

AN ELEMENTARY PROOF OF WEYL'S LIMIT-CLASSIFICATION

J. DAS (néé CHAUDHURI)

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Abstract

It is known [Herman Weyl, 1910] that every linear second-order differential expression L (with real coefficients) is such that $Ly = \lambda y$ ($\text{im } \lambda \neq 0$) has at least one solution belonging to the class $\mathcal{L}^2 = \mathcal{L}^2[0, \infty)$ of functions, the squares of whose moduli are Lebesgue-integrable on $[0, \infty)$. This celebrated result was later proved by E. C. Titchmarsh (1940–1944), using sophisticated analysis of bilinear transformations. The aim of the present note is to prove the same result once again, but using only elementary analysis and school geometry. The power of this method will be appreciated further when one realises the amount of simplifications that can be achieved by this method in case of higher order expressions. This part of the note of course will be taken up in a subsequent paper.

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Herman Weyl, in his classical paper of 1910, proved that every second-order ordinary differential equation of the form

$$(1.1) \quad L[y] = -y^{(2)}(x) + q(x)y(x) = \lambda y(x)$$

(where $y^{(r)}(x) = d^r y/dx^r$, $r = 1, 2$, $q(\cdot)$ is real-valued continuous over any bounded subinterval of $[0, \infty)$), has at least one solution $\psi \equiv \psi(x, \lambda)$ satisfying

$$(1.2) \quad \int_0^\infty |\psi(x, \lambda)|^2 dx < \infty,$$

whenever $\nu = \text{im } \lambda \neq 0$. Such a solution ψ is referred to as an L^2 -solution of (1.1). If two independent solutions of (1.1) are L^2 , (1.1) is said to be in the limit-circle case at infinity, while if only one independent solution is L^2 , the

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other is not L^2 , (1.1) is said to be in the *limit-point case* at infinity. This is the celebrated limit classification of Weyl.

During 1940–1944, E. C. Titchmarsh [2] developed a really sophisticated theory based on the properties of bilinear transformations of the complex plane and established the above mentioned result. He associated the differential equation (1.1) with the boundary conditions

$$(1.3) \quad y(0) \cos \alpha + y^{(1)}(0) \sin \alpha = 0,$$

$$(1.4) \quad y(b) \cos \beta + y^{(1)}(b) \sin \beta = 0.$$

where α, β are arbitrary real numbers. Two linearly independent solutions $\theta = \theta(x) \equiv \theta(x, \lambda), \phi \equiv \phi(x) \equiv \phi(x, \lambda)$ of (1.1) are defined to satisfy the following:

$$(1.5) \quad \theta(0) = \cos \alpha, \quad \theta^{(1)}(0) = \sin \alpha,$$

$$(1.6) \quad \phi(0) = \sin \alpha, \quad \phi^{(1)}(0) = -\cos \alpha.$$

Hence the general solution of (1.1) will be of the form $\theta + l(\lambda)\phi$ and if it is to satisfy the boundary condition (1.4) at $x = b$, we get

$$(1.7) \quad \{\theta(b) + l(\lambda)\phi(b)\} \cos \beta + \{\theta^{(1)}(b) + l(\lambda)\phi^{(1)}(b)\} \sin \beta = 0$$

or

$$(1.8) \quad l(\lambda) = -\frac{\theta(b)z + \theta^{(1)}(b)}{\phi(b)z + \phi^{(1)}(b)},$$

where $z = \cot \beta$. In [2] Titchmarsh interpreted (1.8) as a bilinear transformation T carrying the z -plane to the l -plane. As β is real, by varying $\cot \beta$, we get the real line on the z -plane which, under the bilinear transformation T is mapped onto a circle, C_b say, on the l -plane. Using the fact that a bilinear transformation carries inverse points with respect to a circle into inverse points with respect to the corresponding image circle, Titchmarsh proved that, for any point $l = l(\lambda)$ inside the circle C_b ,

$$(1.9) \quad \int_0^b |\theta + l(\lambda)\phi|^2 dx < -\frac{\text{im } l(\lambda)}{\nu}.$$

