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ALMOST KENMOTSU METRIC AS A CONFORMAL RICCI SOLITON

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Abstract. In the present paper, we characterize $(k,\mu)'$ and generalized $(k,\mu)'$-almost Kenmotsu manifolds admitting the conformal Ricci soliton. It is also shown that a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} does not admit conformal gradient Ricci soliton (g,V,λ) with V collinear with the characteristic vector field ξ. Finally an illustrative example is presented.

1. Introduction

Hamilton [9] introduced the concept of Ricci flow in 1982 and proved its existence. The Ricci flow is an evolution equation for metrics on a Riemannian manifold given by

$$\frac{\partial g}{\partial t} = -2S,$$

where g is the Riemannian metric and S denotes the Ricci tensor.

A self-similar solution to the Ricci flow [9], [14] is called a Ricci soliton [10] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\pounds_V g + 2S = 2\lambda g,$$

where $\pounds_X$ is the Lie derivative, S is the Ricci tensor, g is the Riemannian metric, V is a vector field, and λ is a scalar. The Ricci soliton is denoted by (g,V,λ) and said to be shrinking, steady, and expanding according to whether λ is positive, zero, and negative, respectively.

In [8], Fischer developed the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M where M is considered as a smooth, closed, connected, oriented n-manifold is defined by the equation [8]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \quad \text{and} \quad r = -1,$$

where p is a non-dynamical scalar field which is time dependent, r is the scalar curvature of the manifold, and n is the dimension of the manifold.

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In 2015, Basu and Bhattacharyya [1] introduced the notion of the conformal Ricci soliton equation on Kenmotsu manifold M^{2n+1} as

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{1.1}$$

where λ is constant.

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation. It was later studied by Dutta et al. [7] in Lorentzian α-Sasakian manifolds and Nagaraja and Venu [12] in f-Kenmotsu manifolds.

A conformal Ricci soliton is said to be a conformal gradient Ricci soliton if the vector field V is a gradient of some smooth function on a manifold M. In this case, the conformal gradient Ricci soliton is given by

$$\nabla\nabla f + S = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{1.2}$$

where f is the gradient of the potential vector field V.

The paper is organized as follows.

After preliminaries, in Section 2, we consider a conformal Ricci soliton on $(k,\mu)'$ and generalized $(k,\mu)'$-almost Kenmotsu manifolds. Section 3 deals with a conformal gradient Ricci soliton on $(k,\mu)'$-almost Kenmotsu manifolds. Finally, in Section 4, an example is presented which verifies our theorem.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is said to have a (ϕ,ξ,η)-structure or an almost contact structure if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ, and a 1-form η satisfying ([2], [3]),

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1) easily.

If a manifold M with a (ϕ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on M, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any X, Y on M. The condition for an almost contact metric manifold being normal is equivalent to the vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi,\phi] + 2d\eta\otimes\xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ [2]. Recently in [4], [5], [6], [13], almost contact metric manifolds such that η is closed and $d\Phi = 2\eta\wedge\Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ for any vector fields X,Y. It is well known [11] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t, and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = \mathrm{Ker}(\eta) = \mathrm{Im}(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [13]:

$$h\xi = 0,\ l\xi = 0,\ \mathrm{tr}(h) = 0,\ \mathrm{tr}(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y \tag{2.5}$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([4], [16])

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.6}$$

In [4], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' = \{Z \in T_p(M) : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &\quad + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}. \end{aligned} \tag{2.7}$$

The above notion is called generalized nullity distributions when one allows k, μ to be smooth functions.

Let $X \in \mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue α. Then from (2.5) it is clear that $\alpha^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\alpha = \pm\sqrt{-k-1}$. We denote by $[\alpha]'$ and $[-\alpha]'$ the corresponding eigenspaces related to the non-zero eigenvalue α and $-\alpha$ of h', respectively. In [4], it is proved that in a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$, $k < -1$, $\mu = -2$, and $\mathrm{Spec}(h') = \{0, \alpha, -\alpha\}$ with 0 as a simple eigenvalue and $\alpha = \sqrt{-k-1}$. Also

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{2.8}$$

In [15], Wang and Liu proved that for a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$, the Ricci operator Q of M^{2n+1} is given by

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{2.9}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$. From (2.7), we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{2.10}$$

where $k, \mu \in \mathbb{R}$. Also we get from (2.10)

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{2.11}$$

Contracting X in (2.10), we have

$$S(Y,\xi) = 2nk\eta(Y). \tag{2.12}$$

Using (2.3), we have

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y). \tag{2.13}$$

For further details on almost Kenmotsu manifolds, we refer the reader to go through the references ([15]-[18]).

