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Citation: J. Math. Phys. 57, 062501 (2016); doi: 10.1063/1.4952699
View online: http://dx.doi.org/10.1063/1.4952699
View Table of Contents: http://aip.scitation.org/toc/jmp/57/6
Published by the American Institute of Physics

JOURNAL OF MATHEMATICAL PHYSICS **57**, 062501 (2016)

A spacetime with pseudo-projective curvature tensor

Sahanous Mallick,[1,a)] Young Jin Suh,[2,b)] and Uday Chand De[3,c)]
[1]*Department of Mathematics, Chakdaha College, P.O.-Chakdaha, Dist-Nadia, West Bengal, India*
[2]*Department of Mathematics, Kyungpook National University, Taegu 702-701, South Korea*
[3]*Department of Pure Mathematics, University of Calcutta, 35, B. C. Road, Kolkata 700019, West Bengal, India*

(Received 29 January 2016; accepted 15 May 2016; published online 3 June 2016)

The object of the present paper is to study spacetimes admitting pseudo-projective curvature tensor. At first we prove that a pseudo-projectively flat spacetime is Einstein and hence it is of constant curvature and the energy momentum tensor of such a spacetime satisfying Einstein's field equation with cosmological constant is covariant constant. Next, we prove that if the perfect fluid spacetime with vanishing pseudo-projective curvature tensor obeys Einstein's field equation without cosmological constant, then the spacetime has constant energy density and isotropic pressure, and the perfect fluid always behaves as a cosmological constant and also such a spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field U. Moreover, it is shown that a pseudo-projectively flat spacetime satisfying Einstein's equation without cosmological constant for a purely electromagnetic distribution is an Euclidean space. We also prove that under certain conditions a perfect fluid spacetime with divergence-free pseudo-projective curvature is a Robertson-Walker spacetime and the possible local cosmological structure of such a spacetime is of type I, D or O. We also study dust-like fluid spacetime with vanishing pseudo-projective curvature tensor. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4952699]

I. INTRODUCTION

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study, spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentzian manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity. The Einstein's equations[31] (p. 337), imply that the energy-momentum tensor is of vanishing divergence. This requirement is satisfied if the energy-momentum tensor is covariant-constant.[5] In Ref. 5, M. C. Chaki and Sarbari Ray showed that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime. Several authors studied spacetimes in several ways such as spacetimes with semisymmetric energy momentum tensor by De and Velimirović,[7] m-Projectively flat spacetimes by Zengin,[39] pseudo Z symmetric spacetimes by Mantica and Suh (Refs. 19 and 22), generalized quasi-Einstein spacetimes by Güler and Demirbağ,[12] generalized Robertson-Walker spacetimes by Arslan *et al.*,[3] and many others.

The notion of pseudo-projective curvature tensor was introduced by Prasad[32] and is defined as follows:

[a)]E-mail address: sahanousmallick@gmail.com
[b)]E-mail address: yjsuh@knu.ac.kr
[c)]E-mail address: uc_de@yahoo.com

$$P^\star(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}(\frac{a}{n-1} + b)[g(Y,Z)X - g(X,Z)Y], \quad (1.1)$$

where a and b are constants with $b \neq 0$, R is the Riemannian curvature tensor of type (1,3), S is the Ricci tensor of type (0,2), and r is the scalar curvature of the manifold.

If $a = 1$ and $b = -\frac{1}{n-1}$, then (1.1) reduces to the projective curvature tensor. A semi-Riemannian manifold is called pseudo-projectively flat if $P^\star = 0$ for $n > 3$. The pseudo-projective curvature tensor has been studied by various authors in various ways such as Narain, Prakash, and Prasad,[29] Nagaraja and Somashekhara,[28] Doğru,[8] Jaiswal and Ojha,[15] and many others.

The present paper is organized as follows.

After introduction, in Section II, we characterize a spacetime with vanishing pseudo-projective curvature tensor and some geometric properties of such a spacetime have been obtained. Section III deals with the perfect fluid spacetime with vanishing pseudo-projective curvature tensor. Perfect fluid spacetime with divergence-free pseudo-projective curvature tensor has been studied in Section IV. Finally, we study dust fluid spacetime with vanishing pseudo-projective curvature tensor.

