} e_5 = e_3, \\ \tilde{\nabla}_{e_4} e_1 = 0, & \tilde{\nabla}_{e_4} e_2 = 0, & \tilde{\nabla}_{e_4} e_3 = 0, & \tilde{\nabla}_{e_4} e_4 = -e_5, & \tilde{\nabla}_{e_4} e_5 = e_4, \\ \tilde{\nabla}_{e_5} e_1 = -e_3, & \tilde{\nabla}_{e_5} e_2 = -e_4, & \tilde{\nabla}_{e_5} e_3 = e_1, & \tilde{\nabla}_{e_5} e_4 = e_2, & \tilde{\nabla}_{e_5} e_5 = 0. \end{array}$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$\begin{aligned} & R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = R(e_1, e_5)e_5 = -e_1, \\ & R(e_1, e_2)e_1 = e_2,\ R(e_1, e_3)e_1 = R(e_5, e_3)e_5 = R(e_2, e_3)e_2 = e_3, \\ & R(e_2, e_3)e_3 = R(e_2, e_4)e_4 = R(e_2, e_5)e_5 = -e_2, R(e_3, e_4)e_4 = -e_3, \\ & R(e_2, e_5)e_2 = R(e_1, e_5)e_1 = R(e_4, e_5)e_4 = R(e_3, e_5)e_3 = e_5, \\ & R(e_1, e_4)e_1 = R(e_2, e_4)e_2 = R(e_3, e_4)e_3 = R(e_5, e_4)e_5 = e_4, \end{aligned}$$

$$\begin{aligned} & \tilde{R}(e_1, e_2)e_2 = \tilde{R}(e_1, e_3)e_3 = \tilde{R}(e_1, e_4)e_4 = -e_1, \\ & \tilde{R}(e_1, e_2)e_1 = e_2,\ \tilde{R}(e_1, e_3)e_1 = \tilde{R}(e_2, e_3)e_2 = e_3, \\ & \tilde{R}(e_2, e_3)e_3 = \tilde{R}(e_2, e_4)e_4 = -e_2, \tilde{R}(e_2, e_5)e_5 = e_4 - e_2, \\ & \tilde{R}(e_3, e_4)e_4 = -e_3,\ \tilde{R}(e_2, e_5)e_2 = \tilde{R}(e_1, e_5)e_1 = \tilde{R}(e_4, e_5)e_4 = e_5, \\ & \tilde{R}(e_3, e_5)e_3 = e_5, \tilde{R}(e_1, e_4)e_1 = \tilde{R}(e_2, e_4)e_2 = \tilde{R}(e_3, e_4)e_3 = e_4, \\ & \tilde{R}(e_1, e_5)e_5 = e_3 - e_1, \tilde{R}(e_3, e_5)e_5 = -e_1 - e_3, \tilde{R}(e_4, e_5)e_5 = -e_2 - e_4. \end{aligned}$$

With the help of the above results we get the Ricci tensors as follows:

$$S(e_1,e_1)=S(e_2,e_2)=S(e_3,e_3)=S(e_4,e_4)=S(e_5,e_5)=-4 \tag{9.1}$$

$$\tilde{S}(e_1,e_1)=\tilde{S}(e_2,e_2)=\tilde{S}(e_3,e_3)=\tilde{S}(e_4,e_4)=\tilde{S}(e_5,e_5)=-4. \tag{9.2}$$

Therefore $r=\sum_{i=1}^{5} S(e_i,e_i)=-20$ and $\tilde{r}=\sum_{i=1}^{5}\tilde{S}(e_i,e_i)=-20$.

From (9.2) it can be easily verified that the manifold is Ricci semisymmetric with respect to the quarter-symmetric metric connection. Also from (9.1) it follows that the manifold is Einstein with respect to the Levi-Civita connection. Therefore Theorem 4.2 is verified.

Also $\tilde{r}=r=-20$. Again from the expressions of the curvature tensor we can easily verify that the manifold is ξ-concircularly flat with respect to the quarter-symmetric metric connection. Hence Theorem 6.1 is verified.

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