This is the vital inequality which proves that the circles $\{C_b\}_b$ form a monotonic decreasing set in the sense that if $b' > b$ and $l(\lambda) \in C_{b'}$, then $l(\lambda) \in C_b$ also. The idea of the present paper is to prove the inequality (1.9) directly, using elementary geometry only, without viewing (1.8) as a bilinear transformation. More precisely, it will be shown that (1.8) is actually the equation of the circle C_b and that (1.8) can be exhibited as

$$(1.10) \quad \int_0^b |\theta + l(\lambda)\phi|^2 dx = -\frac{\text{im } l(\lambda)}{\nu},$$

so that the inequality (1.9) holds for any point $l(\lambda)$ inside C_b . It should be remarked here that this method is not only applicable to any higher even order differential equation, it avoids the whole lot of complications involved in dealing with the singular surfaces that are to be associated with higher order equations, as in Everitt [1].

Section 2 states a useful lemma of Titchmarsh with the deductions necessary for our purpose. Section 3 derives the equation of the circle C_b , while Section 4 gives the special form of the equation of the circle C_b . Finally Section 5 indicates that the method given here can be extended to higher even order differential equations.

2. An important lemma

The following Lemma of Titchmarsh ([2], page 23) is vital for our purpose:

LEMMA. *If $F \equiv F(x, \lambda)$ is a solution of $L[y] = \lambda y$ and $G \equiv G(x, \lambda)$ is a solution of $L[y] = \lambda' y$, then*

$$(2.1) \quad (\lambda' - \lambda) \int_a^b FG \, dx = W_a(F, G) - W_b(F, G)$$

where

$$(2.2) \quad W_x(F, G) = F(x)G'(x) - F'(x)G(x),$$

and $0 \leq a \leq b < \infty$.

The following special cases of the lemma can be deduced easily: ($\lambda = \mu + i\nu$, $\bar{\phi} = \phi(x, \bar{\lambda})$, $\bar{\theta} = \theta(x, \bar{\lambda})$):

$$(2.3) \quad W_b(\phi, \bar{\phi}) = 2i\nu \int_0^b |\phi|^2 \, dx,$$

$$(2.4) \quad W_b(\phi, \bar{\theta}) = 1 + 2i\nu \int_0^b \phi \bar{\theta} \, dx,$$

$$(2.5) \quad W_b(\theta, \bar{\theta}) = 2i\nu \int_0^b |\theta|^2 \, dx.$$

3. The circle C_b corresponding to the parameter in the boundary condition

We write

$$(3.1) \quad l(\lambda) = u + iv,$$

$$(3.2) \quad \theta(b) = \theta(b, \lambda) = \xi(b, \lambda) + i\eta(b, \lambda) = \xi + i\eta,$$

$$(3.3) \quad \phi(b) = \phi(b, \lambda) = \chi(b, \lambda) + i\tau(b, \lambda) = \chi + i\tau,$$

where u, v are real numbers and $\xi(b, \lambda), \eta(b, \lambda), \chi(b, \lambda), \tau(b, \lambda)$ are real, even though λ is complex. Also we use the following notation:

$$(3.4) \quad \theta^{(1)}(b) = \xi^{(1)}(b, \lambda) + i\eta^{(1)}(b, \lambda) = \xi' + i\eta',$$

$$(3.5) \quad \phi^{(1)}(b) = \chi^{(1)}(b, \lambda) + i\tau^{(1)}(b, \lambda) = \chi' + i\tau'.$$

Then (1.7) gives, on equating real and imaginary parts,

$$(3.6) \quad (\xi + u\chi - v\tau) \cos \beta + (\xi' + u\chi' - v\tau') \sin \beta = 0,$$

$$(\eta + u\tau + v\chi) \cos \beta + (\eta' + u\tau' + v\chi') \sin \beta = 0$$

whence, eliminating β , we get the locus of the point $l = l(\lambda) = u + iv$ as

$$(3.7) \quad S_b(u, v) = (u^2 + v^2)(\chi\tau' - \chi'\tau) + u(\chi\eta' + \xi\tau' - \chi'\eta - \xi'\tau) + v(\chi'\xi - \eta'\tau + \eta\tau' - \chi\xi') + \xi\eta' - \xi'\eta = 0.$$