II. SPACETIME WITH VANISHING PSEUDO-PROJECTIVE CURVATURE TENSOR

Let V_4 be the spacetime of general relativity, then from Equation (1.1) we have

$$\tilde{P}^\star(X,Y,Z,W) = a\tilde{R}(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) - \frac{r}{4}[\frac{a}{3} + b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \quad (2.1)$$

where $\tilde{P}^\star(X,Y,Z,W) = g(P^\star(X,Y)Z,W)$ and $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.

If $\tilde{P}^\star(X,Y,Z,W) = 0$, then Equation (2.1) leads to

$$aR(X,Y,Z,W) + b[S(Y,Z)g(X,Z) - S(X,Z)g(Y,W) - \frac{r}{4}[\frac{a}{3} + b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0. \quad (2.2)$$

Taking a frame field over X and W, we have from (2.2) that

$$(a+3b)S(Y,Z) = (a+3b)\frac{r}{4}g(Y,Z), \quad (2.3)$$

where S and r denote the Ricci tensor and the scalar curvature of the manifold, respectively.

Thus we can state the following.

Theorem 2.1. *A pseudo-projectively flat spacetime is an Einstein spacetime, provided* $a + 3b \neq 0$.

Again, Equations (2.2) and (2.3) give

$$\tilde{R}(X,Y,Z,W) = \frac{r}{12}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (2.4)$$

Thus we can state the following.

Theorem 2.2. *A pseudo-projectively flat spacetime is a spacetime of constant curvature, provided* $a + 3b \neq 0$.

Remark. The spaces of constant curvature play a significant role in cosmology. The simplest cosmological model is obtained by making the assumption that the universe is isotropic and homogeneous. This is known as cosmological principle. This principle, when translated into the language of differential geometry, asserts that the three dimensional position space is a space of maximal symmetry,[37] that is, a space of constant curvature whose curvature depends upon time. The cosmological solution of Einstein equations which contains a three dimensional spacelike surface of a constant curvature is the Robertson-Walker metrics, while four dimensional space of constant curvature is the de Sitter model of the universe (Refs. 37 and 30).

Let us consider a spacetime satisfying the Einstein's field equation with cosmological constant

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = \kappa T(X,Y), \tag{2.5}$$

where S and r denote the Ricci tensor and scalar curvature, respectively. λ is the cosmological constant, κ is the gravitational constant, and $T(X,Y)$ is the energy momentum tensor.

Using (2.3) and (2.5) we obtain

$$T(X,Y) = \frac{1}{\kappa}[\lambda - \frac{r}{4}]g(X,Y). \tag{2.6}$$

Taking covariant derivative of (2.6) we get

$$(\nabla_Z T)(X,Y) = -\frac{1}{4\kappa}dr(Z)g(X,Y). \tag{2.7}$$

Since pseudo-projectively flat spacetime is Einstein, therefore the scalar curvature r is constant. Hence

$$dr(X) = 0, \tag{2.8}$$

for all X.

Equations (2.7) and (2.8) together yield

$$(\nabla_Z T)(X,Y) = 0.$$

Thus we can state the following.

Theorem 2.3. *In a pseudo-projectively flat spacetime satisfying Einstein's field equation with cosmological constant, the energy momentum tensor is covariant constant.*

Katzin *et al.*[17] were the pioneers in carrying out a detailed study of curvature collineation (CC), in the context of the related particle and field conservation laws that may be admitted in the standard form of general relativity.

The geometrical symmetries of a spacetime are expressed through equation

$$\pounds_\xi A - 2\Omega A = 0, \tag{2.9}$$

where A represents a geometrical/physical quantity, $\pounds_\xi$ denotes the Lie derivative with respect to the vector field ξ, and Ω is a scalar.[17]

One of the most simple and widely used example is the metric inheritance symmetry for $A = g$ in (2.9) and for this case, ξ is the Killing vector field if $\Omega = 0$.

Therefore,

$$(\pounds_\xi g)(X,Y) = 2\Omega g(X,Y). \tag{2.10}$$

A spacetime M is said to admit a symmetry called a curvature collineation (CC) (Refs. 9 and 10) provided there exists a vector field ξ such that

$$(\pounds_\xi R)(X,Y)Z = 0, \tag{2.11}$$

where R is the Riemannian curvature tensor.

Now we shall investigate the role of such symmetry inheritance for the spacetime admitting pseudo-projective curvature tensor.

Let us consider a spacetime admitting pseudo-projective curvature tensor with a Killing vector field ξ is a CC. Then we have

$$(\pounds_\xi g)(X,Y) = 0. \tag{2.12}$$

Again, since M admits a CC we have from (2.11)

$$(\pounds_\xi S)(X,Y) = 0, \tag{2.13}$$

where S is the Ricci tensor of the manifold.

