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*-Ricci soliton on $(\kappa, \mu)'$ -almost Kenmotsu manifolds

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Abstract: Let (M, g) be a non-Kenmotsu $(\kappa, \mu)'$ -almost Kenmotsu manifold of dimension $2n + 1$. In this paper, we prove that if the metric g of M is a *-Ricci soliton, then either M is locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ or the potential vector field is strict infinitesimal contact transformation. Moreover, two concrete examples of $(\kappa, \mu)'$ -almost Kenmotsu 3-manifolds admitting a Killing vector field and strict infinitesimal contact transformation are given.

Keywords: Almost Kenmotsu manifold; *-Ricci soliton; nullity distribution; Lie group

MSC: Primary 53D15; Secondary 53C25.

1 Introduction

On a Riemannian manifold (M, g) if there exists a vector field V and a constant λ satisfying

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric} + \lambda g = 0, \quad (1.1)$$

then it is said that the triple (g, V, λ) , for simplicity, g , defines a Ricci soliton (see Hamilton [1, 2]), where Ric denotes the Ricci tensor. Usually, V and λ are said to be the potential vector field and the soliton constant, respectively. If the potential vector field V is Killing, (1.1) reduces to an Einstein metric (that is, the Ricci tensor is a constant multiple of the Riemannian metric when $\dim M > 2$). The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined by

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\text{Ric}_{ij}(t).$$

Ricci solitons are self-similar solutions to the Ricci flow.

The studying of Ricci solitons on almost contact metric manifolds was introduced by R. Sharma in [3]. In the last decade, a large number of papers were published regarding classification of Ricci solitons on almost contact manifolds. Among others, we refer the readers to [4–7], [8–12] and [13–16] for fruitful results on (almost) Ricci solitons on contact metric, (almost) Kenmotsu and (almost) cosymplectic manifolds, respectively. Recently, a new research interest has appeared regarding the so called *-Ricci soliton which is defined by

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric}^* + \lambda g = 0, \quad (1.2)$$

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where V and λ still denote a vector field (called the potential vector field) and a constant (called the soliton constant). On an almost contact metric manifold (M, ϕ, ξ, η, g) , the *-Ricci tensor Ric^* is defined by

$$\text{Ric}^*(X, Y) = \frac{1}{2} \text{trace}\{Z \rightarrow R(X, \phi Y)\phi Z\} \tag{1.3}$$

for any vector fields X, Y . The *-Ricci tensor Ric^* in almost contact geometry can be regarded in analogy with the usual Ricci tensor in Riemannian geometry. As usual, for simplicity, we say that the metric g of an almost contact metric manifold is a *-Ricci soliton if (1.2) is true.

The notion of *-Ricci tensor was introduced on an almost Hermitian manifold by Tachibana in [17]. Later, such notion was considered on real hypersurfaces of a nonflat complex space form by Kaimakamis and Panagiotidou [18] (see also [19]). Recently, *-Ricci solitons on an almost contact metric manifold (M, ϕ, ξ, η, g) were started to be considered by some authors. More precisely, *-Ricci solitons on Sasakian 3-manifolds and (κ, μ) -contact manifolds were investigated in [20] and [21] respectively. Y. Wang in [22] proved that if the metric of a Kenmotsu 3-manifold represents a *-Ricci soliton, then the manifold is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$.

In this paper, we start to investigate the *-Ricci solitons on almost Kenmotsu manifolds. Because the class of almost Kenmotsu manifolds is rather large, then we have to consider some other special almost Kenmotsu manifolds. By a $(\kappa, \mu)'$ -almost Kenmotsu manifold, we mean that the Reeb vector field of the manifold belongs to the $(\kappa, \mu)'$ -nullity distribution (see [23]). We prove that if the metric of a non-Kenmotsu $(\kappa, \mu)'$ -almost Kenmotsu manifold is a *-Ricci soliton, then the manifold is locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ provided that the potential vector field is not infinitesimal contact transformation. We also construct two concrete examples to illustrate our main results.

2 $(\kappa, \mu)'$ -almost Kenmotsu manifolds

Let (M^{2n+1}, g) be a smooth Riemannian manifold of dimension $2n + 1$. On this manifold if there exist a $(1, 1)$ -type tensor field ϕ , a global vector field ξ and a 1-form η satisfying

$$\phi^2 = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X, Y , then (ϕ, ξ, η, g) is called an almost contact metric structure and M^{2n+1} is called an almost contact metric manifold (see Blair [24]). Usually, ξ and η are called the Reeb or characteristic vector field and an almost contact 1-form respectively.

