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Capillary-gravity waves generated in a viscous fluid

A. K. Pramanik
Department of Applied Mathematics, Calcutta University, Calcutta, India
S. R. Majumdar
Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada

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The linearized initial-value problem of capillary-gravity waves generated by a moving oscillatory surface pressure distribution in a viscous incompressible fluid of infinite depth is solved. It is found that viscosity apart from introducing a damping factor into the amplitude of each wave plays an important role in the critical case. While the solution in the inviscid fluid becomes singular for certain values of the parameters of the problem, the solution in viscous fluid remains valid for all values of the parameters, though the amplitudes are relatively large in the critical case.

I. INTRODUCTION

In this paper we study the two-dimensional capillary-gravity waves generated in a fluid of infinite depth by a surface pressure distribution which oscillates with frequency ω and at the same time moves with a uniform velocity V. The fluid is incompressible but viscous where the coefficient of viscosity ν is assumed to be a small quantity and the surface tension of the fluid T' is also taken into consideration.

For certain limiting values of the parameters this problem has been previously investigated. Thus the problem in a viscous fluid has been treated by Wu and Messick[1] with $\omega = 0$, by Debnath, Bagchi, and Mukhrjee[2] with $V = 0$, while the problem in an inviscid fluid has been discussed by Pramanik.[3] The corresponding ship wave problem in a viscous fluid has been investigated by Cumberbatch.[4] To understand the motivation of our present investigation we at first state some characteristic features of the solution of the corresponding problem in an ideal fluid.[3]

The steady state at far field from the source consists of six progressive waves, four gravity waves, and two capillary waves. These six waves are the contributions of the six roots of the modified frequency equations. There exist two dimensionless parameters a and b involving ω, V, and T' such that the critical case which arises because of the coalescence of some of the roots can be represented by a curve in the (a,b) plane. In each point of this curve two of the roots coincide excepting one point where three roots coincide. This curve divides the whole positive quadrant of the (a,b) plane into seven distinct regions for points (a,b) of which the propagation is different, while the solution becomes singular for (a,b) on the curve separating the regions.

Now, guided partly by the results of the previous papers on viscous fluids and partly by intuition, one may expect that viscosity can resolve this singular behavior of the inviscid problem by separating the double or triple roots into distinct roots. Motivated by this idea we have taken up the investigation of the problem in a viscous fluid with the coefficient of viscosity assumed small. As expected, it is found here that the viscosity really separates the roots, but since the viscosity is very small, a difficulty occurs in that some of the roots are extremely close together. To overcome this difficulty we have suggested a perturbation scheme, and with the help of this scheme we are able to find the distinct roots in all cases.

The problem is formulated as an initial-boundary value problem. The integral representation of the surface elevation is obtained through a joint Fourier–Laplace transform method. Then the asymptotic form of this surface elevation is obtained for large time and distance. This gives a system of progressive waves propagating upstream and downstream similar to the inviscid fluid case, however, with modified amplitudes and phases. As was found by the previous authors it is also found here that viscosity introduces a damping factor into the amplitude of each wave, but the important role of viscosity as stated above is that it resolves the singular behavior of the inviscid problem. Our solution remains valid for all possible values of the parameters. However, the amplitudes are found to be relatively large for (a,b) on the critical curve than those for (a,b) outside the curve. It is found that the amplitude is of the order of $\nu^{1/2}$ for points on the critical curve where two roots coincide, while it is of the order of $\nu^{-1/3}$ where three roots coincide.

Our formulation follows closely the formulation of Debnath *et al.* Since they are concerned with the stationary field there is no case of a singular solution; however, the fact that the motion reduces to a steady state can be inferred from their analysis, so we are mainly concerned with the analysis of the steady state.

II. FORMULATION AND FORMAL SOLUTIONS

For 2-D motion in the x-y plane, a coordinate system is taken with origin on the undisturbed free surface where the y axis is positive opposite to the direction of gravity, and the x axis is positive to the right-hand side. Since the liquid is initially at rest, waves are generated by the surface pressure distribution $p_0(x)\exp(i\omega t)$ which is applied at $t = 0$ and then made to move continually along the positive x-axis direction with a uniform velocity V. For a slow motion in which the velocity components (u,v) are assumed small, the linearized Navier–Stokes equations of motion and the equation of continuity are[5]

$$\frac{\partial u}{\partial t} - V\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u, \quad (1)$$

$$\frac{\partial v}{\partial t} - V\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v, \quad (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \quad (3)$$