3. Conformal Ricci soliton

In this section, we study the conformal Ricci soliton on $(k,\mu)'$ and generalized $(k,\mu)'$-almost Kenmotsu manifolds. Before proving our main theorems, we first prove the following lemmas.

Lemma 3.1. *In a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$, the following relation holds:*

$$(\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = -4n(k+2)g(h'X,Y)\eta(Z).$$

Proof. From (2.9), we have

$$S(X,Y) = -2ng(X,Y) + 2n(k+1)\eta(X)\eta(Y) - 2ng(h'X,Y) \tag{3.1}$$

for any vector fields X, Y on M^{2n+1}.

Taking a covariant derivative of the foregoing equation along any vector field Z we have

$$\begin{aligned} \nabla_Z S(X,Y) &= -2n\nabla_Z g(X,Y) + 2n(k+1)(\nabla_Z \eta(X))\eta(Y) \\ &\quad +2n(k+1)\eta(X)(\nabla_Z \eta(Y)) - 2n\nabla_Z g(h'X,Y). \end{aligned} \tag{3.2}$$

Now, we have

$$(\nabla_Z S)(X,Y) = \nabla_Z S(X,Y) - S(\nabla_Z X,Y) - S(X,\nabla_Z Y).$$

Using (3.1) and (3.2) in the foregoing equation, we obtain

$$\begin{aligned} (\nabla_Z S)(X,Y) &= 2n(k+1)(\nabla_Z \eta)X)\eta(Y) + 2n(k+1)\eta(X)(\nabla_Z \eta)Y \\ &\quad -2ng((\nabla_Z h')X,Y). \end{aligned} \tag{3.3}$$

Now, using (2.8) and (2.13) in (3.3) we obtain

$$\begin{aligned} (\nabla_Z S)(X,Y) &= 2n(k+1)\eta(Y)(g(X,Z) - \eta(X)\eta(Z) + g(h'X,Z)) \\ &\quad +2n(k+1)\eta(X)(g(Y,Z) - \eta(Y)\eta(Z) + g(h'Y,Z)) \\ &\quad +2ng(h'Z + h'^2 Z, X)\eta(Y) + 2n\eta(X)g(h'Z + h'^2 Z, Y). \end{aligned} \tag{3.4}$$

Similarly, we obtain the following:

$$\begin{aligned} (\nabla_X S)(Y,Z) &= 2n(k+1)\eta(Z)(g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)) \\ &\quad +2n(k+1)\eta(Y)(g(X,Z) - \eta(X)\eta(Z) + g(h'X,Z)) \\ &\quad +2ng(h'X + h'^2 X, Y)\eta(Z) + 2n\eta(Y)g(h'X + h'^2 X, Z) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} (\nabla_Y S)(X,Z) &= 2n(k+1)\eta(Z)(g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)) \\ &\quad +2n(k+1)\eta(X)(g(Y,Z) - \eta(Y)\eta(Z) + g(h'Y,Z)) \\ &\quad +2ng(h'Y + h'^2 Y, X)\eta(Z) + 2n\eta(X)g(h'Y + h'^2 Y, Z). \end{aligned} \tag{3.6}$$

Using (3.4)-(3.6), we infer that

$$\begin{aligned} &(\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &= -4n(k+1)\eta(Z)(g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)) \\ &\quad -4ng(h'X + h'^2 X, Y)\eta(Z), \end{aligned} \tag{3.7}$$

where the symmetry of h' is used. Now, using (2.6) and then (2.1) in (3.7) we complete the proof. □

Lemma 3.2. *In a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1}, $(\pounds_X h')Y = 0$ for any $X,\ Y \in [\alpha]'$ or $X,\ Y \in [-\alpha]'$, where* $\mathrm{Spec}(h') = \{0, \alpha, -\alpha\}$.