Taking Lie derivative of (1.1) and then using (2.11), (2.12), and (2.13) we obtain

$$(\pounds_\xi P^\star)(X,Y)Z = 0.$$

Thus we can state the following.

Theorem 2.4. *If a spacetime M admitting the pseudo-projective curvature tensor with ξ as a Killing vector field is CC, then the Lie derivative of the pseudo-projective curvature tensor vanishes along the vector field ξ.*

The well-known symmetry of the energy momentum tensor T is the matter collineation defined by

$$(\pounds_\xi T)(X,Y) = 0,$$

where ξ is the vector field generating the symmetry and $\pounds_\xi$ is the Lie derivative operator along the vector field ξ.

Let ξ be a Killing vector field on the spacetime with vanishing pseudo-projective curvature tensor. Then

$$(\pounds_\xi g)(X,Y) = 0, \tag{2.14}$$

where $\pounds_\xi$ denotes Lie derivative with respect to ξ.

Taking Lie derivatives of both sides of (2.6) with respect to ξ we obtain

$$\frac{1}{\kappa}(\lambda - \frac{r}{4})(\pounds_\xi g)(X,Y) = (\pounds_\xi T)(X,Y). \tag{2.15}$$

In virtue of (2.14), it follows from (2.15) that

$$(\pounds_\xi T)(X,Y) = 0,$$

which implies that the spacetime admits matter collineation.

Conversely, if $(\pounds_\xi T)(X,Y) = 0$, it follows from (2.15) that

$$(\pounds_\xi g)(X,Y) = 0.$$

Hence we can state the following theorem.

Theorem 2.5. *If a spacetime obeying Einstein's field equation has vanishing pseudo-projective curvature tensor, then the spacetime admits matter collineation with respect to a vector field ξ if and only if ξ is a Killing vector field.*

Next, let us suppose that ξ is a conformal Killing vector field. Then we have

$$(\pounds_\xi g)(X,Y) = 2\phi g(X,Y), \tag{2.16}$$

where ϕ is a scalar.

Then from (2.15) we get

$$(\lambda - \frac{r}{4})2\phi g(X,Y) = \kappa(\pounds_\xi T)(X,Y). \tag{2.17}$$

Using (2.6) in (2.17) we obtain

$$(\pounds_\xi T)(X,Y) = 2\phi T(X,Y). \tag{2.18}$$

From (2.18) we can say that the energy-momentum tensor has Lie inheritance property along ξ.

Conversely, if (2.18) holds, then it follows that (2.16) holds, that is, ξ is a conformal Killing vector field. Thus we state the following.

Theorem 2.6. *If a spacetime obeying Einstein's field equation has vanishing pseudo-projective curvature tensor, then a vector field ξ on the spacetime is a conformal Killing vector field if and only if the energy-momentum tensor has the Lie inheritance property along ξ.*

III. PERFECT FLUID SPACETIME WITH VANISHING PSEUDO-PROJECTIVE CURVATURE TENSOR

In this section we consider a perfect fluid spacetime with vanishing pseudo-projective curvature tensor obeying Einstein's field equation without cosmological constant.

The energy momentum tensor T of a perfect fluid is given by[31]

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y), \tag{3.1}$$

where σ is the energy density, p the isotropic pressure, and A is a non-zero 1-form such that $g(X,U) = A(X)$, for all X, U being the velocity vector field of the flow, that is, $g(U,U) = -1$.

Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{3.2}$$

where r is the scalar curvature of the manifold and $\kappa \neq 0$.

In this case Einstein's equation can be written as

$$-(\frac{r}{4} + kp)g(X,Y) = \kappa(\sigma + p)A(X)A(Y). \tag{3.3}$$

Taking a frame field and after contraction over X, Y we obtain

$$r = \kappa(\sigma - 3p). \tag{3.4}$$

In virtue of (2.3) and (3.4) the Ricci tensor of a pseudo-projectively flat spacetime can be written as

$$S(X,Y) = \frac{\kappa(\sigma - 3p)}{4}g(X,Y). \tag{3.5}$$

Let Q be the Ricci operator given by

$$g(QX,Y) = S(X,Y)$$

and

$$S(QX,Y) = S^2(X,Y).$$

Then we obtain that

$$A(QX) = g(QX,U) = S(X,U).$$

Hence we get from (3.5) that

$$S(QX,Y) = \frac{\kappa^2(\sigma - 3p)^2}{16}g(X,Y). \tag{3.6}$$

Taking a frame field and after contraction over X, Y, we obtain from (3.6) that

$$\|Q\|^2 = \frac{\kappa^2(\sigma - 3p)^2}{4}. \tag{3.7}$$

Hence we obtain the following result.