The free-surface boundary conditions are

$$\left.\begin{aligned} & p-\rho g\eta-2\mu\frac{\partial v}{\partial y}=p_0(x)e^{i\omega t}-T'\frac{\partial^2\eta}{\partial x^2}, && (4)\\ & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0, && (5)\\ & v=\frac{\partial\eta}{\partial t}-V\frac{\partial\eta}{\partial x}, && (6)\end{aligned}\right\}\ \text{at } y_0$$

where $\eta(x,t)$ is the equation to the surface elevation at any time t, T' is the surface tension of the liquid, μ is the coefficient of viscosity, and ν is the kinematic viscosity; ν is assumed to be small. The initial conditions are

$$u=v=0,\quad \eta=0 \text{ at } t=0. \quad (7)$$

In addition, we have the boundedness conditions at infinity that u, v, p, and their x derivatives are zero as $|x|\to\infty$ and $u=v=0, p=0$ as $y\to-\infty$.

The above system of equations can be solved to obtain η in the integral form by applying Fourier transformation with respect ot x and Laplace transformation with respect to t. Since this approach is explained in Ref. 2, in the following we simply write down the expression for η:

$$\eta=-\frac{e^{i\omega t}}{\sqrt{2\pi}\rho}\int_{-\infty}^{\infty}\frac{P_0(k)|k|e^{ikx}}{(g|k|+T|k|^3)^{1/2}}\times\int_0^t\exp[-i(\omega-kV-2\nu i|k|^2)\tau]\times\sin[(g|k|+T|k|^3)^{1/2}\tau]d\tau, \quad (8)$$

where

$$P_0(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}p_0(x)e^{-ikx}\,dx.$$

III. STEADY-STATE WAVE INTEGRALS

In this section we shall determine the asymptotic form of η for large t and $|x|$. Now, performing the integration with respect to τ, η can be put in the following form:

$$\eta=(1/\sqrt{2\pi}\rho)(I_1+I_2-e^{i\omega t}I_3),$$

where

$$I_1=\int_{-\infty}^{\infty}\frac{P_0(k)|k|(i\omega-ikV+2\nu|k|^2)\sin[(g|k|+T|k|^3)^{1/2}t]}{(g|k|+T|k|^3)^{1/2}[(g|k|+T|k|^3)+(i\omega-ikV+2\nu|k|^2)^2]}\exp[-2\nu k^2t+i(kVt+kx)]\,dk,$$

$$I_2=\int_{-\infty}^{\infty}\frac{P_0(k)|k|\cos[(g|k|+T|k|^3)^{1/2}t]}{(g|k|+T|k|^3)+(i\omega-ikV+2\nu|k|^2)^2}\exp[-2\nu k^2t+i(kVt+kx)]\,dk,$$

$$I_3=\int_{-\infty}^{\infty}\frac{P_0(k)|k|e^{ikx}}{(g|k|+T|k|^3)-(\omega-kV-2i\nu|k|^2)^2}\,dk.$$

Now, following the approach of Debnath *et al.*,[2] one can find the asymptotic values of the integrals I_1 and I_2 which will give the transient part of the motion. Thus the steady-state value of η is given by

$$\eta_s=-(e^{i\omega t}/\sqrt{2\pi}\rho)I_3. \quad (9)$$

We shall now consider the limit of η_s as $|x|\to\infty$. The reduction of the range of integration over k to the positive k axis only and the substitution $\sqrt{T/g}\,k=\lambda$ reduces the expression for η_s to the following:

$$\eta_s=-[e^{ibt}/2\sqrt{2\pi}\rho(Tg)^{1/2}](I_{31}+I_{32}+I_{33}+I_{34}), \quad (10)$$

where

$$I_{31}=\int_0^{\infty}\frac{\lambda F(\lambda)e^{i\lambda x}}{\sigma(\sigma+a\lambda-b+2ic\lambda^2)}d\lambda,$$

$$I_{32}=\int_0^{\infty}\frac{\lambda F(\lambda)e^{i\lambda x}}{\sigma(\sigma-a\lambda+b-2ic\lambda^2)}d\lambda,$$

$$I_{33}=\int_0^{\infty}\frac{\lambda F(-\lambda)e^{-i\lambda x}}{\sigma(\sigma-a\lambda-b+2ic\lambda^2)}d\lambda,$$

$$I_{34}=\int_0^{\infty}\frac{\lambda F(-\lambda)e^{-i\lambda x}}{\sigma(\sigma+b+a\lambda-2ic\lambda^2)}d\lambda,$$

$$a=\frac{V}{(Tg)^{1/4}},\quad b=\frac{\omega T^{1/4}}{g^{3/4}},\quad c=\frac{\nu g^{1/4}}{T^{3/4}},$$

$$P_0(\sqrt{g/T}\,\lambda)=F(\lambda),\quad \sigma=(\lambda+\lambda^3)^{1/2}.$$

In the above expressions x and t are the dimensionless distance and time. In terms of original variables they are, respectively, equal to $\sqrt{g/T}\,x$ and $(g^{3/4}/T^{1/4})t$.