The fundamental 2-form Φ on an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y . Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost contact manifold. We define on the product $M^{2n+1} \times \mathbb{R}$ an almost complex structure J by

$$J\left(X, f \frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X) \frac{d}{dt}\right),$$

where X denotes a vector field tangent to M^{2n+1} , t is the coordinate of \mathbb{R} and f is a C^∞ -function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of ϕ (see [24]), if $[\phi, \phi] = -2d\eta \otimes \xi$ (or equivalently, the almost complex structure J is integrable), then the almost contact metric structure is said to be normal.

On an almost contact metric manifold if there hold $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, then the manifold is said to be an almost Kenmotsu manifold (see [25]). A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold (see [26]) and this is also equivalent to

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.3}$$

for any vector fields X, Y , where ∇ denotes the Levi-Civita connection of the metric g . On an almost Kenmotsu manifold, we set $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $h' = h \circ \phi$, where \mathcal{L} is the Lie derivative. It is easily seen that both the above two operators are symmetric. The following formulas can be seen in [23, 27]:

$$h\xi = 0, \operatorname{tr}h = \operatorname{tr}(h') = 0, h\phi + \phi h = 0, \quad (2.4)$$

$$\nabla\xi = h' + \operatorname{id} - \eta \otimes \xi. \quad (2.5)$$

If the Reeb vector field ξ of an almost Kenmotsu manifold M^{2n+1} belongs to the so called $(\kappa, \mu)'$ -nullity distribution, i.e.,

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \quad (2.6)$$

for some constants κ and μ , then M^{2n+1} is said to be a $(\kappa, \mu)'$ -almost Kenmotsu manifold (see [10, 23]). It follows from (2.6) that

$$R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X) + \mu(g(h'X, Y)\xi - \eta(Y)h'X) \quad (2.7)$$

for any vector fields X, Y .

3 \star -Ricci solitons on $(\kappa, \mu)'$ -almost Kenmotsu manifolds

Given a $(\kappa, \mu)'$ -almost Kenmotsu manifold M^{2n+1} , it has been proved in [23, Proposition 4.1] that $\mu = -2$. Also, we have

$$h^2 = h'^2 = (\kappa + 1)\phi^2. \quad (3.1)$$

In view of symmetry of h' and $h'\xi = 0$, we denote by X an eigenvector field of h' orthogonal to ξ with corresponding eigenvalue θ . By (2.1) and (3.1), it follows that $\theta^2 = -(\kappa + 1)$ and hence we have $\kappa \leq -1$. The equality holds if and only if $h = 0$ and in this case the manifold is called a C -almost Kenmotsu manifold because in this context the Reeb foliation is conformal (see [28]). Throughout this paper, we consider those $(\kappa, \mu)'$ -almost Kenmotsu manifolds with $\kappa < -1$, i.e., $h \neq 0$ everywhere.

Lemma 3.1 ([29]). *On a $(\kappa, \mu)'$ -almost Kenmotsu manifold with $\kappa < -1$ the Ricci operator is given by*

$$Q = -2n\operatorname{id} + 2n(\kappa + 1)\eta \otimes \xi - 2nh', \quad (3.2)$$

where the Ricci operator is defined by $\operatorname{Ric}(X, Y) = g(QX, Y)$.

Lemma 3.2. *On a $(\kappa, \mu)'$ -almost Kenmotsu manifold with $\kappa < -1$ the \star -Ricci tensor is given by*

$$\operatorname{Ric}^\star(X, Y) = -(\kappa + 2)(g(X, Y) - \eta(X)\eta(Y)) \quad (3.3)$$

for any vector fields X, Y .

Proof. On a non-Kenmotsu $(\kappa, \mu)'$ -almost Kenmotsu manifold, using [23, Proposition 4.2.] we compute the curvature tensor of the manifold as the following

$$\begin{aligned} R(X, Y)Z = & \kappa\eta(Z)(\eta(Y)X - \eta(X)Y) + \kappa(g(Y, Z)\eta(X) - g(X, Z)\eta(Y))\xi \\ & + (g(Y - h'Y, Z)\eta(X) - g(X - h'X, Z)\eta(Y))\xi \\ & + \eta(Z)(\eta(Y)(X - h'X) - \eta(X)(Y - h'Y)) \\ & - (g(Y + h'Y, Z)(X + h'X) - g(X + h'X, Z)(Y + h'Y)) \end{aligned} \quad (3.4)$$

for any vector fields X, Y, Z . We remark that expression of curvature tensor R of non-Kenmotsu α -almost Kenmotsu manifolds satisfying $(\kappa, \mu)'$ -nullity condition was obtain by D. Dileo [30, Corollary 2]. However, her formula contains a little error. The application of (3.4) together with the definition of \star -Ricci tensor (1.3) and (2.4) give (3.3). \square

Notice that (3.2) can be deduced directly from (3.4).