Theorem 3.1. *If a pseudo-projectively flat perfect fluid spacetime obeys Einstein's field equation without cosmological constant, then the square of the length of the Ricci operator of the spacetime is* $\frac{\kappa^2(\sigma-3p)^2}{4}$.

Now putting $X = Y = U$ in (3.3) we obtain

$$r = 4\kappa\sigma. \tag{3.8}$$

Equations (3.4) and (3.8) together give $\sigma + p = 0$. Therefore Equation (3.1) in this case takes the form

$$T(X,Y) = pg(X,Y). \tag{3.9}$$

Since the scalar curvature r of a pseudo-projectively flat spacetime is constant, therefore from (3.8) it follows that $\sigma = constant$ and hence from $\sigma + p = 0$ we obtain $p = constant$. Now $\sigma + p = 0$

means the fluid behaves as a cosmological constant.[38] This is also termed as phantom barrier.[6] Now in a cosmology we know such a choice $\sigma = -p$ leads to rapid expansion of the spacetime which is now termed as inflation.[2]

Thus we can state the following.

Theorem 3.2. *If a perfect fluid spacetime with vanishing pseudo-projective curvature tensor obeying Einstein's equation without cosmological constant, then the spacetime has constant energy density and isotropic pressure and the spacetime represents inflation and also the fluid behaves as a cosmological constant.*

We know[34] that if the Ricci tensor S of type (0,2) of the spacetime satisfies condition

$$S(X,X) > 0, \tag{3.10}$$

for every timelike vector field X, then (3.10) is called the timelike convergence condition.

Equations (3.1) and (3.2) together yield

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa[(\sigma + p)A(X)A(Y) + pg(X,Y)]. \tag{3.11}$$

Putting $X = Y = U$ in (3.11) and using (3.4) we obtain

$$S(U,U) = \frac{\kappa(\sigma + 3p)}{2}. \tag{3.12}$$

Since the spacetime under consideration satisfies the timelike convergence condition and $\kappa > 0$, we have

$$\sigma + 3p > 0. \tag{3.13}$$

The inequality (3.13) shows that the spacetime under consideration obeys cosmic strong energy condition.

Thus we can state the following.

Theorem 3.3. *If a pseudo-projectively flat perfect fluid spacetime satisfying Einstein's equation without cosmological constant obeys the timelike convergence condition, then such a spacetime also satisfies cosmic strong energy condition.*

Let us suppose that the scalar curvature r of the spacetime be positive. Then from (3.4) we have

$$\sigma > 3p. \tag{3.14}$$

Now Equations (3.13) and (3.14) together yield $\sigma > 0$. This means that the spacetime under consideration consists of pure matter.

Thus we have the following result.

Theorem 3.4. *If a pseudo-projectively flat perfect fluid spacetime satisfying Einstein's equation without cosmological constant obeys the timelike convergence condition, then such a spacetime contains pure matter, provided the scalar curvature r is positive.*

Taking a frame field after contraction over X and Y we get from (3.2) that

$$r = -\kappa t, \tag{3.15}$$

where $t = trace\ T$.

Therefore, Equation (3.2) can be written as

$$S(X,Y) = \kappa[T(X,Y) - \frac{t}{2}g(X,Y)]. \tag{3.16}$$

Einstein's field equation without cosmological constant for a purely electromagnetic distribution takes the form[1]

$$S(X,Y) = \kappa T(X,Y). \tag{3.17}$$

In Ref 38, p. 61, this kind of spacetime is called "pure radiation field" (or null dust) and its energy momentum tensor is given by $T_{kl} = \Phi^2 X_k X_l$, being $X_k X^k = 0$.

Using (3.16) and (3.17) we obtain $t = 0$. Thus from (3.15) we get $r = 0$. Hence from (2.4) we obtain $\tilde{R}(X,Y,Z,W) = 0$ which means that the spacetime is flat.

Thus we can state the following.

Theorem 3.5. *A pseudo-projectively flat spacetime satisfying Einstein's equation without cosmological constant for a purely electromagnetic distribution is an Euclidean space.*

Remark 1. This theorem points out towards a condition under which a semi-Riemannian space can be reduced to an Euclidean space.