Now the ultimate steady-state waves at far field depend upon the poles of the above integrands. In the next section we first locate and characterize these poles.

A. The location of the poles

The relevant poles are the solutions of the following four equations:

$$\sigma-a\lambda-b+2ic\lambda^2=0, \quad (11)$$

$$\sigma-a\lambda+b-2ic\lambda^2=0, \quad (12)$$

$$\sigma+a\lambda-b+2ic\lambda^2=0, \quad (13)$$

$$\sigma+a\lambda+b-2ic\lambda^2=0. \quad (14)$$

These are called the modified frequency equations. The roots of these equations will be determined by the method of perturbation series expansion in powers of c. When $c=0$, the roots are the solutions of the following equation:

$$\sigma=a\lambda+b,\quad \sigma=a\lambda-b,\quad \sigma=-a\lambda+b,\quad \sigma=-a\lambda-b. \quad (15)$$

The real positive roots of (15) are determined by Pramanik[3] as points of intersection in the (m,λ) plane of the curve $m=\sigma(\lambda)$ and the straight lines $m=a\lambda+b$,

$m = a\lambda - b$, and $m = -a\lambda + b$. Without going into details we simply state a few results which we shall need in our subsequent discussions. The first equation has, at most, three roots: λ_{01}, λ_{02}, and λ_{03} ($\lambda_{01} < \lambda_{02} < \lambda_{03}$); the second has two such roots: λ_{04} and λ_{05} ($\lambda_{04} < \lambda_{05}$); and the third equation always has one root: λ_{06} ($\lambda_{06} < \lambda_{04}$). The critical cases $\lambda_{01} = \lambda_{02}$, $\lambda_{02} = \lambda_{03}$, and $\lambda_{04} = \lambda_{05}$ are, respectively, represented in the (a,b) plane by three segments of a curve c_1, c_2, and c_3 (Fig. 1). The distribution of roots for (a,b) in various regions can be easily determined as is done in Ref. 3. Now, for any point (a,b) on the curves c_1, c_2, c_3 as shown in Fig. 1, excepting the point A_0, two of the roots coincide, while for (a,b) at A_0, three roots coincide. The conditions for these two cases are, respectively, given by the following two equations:

$$\sigma'(\lambda_{0n}) = a, \quad n = 1,2, \text{ or } 3 \quad \text{and} \quad n = 4 \text{ or } 5 \tag{16}$$

$$\left.\begin{aligned}\sigma'(\lambda_{0n}) &= a\\ \sigma''(\lambda_{0n}) &= 0\end{aligned}\right\} \quad n = 1,2, \text{ or } 3. \tag{17}$$

We shall also use the notations $a(\lambda_{00}) = a_0$ and $a(\lambda'_{00}) = a_1$, where λ_{00} is the point of inflexion on $m = \sigma(\lambda)$ and λ'_{00} is the point at which $m = a\lambda$ is tangent to $m = \sigma(\lambda)$.

Now corresponding to the six roots λ_{0n} for $c = 0$, let the roots of the equations for $c \neq 0$ be denoted by λ_n, $n = 1$ to 6, where λ_1, λ_2, and λ_3 are the roots of (11), λ_4 and λ_5 are the roots of (12), and λ_6 that of (13). We further assume $\mathrm{Re}(\lambda_1) < \mathrm{Re}(\lambda_2) < \mathrm{Re}(\lambda_3)$, $\mathrm{Re}(\lambda_6) < \mathrm{Re}(\lambda_4) < \mathrm{Re}(\lambda_5)$. To find these roots, which are valid for all points in the (a,b) plane, we assume a perturbation power series in powers $c^{1/6}$ for λ_n in the following form:

$$\lambda_n = \lambda_{0n} + \lambda_{1n}c^{1/6} + \lambda_{3n}c^{1/3} + \lambda_{3n}c^{1/2} + \lambda_{1n}c^{2/3} + \lambda_{1n}c^{5/6} + \lambda_{6n}c + \cdots$$
$$n = 1 \text{ to } 6. \tag{18}$$

Substituting this expression into Eq. (11) and equating the coefficients of various powers of $c^{1/6}$ up to c from both sides, we shall get six equations for the determination of the relevant coefficients. These equations are written down in terms of various derivatives of the function $\sigma(\lambda)$ as follows:

$$[\sigma'(\lambda_{0n}) - a]\lambda_{1n} = 0, \tag{19}$$

$$\sigma''(\lambda_{0n})\lambda_{1n}^2 + 2[\sigma'(\lambda_{0n}) - a]\lambda_{2n} = 0, \tag{20}$$

$$\sigma'''(\lambda_{0n})\lambda_{1n}^3 + 6\sigma''(\lambda_{0n})\lambda_{1n}\lambda_{2n} + 6[\sigma'(\lambda_{0n}) - a]\lambda_{3n} = 0, \tag{21}$$