Proof. We consider a local orthonormal basis $\{\xi, e_i, \phi e_i\}$, $i = 1, 2, \dots, n$ with $e_i \in [\alpha]'$ for M^{2n+1} and for any $X,\ Y \in [\alpha]'$, we have

$$\begin{aligned}\nabla_X Y &= \sum_i g(\nabla_X Y, e_i)e_i + g(\nabla_X Y, \xi)\xi \\ &= \sum_i g(\nabla_X Y, e_i)e_i - (1+\alpha)g(X,Y)\xi.\end{aligned}$$

For details of the above equation, see Proposition 4.1 of [4]. Now,

$$\begin{aligned}(\pounds_X h')Y &= \pounds_X h'Y - h'(\pounds_X Y) \\ &= \alpha \pounds_X Y - h'(\pounds_X Y) \\ &= \alpha(\nabla_X Y - \nabla_Y X) - h'(\nabla_X Y - \nabla_Y X) \\ &= \alpha(\nabla_X Y - \sum_i g(\nabla_X Y, e_i)e_i) - \alpha(\nabla_Y X - \sum_i g(\nabla_Y X, e_i)e_i) \\ &= -\alpha(1+\alpha)g(X,Y)\xi + \alpha(1+\alpha)g(X,Y)\xi \\ &= 0.\end{aligned}$$

Similarly, one can prove the same when $X,\ Y \in [-\alpha]'$. Hence, the proof is complete. □

Theorem 3.3. *A $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$ admitting conformal Ricci soliton (g, V, λ) is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ or the conformal Ricci soliton* (i) *expanding,* (ii) *steady, or* (iii) *shrinking according to whether the non-dynamical scalar field p is*

(i) $p < -4nk - \frac{2}{2n+1}$,
(ii) $p = -4nk - \frac{2}{2n+1}$,
(iii) $p > \frac{8n^2+4n-2}{2n+1}$.

Proof. From (1.1) we have

$$(\pounds_V g)(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X,Y). \tag{3.8}$$

Differentiating the above equation covariantly along any vector field Z we get

$$(\nabla_Z \pounds_V g)(X,Y) = -2(\nabla_Z S)(X,Y). \tag{3.9}$$

It is well known that ([19, p. 23])

$$(\pounds_V \nabla_X g - \pounds_X \nabla_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y).$$

Since g is parallel with respect to the Levi-Civita connection ∇, then the above relation becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{3.10}$$

Since $\pounds_V \nabla$ is symmetric, then it follows from (3.10) that

$$\begin{aligned}g((\pounds_V \nabla)(X,Y),Z) &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) \\ &\quad -\frac{1}{2}(\nabla_Z \pounds_V g)(X,Y).\end{aligned} \tag{3.11}$$

Using (3.9) in (3.11) we have

$$g((\pounds_V\nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{3.12}$$

Now using Lemma 3.1 in (3.12) we have

$$g((\pounds_V\nabla)(X,Y),Z) = -4n(k+2)g(h'X,Y)\eta(Z),$$

which implies

$$(\pounds_V\nabla)(X,Y) = -4n(k+2)g(h'X,Y)\xi. \tag{3.13}$$

Substituting $Y = \xi$ in (3.13) we get $(\pounds_V\nabla)(X,\xi) = 0$. From this we obtain $\nabla_Y(\pounds_V\nabla)(X,\xi) = 0$. This gives

$$(\nabla_Y\pounds_V\nabla)(X,\xi) + (\pounds_V\nabla)(\nabla_Y X,\xi) + (\pounds_V\nabla)(X,\nabla_Y\xi) = 0. \tag{3.14}$$

Using $(\pounds_V\nabla)(X,\xi) = 0$, (3.12), and (2.3) in (3.14) we infer that

$$(\nabla_Y\pounds_V\nabla)(X,\xi) = 4n(k+2)(g(h'X,Y) + g(h'^2X,Y))\xi. \tag{3.15}$$

Using the foregoing equation in the following formula ([19, p. 23])

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z),$$

we obtain

$$(\pounds_V R)(X,\xi)\xi = (\nabla_X\pounds_V\nabla)(\xi,\xi) - (\nabla_\xi\pounds_V\nabla)(X,\xi) = 0. \tag{3.16}$$