Next we note that

$$(3.8) \quad W_b(\phi, \bar{\phi}) = \phi(b)\bar{\phi}^{(1)}(b) - \phi^{(1)}(b)\bar{\phi}(b) = -2i(\chi\tau' - \chi'\tau)$$

so that

$$(3.9) \quad \chi\tau' - \chi'\tau = -\nu \int_0^b |\phi|^2 dx$$

Since ϕ is a non-trivial solution of (1.1), we have $\chi\tau' - \chi'\tau \neq 0$ unless $\nu = 0$. Thus, as β varies, the point $l = l(\lambda) = u + iv$ describes a circle, C_b say, on the (u, v) -plane and the equation of C_b is given by

$$(3.10) \quad S_b(u, v) = 0.$$

4. The equation of the circle C_b and its limit as $b \rightarrow \infty$

In this section we establish that

$$(4.1) \quad S_b(u, v) \equiv -\nu \int_0^b |\theta + l\phi|^2 dx - \text{im } l(\lambda) = 0.$$

PROOF. It is easy to see, using (2.3)–(2.5), that

$$\begin{aligned}
 \int_0^b |\theta + l\phi|^2 dx &= \int_0^b |\theta|^2 dx + l \int_0^b \phi \bar{\theta} dx + \bar{l} \int_0^b \bar{\phi} \theta dx + |l|^2 \int_0^b |\phi|^2 dx \\
 (4.2) \qquad &= (u^2 + v^2) \frac{W_b(\phi, \bar{\phi})}{2i\nu} + \frac{u}{2i\nu} \{W_b(\phi, \bar{\theta}) - W_b(\phi, \theta)\} \\
 &\quad + \frac{v}{2\nu} \{W_b(\phi, \bar{\theta}) + W_b(\bar{\phi}, \theta)\} - W_b(\bar{\phi}, \theta) - \frac{v}{\nu} + \frac{W_b(\theta, \bar{\theta})}{2i\nu}.
 \end{aligned}$$

Using the definitions of $W_x(F, G)$, we can immediately deduce that

$$(4.3) \qquad W_b(\phi, \bar{\theta}) - W_b(\bar{\phi}, \theta) = -2i(\chi\eta' - \chi'\eta + \xi\tau' - \xi'\tau),$$

$$(4.4) \qquad W_b(\phi, \bar{\theta}) + W_b(\bar{\phi}, \theta) = -2(\chi'\xi - \chi\xi' + \eta\tau' - \eta'\tau),$$

$$(4.5) \qquad W_b(\theta, \bar{\theta}) = -2i(\xi\eta' - \xi'\eta).$$

From (4.2), (3.8), (4.3)–(4.5), it is then easy to prove (4.1). Now, we apply elementary geometry only, to infer that a point $u + iv$ will lie inside the circle C_b , given by $S_b(u, v) = 0$, if $S_b(u, v) < 0$ when $\chi\tau' - \chi'\tau > 0$ (that is, $\nu < 0$) and if $S_b(u, v) > 0$ when $\chi\tau' - \chi'\tau < 0$ (that is, $\nu > 0$); that is, if $\int_0^b |\theta + l\phi|^2 + \text{im } l/\nu < 0$ for all $\nu \neq 0$, whether positive or negative.

5. Remarks

The method adopted in this paper does not involve the coefficients of the differential equation explicitly. Hence the differential equation can be replaced by

$$M[y] = -(py^{(1)})^{(1)} + qy = \lambda y,$$

where p is real-valued and continuous on $[0, \infty)$ and $p(x) \neq 0$ for and $x \in [0, \infty)$.

The method of this proof for the limit-classification can be used in the case of higher even order differential equations. The increase in the order of the differential equation will increase the dimension of the surface $S_b(u, v) = 0$ and hence matrix techniques must be used. These techniques will be developed in a subsequent paper, where it will be shown how this method nicely avoids all the complications involved in determining the monotonic nature of the singular surfaces.

References

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Department of Pure Mathematics
Calcutta University
35, Ballygunge Circular Road
Calcutta - 700 019
India