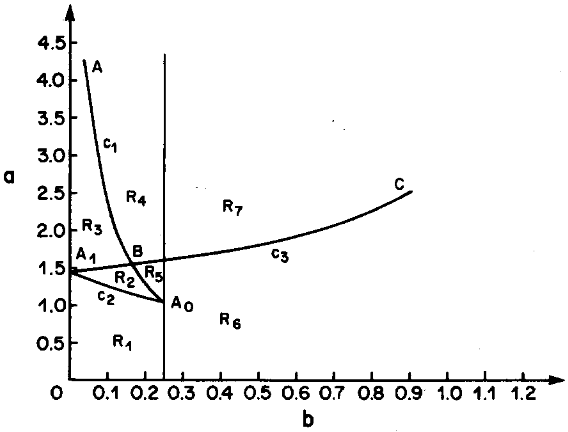

FIG. 1. The critical curve in the (a,b) plane.

$$\sigma^{iv}(\lambda_{0n})\lambda_{1n}^4 + 12\sigma'''(\lambda_{0n})\lambda_{1n}^2\lambda_{2n} + 12\sigma''(\lambda_{0n})\lambda_{2n}^2 + 24\sigma''(\lambda_{0n})\lambda_{1n}\lambda_{3n} + 24[\sigma'(\lambda_{0n}) - a]\lambda_{4n} = 0, \tag{22}$$

$$\sigma^{v}(\lambda_{0n})\lambda_{1n}^5 + 20\sigma^{iv}(\lambda_{0n})\lambda_{1n}^3\lambda_{2n} + 60\sigma'''(\lambda_{0n})\lambda_{1n}\lambda_{2n}^2 + 60\sigma'''(\lambda_{0n})\lambda_{1n}^2\lambda_{3n} + 120\sigma''(\lambda_{0n})\lambda_{2n}\lambda_{3n} + 120\sigma''(\lambda_{0n})\lambda_{1n}\lambda_{4n} + 120[\sigma'(\lambda_{0n}) - a]\lambda_{5n} = 0, \tag{23}$$

$$\begin{aligned}&\sigma^{vi}(\lambda_{0n})\lambda_{1n}^6 + 30\sigma^{v}(\lambda_{0n})\lambda_{1n}^4\lambda_{2n} + 180\sigma^{iv}(\lambda_{0n})\lambda_{1n}^2\lambda_{2n}^2\\ &+ 120\sigma^{iv}(\lambda_{0n})\lambda_{1n}^3\lambda_{3n} + 120\sigma'''(\lambda_{0n})\lambda_{2n}^3\\ &+ 720\sigma'''(\lambda_{0n})\lambda_{1n}\lambda_{2n}\lambda_{3n} + 360\sigma'''(\lambda_{0n})\lambda_{1n}^2\lambda_{4n}\\ &+ 360\sigma''(\lambda_{0n})\lambda_{3n}^2 + 720\sigma''(\lambda_{0n})\lambda_{2n}\lambda_{4n}\\ &+ 720\sigma''(\lambda_{0n})\lambda_{1n}\lambda_{5n} + 720[\sigma'(\lambda_{0n}) - a]\lambda_{6n}\\ &= -1440i\lambda_{0n}^2.\end{aligned} \tag{24}$$

To determine the coefficients λ_{mn} from these equations, we consider the following three cases:

(i) The point (a,b) is within the region R_3; that is, all of λ_{01}, λ_{02}, and λ_{03} are real, positive, and distinct. In this case we find from (19), $\lambda_{1n} = 0$, and then from (20), (21), (22), and (23) we get, successively, $\lambda_{2n} = 0$, $\lambda_{3n} = 0$, $\lambda_{4n} = 0$, and $\lambda_{5n} = 0$, and then Eq. (24) gives

$$\lambda_{6n} = -2i\lambda_{0n}^2/[\sigma'(\lambda_{0n}) - a], \quad n = 1,2,3.$$

(ii) The point (a,b) is on the curves c_1 and c_2 excepting the point A_0. In this case, using the condition (16) we see from (20) that $\lambda_{1n} = 0$. Then (21) is automatically satisfied, and (22) gives $\lambda_{2n} = 0$. Then (23) is automatically satisfied, and (24) gives

$$\lambda_{3n}^2 = -4i\lambda_{0n}^2/\sigma''(\lambda_{0n}), \quad n = 1 \text{ or } 2 \text{ and } 2 \text{ or } 3.$$