Now, substituting $Y = \xi$ in (3.8) and using (2.12) we have

$$(\pounds_V g)(X,\xi) = [2\lambda - (p + \frac{2}{2n+1}) - 4nk]\eta(X). \tag{3.17}$$

Lie-differentiating $g(X,\xi) = \eta(X)$ along V and using (3.17) we obtain

$$(\pounds_V\eta)X - g(X,\pounds_V\xi) - [2\lambda - (p + \frac{2}{2n+1}) - 4nk]\eta(X) = 0. \tag{3.18}$$

From (3.18), after putting $X = \xi$ we can easily obtain that

$$\eta(\pounds_V\xi) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - 2nk]. \tag{3.19}$$

From (2.10), we have

$$R(X,\xi)\xi = k(X - \eta(X)\xi) - 2h'X. \tag{3.20}$$

Now, using (3.18)-(3.20) and (2.10)-(2.11) we obtain

$$\begin{aligned}(\pounds_V R)(X,\xi)\xi &= \pounds_V R(X,\xi)\xi - R(\pounds_V X,\xi)\xi - R(X,\pounds_V\xi)\xi - R(X,\xi)\pounds_V\xi \\ &= k[2\lambda - (p + \frac{2}{2n+1}) - 4nk](X - \eta(X)\xi) - 2(\pounds_V h')X \\ &\quad -2[2\lambda - (p + \frac{2}{2n+1}) - 4nk]h'X - 2n\eta(X)h'(\pounds_V\xi) \\ &\quad -2g(h'X,\pounds_V\xi)\xi. \end{aligned} \tag{3.21}$$

Equating (3.16) and (3.21) and then taking an inner product with Y yields

$$\begin{aligned} & k[2\lambda - (p + \frac{2}{2n+1}) - 4nk](g(X,Y) - \eta(X)\eta(Y)) \\ & -2g((\pounds_V h')X,Y) - 2[2\lambda - (p + \frac{2}{2n+1}) - 4nk]g(h'X,Y) \\ & -2n\eta(X)g(h'(\pounds_V\xi),Y) - 2g(h'X,\pounds_V\xi)\eta(Y) = 0. \end{aligned}$$

Replacing X by ϕX in the above equation, we infer that

$$k[2\lambda-(p+\frac{2}{2n+1})-4nk]g(\phi X,Y)-2g((\pounds_V h')\phi X,Y)$$
$$-2[2\lambda-(p+\frac{2}{2n+1})-4nk]g(h'\phi X,Y)=0. \tag{3.22}$$

Letting $X\in[-\alpha]'$ and $V\in[\alpha]'$, then $\phi X\in[\alpha]'$. Then from (3.22), we have

$$(k-2\alpha)[2\lambda-(p+\frac{2}{2n+1})-4nk]g(\phi X,Y)-2g((\pounds_V h')\phi X,Y)=0. \tag{3.23}$$

Since, $V,\ \phi X\in[\alpha]'$, using Lemma 3.2 we have $(\pounds_V h')\phi X=0$. Therefore, equation (3.23) reduces to

$$(k-2\alpha)[2\lambda-(p+\frac{2}{2n+1})-4nk]g(\phi X,Y)=0,$$

which implies either $k=2\alpha$ or $2\lambda=(p+\frac{2}{2n+1})+4nk$.

Case 1. If $k=2\alpha$, then from $\alpha^2=-(k+1)$ we get $\alpha=-1$ and hence $k=-2$. Then from Proposition 4.2 of [4], we have

$$R(X_\alpha,Y_\alpha)Z_\alpha=0$$

and

$$R(X_{-\alpha},Y_{-\alpha})Z_{-\alpha}=-4[g(Y_{-\alpha},Z_{-\alpha})X_{-\alpha}-g(X_{-\alpha},Z_{-\alpha})Y_{-\alpha}]$$

for any $X_\alpha,Y_\alpha,Z_\alpha\in[\alpha]'$ and $X_{-\alpha},Y_{-\alpha},Z_{-\alpha}\in[-\alpha]'$. Also noticing $\mu=-2$ it follows from Proposition 4.3 of [4] that $K(X,\xi)=-4$ for any $X\in[-\alpha]'$ and $K(X,\xi)=0$ for any $X\in[\alpha]'$. Again from Proposition 4.3 of [4] we see that $K(X,Y)=-4$ for any $X,Y\in[-\alpha]'$ and $K(X,Y)=0$ for any $X,Y\in[\alpha]'$. As is shown in [4] the distribution $[\xi]\oplus[\alpha]'$ is integrable with totally geodesic leaves and the distribution $[-\alpha]'$ is integrable with totally umbilical leaves by $H=-(1-\alpha)\xi$, where H is the mean curvature tensor field for the leaves of $[-\alpha]'$ immersed in M^{2n+1}. Here $\alpha=-1$; then the two orthogonal distributions $[\xi]\oplus[\alpha]'$ and $[-\alpha]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.