Remark 2. Since for a pure radiation field $t = 0$, here we obtain from (3.15) that $r = 0$ and consequently from (3.4) we obtain $\sigma = 3p$. In Ref. 38, p. 63, pure radiation field may be considered as representing the incoherent superposition of waves with random phases and polarization but the same propagation direction. This is also used to describe perfect fluids with $\sigma = 3p$ (incoherent radiation) (Ref. 38, p. 66).

In a pseudo-projectively flat perfect fluid spacetime, from (2.4) it follows that the curvature tensor R is given by

$$R(X,Y)Z = \frac{r}{12}[g(Y,Z)X - g(X,Z)Y], \tag{3.18}$$

where r is the scalar curvature of the spacetime.

Since pseudo-projectively flat spacetime is Einstein space, it follows that $r =$ constant. Let $U^{\perp}$ denote the 3-dimensional distribution in a quasi-conformally flat perfect fluid spacetime orthogonal to U.

Then

$$R(X,Y)Z = \frac{r}{12}[g(Y,Z)X - g(X,Z)Y], \tag{3.19}$$

for all $X,Y,Z \in U^{\perp}$ and

$$R(X,U)U = -\frac{r}{12}X, \tag{3.20}$$

for every X$\in U^{\perp}$.

According to Karchar[16] a Lorentzian manifold is called infinitesimally spatially isotropic relative to timelike unit vector field U if its curvature tensor R satisfies the relation

$$R(X,Y)Z = l[g(Y,Z)X - g(X,Z)Y], \tag{3.21}$$

for all $X,Y,Z \in U^{\perp}$ and $R(X,U)U = mX$ for all X$\in U^{\perp}$, where l, m are real valued function on the manifold. So by virtue of (3.19) and (3.20) we can state the following.

Theorem 3.6. *A pseudo-projectively flat perfect fluid spacetime obeying the Einstein's field equation without cosmological constant and having the vector field U as the velocity vector field is infinitesimally spatially isotropic relative to the unit timelike vector field U.*

IV. PERFECT FLUID SPACETIMES WITH DIVERGENCE-FREE PSEUDO-PROJECTIVE CURVATURE TENSOR

In this section we consider a perfect fluid spacetime with divergence-free pseudo-projective curvature tensor.

Definition. A spacetime is said to be pseudo-projective conservative if

$$(div P^{*})(X,Y,Z) = 0,$$

where "*div*" denotes the divergence.

Definition. A symmetric (0,2) type tensor field E on a semi-Riemannian manifold (M^n, g) is said to be of Codazzi type if it satisfies the equation

$$(\nabla_X E)(Y,Z) = (\nabla_Y E)(X,Z),$$

for arbitrary vector fields X, Y, and Z.

From Equation (1.1) we obtain for a spacetime that

$$(divP^*)(X,Y,Z) = (a+b)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ -\frac{1}{4}(\frac{a}{3}+b)[g(X,Z)dr(Y) - g(Y,Z)dr(X)]. \tag{4.1}$$

Equation (4.1) is a particular case for the expression of the divergence of some generalized curvature (that is, curvature tensors satisfying $K_{ijkl} = -K_{jikl} = -K_{ijlk}$, $K_{ijkl} + K_{jkil} + K_{kijl} = 0$ as defined in Ref. 18), namely,

$$\nabla_m K^m_{jkl} = A\nabla_m R^m_{jkl} + B[(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}],$$

where A and B are constants (see Ref. 27 Proposition 4.6., Ref. 20 Theorem 2.2., and Ref. 21 Theorem 3.7.). In these papers the authors proved that if $\nabla_m K^m_{jkl} = 0$ and the condition $B \neq \frac{A}{2(n-1)}$ is satisfied, then the scalar curvature is a covariant constant $\nabla_l R = 0$. Thus $dr(X) = 0$ can be derived from $divP^* = 0$, provided $a \neq -\frac{5b}{3}$. However, $divP^* = 0$ and $dr(X) = 0$ imply $divC = 0$, where C is the conformal curvature tensor.

The conditions $divC = 0$ and $dr(X) = 0$ are equivalent to have a "Yang Pure Space" (see Ref. 13 Eq. (2)). In Ref. 13, Theorem 4.1 the authors proved that a 4-dimensional perfect fluid spacetime with $p + \sigma \neq 0$ is a Yang pure space if and only if it is a Robertson-Walker spacetime. For results about perfect fluids with divergence-free Weyl tensor see for example Refs. 35 and 26.