(iii) The point (a,b) is at A_0. In this case, using the conditions (17) we see that (19) and (20) are automatically satisfied, but (21) gives $\lambda_{1n} = 0$. Then (22) and (23) are automatically satisfied, and (24) gives

$$\lambda_{2n}^3 = -12i\lambda_{0n}^2/\sigma'''(\lambda_{0n}), \quad n = 1,2, \text{ or } 3.$$

To determine the roots λ_4 and λ_5, substituting (18) into the Eq. (12) we shall get six equations similar to the above equations from which the relevant coefficients can be determined for the two cases when λ_{04} and λ_{05} are distinct, and when they are coincident. The whole set of roots of the Eqs. (11), (12), and (13) can be put in the following way:

(i) For any point (a,b) within the region R_3,

$$\lambda_n = \lambda_{0n} + c\lambda_{6n}, \quad n = 1 \text{ to } 6, \tag{25}$$

where

$$\begin{aligned}\lambda_{6n} &= -\frac{4i\lambda_{0n}^2}{(\lambda_{0n} + \lambda_{0n}^3)^{-1/2}(1 + 3\lambda_{0n}^2) - 2a}, \quad n = 1,2,3\\ &= \frac{4i\lambda_{0n}^2}{(\lambda_{0n} + \lambda_{0n}^3)^{-1/2}(1 + 3\lambda_{0n}^2) - 2a}, \quad n = 4,5\\ &= -\frac{4i\lambda_{0n}^2}{(\lambda_{0n} + \lambda_{0n}^3)^{-1/2}(1 + 3\lambda_{0n}^2) + 2a}, \quad n = 6.\end{aligned}$$

(ii) For any point (a,b) on the curve c_1 excepting the point A_0. Here

$$\lambda_{01} = \lambda_{02},$$

$$\lambda_n = \lambda_{0n} + c\lambda_{6n}, \quad n = 3,4,5,6 \qquad (26)$$

$$= \lambda_{0n} + c^{1/2}\lambda_{3n}, \quad n = 1,2,$$

where

$$\lambda_{31} = -H(\lambda_{01}), \quad \lambda_{32} = H(\lambda_{02}),$$

$$H(\lambda) = \frac{2\sqrt{2}(1+i)\lambda}{[(\lambda+\lambda^3)^{-3/2}(1+3\lambda^2)^2 - 24(\lambda+\lambda^3)^{-1/2}\lambda\,]^{1/2}}.$$

Here the roots λ_4 and λ_5 are not present for points (a,b) on the portion of c_1 from the point B to A_0 (see the remarks below).

(iii) For any point (a,b) on c_2 excepting the point A_0. Here

$$\lambda_{02} = \lambda_{03},$$

$$\lambda_n = \lambda_{0n} + c\lambda_{6n}, \quad n = 1,6 \qquad (27)$$

$$= \lambda_{0n} + c^{1/2}\lambda_{3n}, \quad n = 2,3,$$

where

$$\lambda_{32} = -H_1(\lambda_{02}), \quad \lambda_{33} = H_1(\lambda_{03}),$$

$$H_1(\lambda) = \frac{2\sqrt{2}(1-i)\lambda}{[-(\lambda+\lambda^3)^{-3/2}(1+3\lambda^2) + 24(\lambda+\lambda^3)^{-1/2}\lambda\,]^{1/2}}.$$

(iv) For any point (a,b) on c_3. Here

$$\lambda_{04} = \lambda_{05},$$

$$\lambda_n = \lambda_{0n} + c\lambda_{6n}, \quad n = 1,2,3,6 \qquad (28)$$

$$= \lambda_{0n} + c^{1/2}\lambda_{3n}, \quad n = 4,5,$$

where

$$\lambda_{34} = -H_2(\lambda_{04}), \quad \lambda_{35} = H_2(\lambda_{05}),$$

$$H_2(\lambda) = \frac{2\sqrt{2}(1+i)\lambda}{[-(\lambda+\lambda^3)^{-3/2}(1+3\lambda^2) + 24(\lambda+\lambda^3)^{-1/2}\lambda\,]^{1/2}}.$$

Here the roots λ_1 and λ_2 are not present for points (a,b) on the portion BC.