Case 2. Let $2\lambda=(p+\frac{2}{2n+1})+4nk$. Now, the conformal Ricci soliton is expanding, steady, or shrinking according to whether $\lambda<0$, $\lambda=0$, or $\lambda>0$, respectively. Therefore, the conformal Ricci soliton is expanding when $p<-4nk-\frac{2}{2n+1}$, steady when $p=-4nk-\frac{2}{2n+1}$,and shrinking when $p>\frac{8n^2+4n-2}{2n+1}$, where the fact $k\leq-1$ is used in the case of shrinking. This completes the proof. □

Theorem 3.4. *If (g,ξ,λ) is a conformal Ricci soliton in a generalized $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1}, then M^{2n+1} is η-Einstein and $\lambda=\frac{p}{2}+\frac{1}{2n+1}+2nk$.*

Proof. Since (g,ξ,λ) is a conformal Ricci soliton in M^{2n+1}, we have from (1.1)

$$(\pounds_\xi g)(X,Y)+2S(X,Y)=[2\lambda-(p+\frac{2}{2n+1})]g(X,Y). \tag{3.24}$$

Now, using (2.3) we obtain

$$\begin{aligned}(\pounds_\xi g)(X,Y)&=g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)\\&=2g(X,Y)-2\eta(X)\eta(Y)-2g(\phi hX,Y).\end{aligned} \tag{3.25}$$

Substituting (3.25) in (3.24) we get

$$\begin{aligned} & 2g(X,Y) - 2\eta(X)\eta(Y) - 2g(\phi hX,Y) + 2S(X,Y) \\ = & [2\lambda - (p + \frac{2}{2n+1})]g(X,Y). \end{aligned} \tag{3.26}$$

From (2.9), we get

$$g(\phi hX,Y) = \frac{1}{2n}S(X,Y) + g(X,Y) - (k+1)\eta(X)\eta(Y). \tag{3.27}$$

Now, substituting (3.27) in (3.26) we get

$$S(X,Y) = \frac{n[2\lambda - (p + \frac{2}{2n+1})]}{2n-1}g(X,Y) - \frac{2nk}{2n-1}\eta(X)\eta(Y),$$

which shows that the manifold is η-Einstein.

Putting $X = Y = \xi$ in the foregoing equation, we obtain

$$2nk = \frac{n[2\lambda - (p + \frac{2}{2n+1})]}{2n-1} - \frac{2nk}{2n-1}.$$

From above, it follows that $\lambda = \frac{p}{2} + \frac{1}{2n+1} + 2nk$. □

4. Conformal gradient Ricci soliton

In this section we consider a conformal gradient Ricci soliton in the framework of $(k,\mu)'$-almost Kenmotsu manifolds. If V is any vector field collinear with ξ, then there is a smooth function b on M such that $V = b\xi$. In this case, $h'V = 0$. Here we prove the following theorem.

Theorem 4.1. *A $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} does not admit conformal gradient Ricci soliton (g,V,λ) with V collinear with the characteristic vector field ξ.*

The proof of the above theorem relies on the following lemma.

Lemma 4.2. *In a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} admitting conformal gradient Ricci soliton (g,V,λ), the following relation holds:*

$$R(X,Y)Df = 2n(k+2)(\eta(X)h'Y - \eta(Y)h'X), \tag{4.1}$$

where $f : M^{2n+1} \to \mathbb{R}$ is a smooth function such that $V = Df$, D is the gradient operator.