Thus we can state the following theorem.

Theorem 4.1. *A perfect fluid spacetime with divergence-free pseudo-projective curvature tensor is a Robertson-Walker spacetime, provided* $a \neq -\frac{5b}{3}$.

Now using $divP^* = 0$ and $dr(X) = 0$ in (4.1) we obtain

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0. \tag{4.2}$$

Hence using (3.2) and (4.2) we have

$$(\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z) = 0. \tag{4.3}$$

Thus we can state the following.

Theorem 4.2. *In a perfect fluid spacetime with divergence-free pseudo-projective curvature tensor the energy momentum tensor is of Codazzi type, provided* $a \neq -\frac{5b}{3}$.

But it is proved that if in a perfect fluid spacetime the energy momentum tensor is of Codazzi type then each of the shear and vorticity of the fluid vanishes and its velocity vector field is hypersurface orthogonal, i.e., its velocity vector field is proportional to the gradient vector field of the energy density (Refs. 11 and 33).

Hence from the above results, we can state the following.

Theorem 4.3. *In a perfect fluid spacetime with divergence-free pseudo-projective curvature tensor each of the shear and vorticity of the fluid vanishes and its velocity vector field is hypersurface orthogonal, i.e., its velocity vector field is proportional to the gradient vector field of the energy density, provided* $a \neq -\frac{5b}{3}$.

It has been proved by Barnes[4] that if a perfect fluid spacetime is shear-free and vorticity-free, the velocity vector field U is hypersurface orthogonal, and the energy density is constant over a hypersurface orthogonal to U, then the possible local cosmological structures of the spacetime are of Petrov type I, D, or O.

In view of Theorem 4.3 and the result of Barnes leads to the following theorem.

Theorem 4.4. *If a perfect fluid spacetime is of divergence-free pseudo-projective curvature tensor, then the possible local cosmological structure of such a spacetime is of type I, D, or O, provided* $a \neq -\frac{5b}{3}$.

Remark 3. Theorem 4.4 can be derived in an alternative and shorter way. Since the energy momentum tensor is a Codazzi tensor, from Equation (7) in Ref. 23 (see also Refs. 24 and 25) we have

$$T_{im}C^m_{jkl} + T_{jm}C^m_{kil} + T_{km}C^m_{ijl} = 0$$

and the manifold is named "Weyl compatible" (see Ref. 19). From Equation (3.1) it is

$$u_i u_m C^m_{jkl} + u_j u_m C^m_{kil} + u_k u_m C^m_{ijl} = 0$$

and the spacetime is "purely electric" (see Refs. 14 and 19 Theorem 3.3. and Ref. 25). It is well known that purely electric spacetimes are of Petrov types I, D, or O (conformally flat) (see Ref. 38, or Ref. 19 Theorem 3.4. and Ref. 14)

V. DUST FLUID SPACETIME WITH VANISHING PSEUDO-PROJECTIVE CURVATURE TENSOR

In a dust or pressureless fluid spacetime, the energy momentum tensor is of the form[36]

$$T(X,Y) = \sigma A(X)A(Y), \tag{5.1}$$

where σ is the energy density of the dust-like matter and A is a non-zero 1-form such that $g(X,U) = A(X)$, for all X, U being the velocity vector field of the flow, that is, $g(U,U) = -1$.

Using (2.6) and (5.1) we obtain

$$(\lambda - \frac{r}{4})g(X,Y) = \kappa\sigma A(X)A(Y). \tag{5.2}$$

A frame field after contraction over X and Y leads to

$$\lambda = \frac{r}{4} - \frac{\kappa\sigma}{4}. \tag{5.3}$$

Again, if we put $X = Y = U$ in (5.2), we get

$$\lambda = \frac{r}{4} - \kappa\sigma. \tag{5.4}$$

Thus combining the Equations (5.3) and (5.4), we finally obtain that

$$\sigma = 0. \tag{5.5}$$

Thus from (5.1) and (5.5) we conclude that

$$T(X,Y) = 0.$$

This means that the spacetime is devoid of the matter. Thus we can state the following.

Theorem 5.1. *A pseudo-projectively flat dust fluid spacetime satisfying Einstein's field equation with cosmological constant is vacuum.*

ACKNOWLEDGMENTS

The second author was supported by Grant Project No. NRF-2015-R1A2A1A-01002459 from National Research Foundation. The authors wish to express their sincere thanks and gratitude to the referee for his/her valuable suggestions towards the improvement of the paper.

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