(v) For the point (a,b) at A_0. Here

$$\lambda_{01} = \lambda_{02} = \lambda_{03},$$

$$\lambda_n = \lambda_{0n} + c\lambda_{1n}, \quad n = 6 \qquad (29)$$

$$= \lambda_{0n} + c^{1/3}\lambda_{2n}, \quad n = 1,2,3,$$

where

$$\lambda_{21} = -1/2(\sqrt{3}+i)G(\lambda_{01}), \quad \lambda_{22} = iG(\lambda_{02}), \quad \lambda_{23} = 1/2(\sqrt{3}-i)G(\lambda_{03}),$$

$$G(\lambda) = \frac{2(2\lambda)^{2/3}}{[(\lambda+\lambda^3)^{-5/2}(1+3\lambda^2)^3 - 12(\lambda+\lambda^3)^{-3/2}(1+3\lambda^2) + 8(\lambda+\lambda^3)^{-1/2}]^{1/3}}.$$

It is to be noted that whenever λ_{0n} are real and positive the corresponding roots λ_n are all distinct, even though there may be some repeated roots in the set $\{\lambda_{0n}\}$. Now for points (a,b) in regions other than R_3, some of the roots λ_{0n} are complex. We need not determine these roots λ_n, since as we shall see later on that they will not contribute to the ultimate wave system. The roots λ_n in other regions corresponding to the real positive λ_{0n} that are present, have the same expressions as given in (25).

Now, for our subsequent development we need the sign of $\text{Im}(\lambda_n)$, so we determine these signs. For any point (a,b) within the regions R_n, the sign of $\text{Im}(\lambda_n)$ is determined by the sign of $[\sigma'(\lambda_{0n}) - a]$, where σ' denotes the derivative of σ with respect to λ. These signs can be determined by the following geometrical considerations. We note $\sigma'(\lambda_{0n})$ decreases monotonically from the value infinity to the minimum value $\sigma'(\lambda_{00}) = a_0$ as λ_{0n} increases from zero to λ_{00} and then increases from this value to infinity as λ_{0n} still increases. Now, to determine the sign of $[\sigma'(\lambda_{01}) - a]$, let us make b fixed and increase a from zero. In this process in the (m,λ) plane the straight line $m = a\lambda + b$ is rotated about a fixed point on the m axis, while in the (a,b) plane (Fig. 1) we are moving along a straight line parallel to the a axis. In this process λ_{01} monotonically increases and $\sigma'(\lambda_{01})$ decreases monotonically up to the point where we meet the curve c_1, at which point $\sigma'(\lambda_{01}) = a$ and thereafter λ_{01} does not exist. It then follows that $a < \sigma'(\lambda_{01})$ whenever λ_{01} exists. For sign of $[\sigma'(\lambda_{02}) - a]$, we note that λ_{02} begins to exist only when we reach the curve c_2, at which $\sigma'(\lambda_{02}) = a$. Thereafter $\sigma'(\lambda_{02})$ at first decreases, and the minimum value it takes is $\sigma'(\lambda_{00}) = a_0$, and then $\sigma'(\lambda_{02})$ begins to increase up to the curve c_1 where again $\sigma'(\lambda_{02}) = a$. After this point λ_{02} does not exist, so whenever λ_{02} exists, $\sigma'(\lambda_{02}) < a$. Now, when we increase a from zero, λ_{03} begins to exist from the point where we meet c_2. Thereafter both a and $\sigma'(\lambda_{03})$ increase, but here we cannot know the relative magnitudes of a and $\sigma'(\lambda_{03})$. However, that $a < \sigma'(\lambda_{03})$ can be ascertained by the following consideration.

Now we increase b and keep a fixed. In this process in the (m,λ) plane the straight line $m = a\lambda + b$ is moving parallel to itself while in the (a,b) plane it moves parallel to the b axis. For $a_0 < a < a_1$, we at first meet the curve c_2 at which λ_{03} begins to exist and $\sigma'(\lambda_{03}) = a$. After that $\sigma'(\lambda_{03})$ increases continually and a is fixed, so that $a < \sigma'(\lambda_{03})$. For $a > a_1$, λ_{03} always exists and for all λ_{03}, $a < \sigma'(\lambda_{03})$. If $a > \sigma'(\lambda_{03})$ for some values of λ_{03}, then since a is fixed and $\sigma'(\lambda_{03})$ increases montonically to infinity, the case $a = \sigma'(\lambda_{03})$ would have occurred for some λ_{03}, which is not true. The signs of $[\sigma'(\lambda_{04}) - a]$ and $[\sigma'(\lambda_{05}) - a]$ can be determined by similar considerations. Again, the signs of $\text{Im}(\lambda_n)$ for points (a,b) on the critical curves are easy to determine since in all expressions the quantity within the radical sign is positive. Thus we arrive at the following results valid for all (a,b):

$$\text{Im}(\lambda_1) < 0, \quad \text{Im}(\lambda_2) > 0, \quad \text{Im}(\lambda_3) < 0,$$

$$\text{Im}(\lambda_4) < 0, \quad \text{Im}(\lambda_5) > 0, \quad \text{Im}(\lambda_6) < 0. \qquad (30)$$