Proof. From (1.2) we can write

$$\nabla_X Df = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - QX. \tag{4.2}$$

Taking the covariant derivative of the above equation along Y we get

$$\nabla_Y\nabla_X Df = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_Y X - \nabla_Y QX. \tag{4.3}$$

Interchanging X and Y in (4.3) we get

$$\nabla_X\nabla_Y Df = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_X Y - \nabla_X QY. \tag{4.4}$$

Again, from (4.2) we obtain

$$\nabla_{[X,Y]} Df = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})](\nabla_X Y - \nabla_Y X) - Q(\nabla_X Y - \nabla_Y X). \tag{4.5}$$

Using (4.3)-(4.5) in the following:

$$R(X,Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df$$

we obtain

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{4.6}$$

Now, using (2.8), (2.9), and (2.13) we obtain

$$\begin{aligned}(\nabla_Y Q)X &= \nabla_Y QX - Q(\nabla_Y X)\\ &= 2n(k+1)(g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y))\xi\\ &\quad +2n(k+1)\eta(X)(Y - \eta(Y)\xi - \phi hY) + 2ng(h'Y + h'^2Y, X)\xi\\ &\quad +2n\eta(X)(h'Y + h'^2Y).\end{aligned} \tag{4.7}$$

Interchanging X and Y in the above equation we obtain $(\nabla_X Q)Y$. Now, substituting $(\nabla_Y Q)X$ and $(\nabla_X Q)Y$ in (4.6) and using (2.6) we complete the proof. □

Proof of Theorem 4.1. Putting $X = \xi$ in (4.1) we have

$$R(\xi,Y)Df = 2n(k+2)h'Y,$$

which implies

$$g(R(\xi,Y)Df, X) = 2n(k+2)g(h'Y,X). \tag{4.8}$$

Again, using (2.11) we have

$$\begin{aligned}g(R(\xi,Y)Df,X) &= -g(R(\xi,Y)X, Df)\\ &= -kg(X,Y)(\xi f) + k\eta(X)(Yf)\\ &\quad +2g(h'X,Y)(\xi f) - 2\eta(X)((h'Y)f).\end{aligned} \tag{4.9}$$

From (4.8) and (4.9) we get

$$\begin{aligned}&-kg(X,Y)(\xi f) + k\eta(X)(Yf) + 2g(h'X,Y)(\xi f) - 2\eta(X)((h'Y)f)\\ =\ & 2n(k+2)g(h'Y,X).\end{aligned}$$

Antisymmetrizing the foregoing equation we obtain

$$k\eta(X)(Yf) - k\eta(Y)(Xf) - 2\eta(X)((h'Y)f) + 2\eta(Y)((h'X)f) = 0. \tag{4.10}$$

Now, $(h'X)f = g(h'X, Df) = g(X, h'(Df)) = 0$ for any vector field X as $h'V = h'(Df) = 0$ by hypothesis. Hence, from (4.10) we get

$$\eta(X)(Yf) - \eta(Y)(Xf) = 0,$$

as $k \leq -1$. Putting $X = \xi$ in the above equation we obtain

$$Df = (\xi f)\xi. \tag{4.11}$$

Differentiating (4.11) covariantly along X, we obtain

$$\nabla_X Df = (X(\xi f))\xi + (\xi f)(X - \eta(X)\xi - \phi hX). \tag{4.12}$$

Equating (4.2) and (4.12) we obtain

$$QX = ([\lambda - (\frac{p}{2} + \frac{1}{2n+1})] + (\xi f))X + ((\xi f)\eta(X) - X(\xi f))\xi + (\xi f)\phi hX. \tag{4.13}$$

Comparing (2.9) and (4.13) we have the following:

$$[\lambda - (\frac{p}{2} + \frac{1}{2n+1})] + (\xi f) = -2n, \tag{4.14}$$

$$(\xi f)\eta(X) - X(\xi f) = 2n(k+1)\eta(X), \tag{4.15}$$

$$(\xi f) = 2. \tag{4.16}$$

Using (4.16) in (4.14) we get $2\lambda - (p + \frac{2}{2n+1}) = -4n - 4$ which implies $\lambda = \frac{p}{2} + \frac{1}{2n+1} - 2n - 2$. Again using (4.16) in (4.15) we get $2\eta(X) = 2n(k+1)\eta(X)$ for any vector field X which implies $k = -1 + \frac{1}{n} > -1$ which is a contradiction as $k \leq -1$. Hence, a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} does not admit conformal gradient Ricci soliton (g, V, λ) such that the potential vector field V is collinear with the characteristic vector field ξ.