B. Asymptotic values of the integrals

Now we evaluate the integrals in η_s asymptotically for $|x| \to \infty$. We at first confine our attention to those values of

(a,b) which are in the region R_3, including its boundaries. We note that for this case all the six roots λ_n are present and distinct. We consider the integral I_{33}. We can write

$$\sigma - a\lambda - b + 2ic\lambda^2 \equiv q(\lambda)(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3),$$

where

$$q(\lambda_1) = [\sigma'(\lambda_1) - a + 4ic\lambda_1]/(\lambda_1-\lambda_2)(\lambda_1-\lambda_3), \quad (31)$$

and two similar expressions for $q(\lambda_2)$ and $q(\lambda_3)$.

Then by a partial fraction development the integral I_{33} can be broken up into the following three integrals:

$$\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\int_0^\infty \frac{F(-\lambda)\lambda}{\sigma q(\lambda)}\frac{e^{-i\lambda x}}{\lambda-\lambda_1}d\lambda + \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\int_0^\infty \frac{F(-\lambda)\lambda}{\sigma q(\lambda)}\frac{e^{-i\lambda x}}{\lambda-\lambda_2}d\lambda + \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\int_0^\infty \frac{F(-\lambda)\lambda}{\sigma q(\lambda)}\frac{e^{-i\lambda x}}{\lambda-\lambda_3}d\lambda. \quad (32)$$

Now we may write

$$\frac{F(-\lambda)\lambda}{q(\lambda)\sigma} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_1(\alpha)e^{i\lambda\alpha}\,d\alpha, \quad (33)$$

where

$$F_1(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(-\lambda)\lambda}{\sigma q(\lambda)}e^{-i\lambda\alpha}\,d\lambda.$$

Substituting this, and then by a change in the order of integration with respect to λ and α, the second integral in (32) is written as follows:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F_1(\alpha)d\alpha\int_0^\infty \frac{e^{-i\lambda\alpha(x-\alpha)}}{\lambda-\lambda_2}d\lambda &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F_1(\alpha)d\alpha\left(\int_0^\infty\frac{\cos\lambda|x-\alpha|}{\lambda-\lambda_2}d\lambda - i\,\mathrm{sgn}(x-\alpha)\int_0^\infty\frac{\sin\lambda|x-\alpha|}{\lambda-\lambda_2}d\lambda\right)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F_1(\alpha)d\alpha\,\{[\cos z\,\mathrm{ci}(-z)+\sin z\,\mathrm{si}(-z)] - i\,\mathrm{sgn}(x-\alpha)[\sin z\,\mathrm{ci}(-z)-\cos z\,\mathrm{si}(-z)]\}, \quad (34)\end{aligned}$$

where $z=\lambda_2|x-\alpha|$ and si and ci are the sine and cosine integrals. Now since $\mathrm{Im}(\lambda_2)>0$, we get the following formula[6,7]:

$$\mathrm{ci}(-z) = \pi i + \mathrm{ci}(z), \quad \mathrm{si}(-z) = -\pi - \mathrm{si}(z).$$

Using these formulas and the asymptotic expansions for the sine and the cosine integrals, we obtain

$$\int_0^\infty \frac{e^{-i\lambda(x-\alpha)}}{\lambda-\lambda_2}d\lambda \simeq \pi i[1-\mathrm{sgn}(x-a)] \times\exp(-i\lambda_2|x-\alpha|), \quad |x|\to\infty. \quad (35)$$

Substituting this value and using the relations in (31) and (33), the second integral in (32) finally reduces to

$$\begin{aligned}\frac{2\pi i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F_1(a)\exp[-i\lambda_2(x-\alpha)]\,da &\simeq \frac{2\pi i\lambda_2F(-\lambda_2)(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{\sigma(\lambda_2)[\sigma'(\lambda_2)-a+4ic\lambda_2^2]}, \quad x\to-\infty\\ &\simeq 0, \quad x\to\infty.\end{aligned}$$

The remaining two integrals in (32) can be evaluated in a similar way. The integrals I_{31} and I_{32}, having poles, can each be broken up into several integrals which have one pole. They can then be evaluated by the above method.