5. Example of a 5-dimensional almost Kenmotsu manifold

We consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. Let ξ, e_2, e_3, e_4, e_5 be five vector fields in $\mathbb{R}^5$ which satisfies [4]

$$[\xi, e_2] = -2e_2,\ [\xi, e_3] = -2e_3,\ [\xi, e_4] = 0,\ [\xi, e_5] = 0,$$

$[e_i, e_j] = 0$, where $i, j = 2, 3, 4, 5$.

Let g be the Riemannian metric defined by

$$g(\xi, \xi) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1$$

and $g(\xi, e_i) = g(e_i, e_j) = 0$ for $i \neq j$; $i, j = 2, 3, 4, 5$.

Let η be the 1-form defined by $\eta(Z) = g(Z, \xi)$, for any $Z \in T(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi) = 0,\ \phi(e_2) = e_4,\ \phi(e_3) = e_5,\ \phi(e_4) = -e_2,\ \phi(e_5) = -e_3.$$

Using the linearity of ϕ and g, we have

$$\eta(\xi) = 1,\ \phi^2(Z) = -Z + \eta(Z)\xi,\ g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$$

for any $Z, U \in T(M)$.

Moreover, $h'\xi = 0,\ h'e_2 = e_2,\ h'e_3 = e_3,\ h'e_4 = -e_4,\ h'e_5 = -e_5$.

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\nabla_\xi \xi = 0,\ \nabla_\xi e_2 = 0,\ \nabla_\xi e_3 = 0,\ \nabla_\xi e_4 = 0,\ \nabla_\xi e_5 = \xi,$$

$$\nabla_{e_2} \xi = 2e_2,\ \nabla_{e_2} e_2 = -2\xi,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,$$

$$\nabla_{e_3} \xi = 2e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -2\xi,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0,$$

$$\nabla_{e_4} \xi = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = 0,\ \nabla_{e_4} e_5 = 0,$$

$$\nabla_{e_5} \xi = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = 0.$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h'X$$

for any $X \in T(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$R(\xi, e_2)\xi = 4e_2,\ R(\xi, e_2)e_2 = -4\xi,\ R(\xi, e_3)\xi = 4e_3,\ R(\xi, e_3)e_3 = -4\xi,$$

$$R(\xi, e_4)\xi = R(\xi, e_4)e_4 = R(\xi, e_5)\xi = R(\xi, e_5)e_5 = 0,$$

$$R(e_2, e_3)e_2 = 4e_3,\ R(e_2, e_3)e_3 = -4e_2,\ R(e_2, e_4)e_2 = R(e_2, e_4)e_4 = 0,$$

$$R(e_2, e_5)e_2 = R(e_2, e_5)e_5 = R(e_3, e_4)e_3 = R(e_3, e_4)e_4 = 0,$$

$$R(e_3, e_5)e_3 = R(e_3, e_5)e_5 = R(e_4, e_5)e_4 = R(e_4, e_5)e_5 = 0.$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution with $k = -2$ and $\mu = -2$. Therefore, from $\alpha^2 = -(k+1)$, we get $\alpha = \pm 1$. Without lose of generality we consider $\alpha = -1$. Then by the same argument as in Theorem 3.3 we can say that the manifold is locally isometric to $\mathbb{H}^3(-4) \times \mathbb{R}^2$.

Using the expressions of the curvature tensor R we have

$$R(X, Y)Z = -4[g(Y, Z)X - g(X, Z)Y].$$

From the above equation we obtain

$$S(Y, Z) = -16g(Y, Z), \text{ which implies } r = -80.$$

Now, it is easy to see that

$$(\pounds_\xi g)(\xi, \xi) = (\pounds_\xi g)(e_4, e_4) = (\pounds_\xi g)(e_5, e_5) = 0,$$

$$(\pounds_\xi g)(e_2, e_2) = (\pounds_\xi g)(e_3, e_3) = 4.$$

Consider $V = \xi$ and then tracing (1.1) we obtain $\lambda = \frac{p}{2} + \frac{1}{5} + \frac{76}{5}$. Hence, (g, ξ, λ) is a conformal Ricci soliton on M. Thus Theorem 3.3 is verified.

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