IV. THE STEADY-STATE WAVES

The steady-state waves at far field are the combinations of the asymptotic values of the three integrals I_{31}, I_{32}, and I_{33}. In the following we write down the waves for the case when the point (a,b) is in the region R_3 including its boundaries.

$$\begin{aligned}\eta_s &= \frac{i\pi^{1/2}}{\rho(2gT)^{1/2}}\Big(\frac{\lambda_1F(-\lambda_1)\exp[i(bt-\lambda_1x)]}{\sigma(\lambda_1)[\sigma'(\lambda_1)-a+4ic\lambda_1]} + \frac{\lambda_3F(-\lambda_3)\exp[i(bt-\lambda_3x)]}{\sigma(\lambda_3)[\sigma'(\lambda_3)-a+4ic\lambda_3]} - \frac{\lambda_5F(\lambda_5)\exp[i(bt+\lambda_5x)]}{\sigma(\lambda_5)[\sigma'(\lambda_5)-a-4ic\lambda_5]}\Big), \quad x\to\infty \quad (36)\\ &= \frac{i\pi^{1/2}}{\rho(2gT)^{1/2}}\Big(\frac{\lambda_4F(\lambda_4)\exp[i(bt+\lambda_4x)]}{\sigma(\lambda_4)[\sigma'(\lambda_4)-a-4ic\lambda_4]} - \frac{\lambda_2F(-\lambda_2)\exp[i(bt-\lambda_2x)]}{\sigma(\lambda_2)[\sigma'(\lambda_2)-a+4ic\lambda_4]} + \frac{\lambda_6F(\lambda_6)\exp[i(bt+\lambda_6x)]}{\sigma(\lambda_6)[\sigma'(\lambda_6)+a+4ic\lambda_6]}\Big), \quad x\to-\infty. \quad (37)\end{aligned}$$

Thus we see that when the point (a,b) is in the region R_3 including its boundaries, the steady state consists of six progressive waves, three in the downstream side and three in the upstream. These six waves correspond to six roots λ_n of the modified frequency equations. Among these, the four waves given by the roots $\lambda_1, \lambda_2, \lambda_4$, and λ_6 are the gravity waves and the remaining two are the capillary waves. Each of these waves contains a damping factor in x in its amplitude. This damping factor is of the form $\exp(-|x|\lambda c)$ or $\exp(-|x|\lambda c^{1/2})$, where λ is a positive constant, depending on whether the point (a,b) is within the region or on the critical curves with the exception of point A_0, for which the damping factor is of the form $\exp(-|x|\lambda c^{1/3})$. Now for (a,b) in regions other than R_3, where some of the quantities λ_{0n} become complex, this damping factor would be of the form $e^{-|x|\lambda}$. This factor is negligibly small compared to the damping factors noted above since c is assumed small and the corresponding wave becomes insignificant compared to the other waves. Thus we see that the root of the modified frequency equation whose zeroth approximation is real can contribute to the ultimate wave system. The waves for (a,b) in other regions are then easy to determine. These are the same waves given in (36) and (37), only the wave corresponding to a pole λ_n, for which λ_{0n} is complex in a region, is to be deleted for that region. The direction of the propagation of the other waves remains the same.

Thus we see that when viscosity of the liquid is taken into account, the wave amplitudes remain finite for all values of the parameters a and b, whereas in an ideal fluid they become infinite for (a,b) on the critical curves. However, here it can also be verified that the amplitudes are relatively large when (a,b) is on the critical curves. For example, let us consider the amplitude of the first wave in (36). The denominator of the amplitude of this wave contains the factor $[\sigma'(\lambda_1) - a + 4ic\lambda_1]$. Now, for (a,b) on c_1, excepting A_0, and substituting the value of λ_1 from (26), we see that this factor has the following value:

$$\sigma'(\lambda_{01} + c^{1/2}\lambda_{31}) - a + 4ic(\lambda_{01} + c^{1/2}\lambda_{31})$$
$$\simeq [\sigma'(\lambda_{01}) - a] + c^{1/2}\sigma''(\lambda_{01}) + 4ic\lambda_{01}$$
$$\simeq c^{1/2}\sigma''(\lambda_{01}),$$

where we have used the relation (16). Thus the amplitude becomes of the order $c^{-1/2}$, which is actually large compared to the amplitudes of the waves for (a,b) within the regions. Similarly, it is easy to verify that this wave has the amplitude of the order of $c^{-1/3}$ for (a,b) at A.

As stated above, the solution in an ideal fluid becomes singular for (a,b) on the critical curves. This is caused by the occurrence of familiar resonance between the free natural waves and the forcing terms, as a result of which infinite energy radiation occurs from the source. This radiation of infinite energy has been shown by Wu[8] in the case of a fluid without surface tension, and it is easy to verify in the case with surface tension. Now, when the viscosity is taken into account some part of the energy is dissipated. As a result, infinite energy radiation is not possible in any case. However, since our coefficient of viscosity is very small, the energy radiation in the critical case is still large, making some of the wave amplitudes still considerably large.

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