Lemma 3.3. *If a triple (g, V, λ) on a non-Kenmotsu $(\kappa, \mu)'$ -almost Kenmotsu manifold is a *-Ricci soliton, then we have*

$$\begin{aligned}
 & (\mathcal{L}_V R)(X, Y, Z) \\
 &= 2(\kappa + 2)(g(X + h'X, Z)(Y + h'Y) - g(Y + h'Y, Z)(X + h'X)) \\
 & \quad + 2(\kappa + 2)(\eta(Y)\eta(Z)(X + h'X) - \eta(X)\eta(Z)(Y + h'Y)) \\
 & \quad + 2(\kappa + 2)(g(\kappa Y - 2h'Y, Z)\eta(X) - g(\kappa X - 2h'X, Z)\eta(Y)) \\
 & \quad + 2(\kappa + 2)(g(Y + h'Y, Z)\eta(X) - g(X + h'X, Z)\eta(Y))\xi
 \end{aligned} \tag{3.5}$$

for any vector fields X, Y, Z .

Proof. Let M^{2n+1} be a $(\kappa, \mu)'$ -almost Kenmotsu manifold with $\kappa < -1$ whose metric g is a *-Ricci soliton for certain potential vector field V and constant λ . From (1.1) and (3.3) we have

$$(\mathcal{L}_V g)(X, Y) = 2(\kappa + 2 - \lambda)g(X, Y) - 2(\kappa + 2)\eta(X)\eta(Y) \tag{3.6}$$

for any vector fields X, Y . Taking the covariant derivative of the above relation gives

$$\begin{aligned}
 (\nabla_X \mathcal{L}_V g)(Y, Z) &= -2(\kappa + 2)(g(X + h'X, Y)\eta(Z) \\
 & \quad + g(X + h'X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)).
 \end{aligned} \tag{3.7}$$

From Yano [31, pp. 23] we have the following formula

$$(\nabla_Z \mathcal{L}_V g - \mathcal{L}_V \nabla_Z g + \nabla_{[V, Z]}g)(X, Y) = g((\mathcal{L}_V \nabla)(Z, X), Y) + g((\mathcal{L}_V \nabla)(Z, Y), X).$$

On Riemannian manifold (M, g) , because the metric g is parallel, it follows that

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = g((\mathcal{L}_V \nabla)(Z, X), Y) + g((\mathcal{L}_V \nabla)(Z, Y), X). \tag{3.8}$$

In view of the symmetry of the $(1, 2)$ -type tensor filed $\mathcal{L}_V \nabla$, i.e., $(\mathcal{L}_V \nabla)(X, Y) = (\mathcal{L}_V \nabla)(Y, X)$, interchanging cyclicly the roles of X, Y, Z in (3.8) we obtain

$$\begin{aligned}
 g((\mathcal{L}_V \nabla)(X, Y), Z) &= \frac{1}{2}(\nabla_X \mathcal{L}_V g)(Y, Z) \\
 & \quad + \frac{1}{2}(\nabla_Y \mathcal{L}_V g)(Z, X) - \frac{1}{2}(\nabla_Z \mathcal{L}_V g)(X, Y).
 \end{aligned} \tag{3.9}$$

Putting (3.7) into (3.9) we obtain

$$(\mathcal{L}_V \nabla)(X, Y) = -2(\kappa + 2)g(X + h'X, Y)\xi + 2(\kappa + 2)\eta(X)\eta(Y)\xi. \tag{3.10}$$

Taking the covariant derivative of (3.10) we get

$$\begin{aligned}
 (\nabla_X \mathcal{L}_V \nabla)(Y, Z) &= -2(\kappa + 2)g((\nabla_X h')Y, Z)\xi \\
 & \quad - 2(\kappa + 2)g(Y + h'Y, Z)(X - \eta(X)\xi + h'X) \\
 & \quad + 2(\kappa + 2)g(X - \eta(X)\xi + h'X, Y)\eta(Z)\xi \\
 & \quad + 2(\kappa + 2)g(X - \eta(X)\xi + h'X, Z)\eta(Y)\xi \\
 & \quad + 2(\kappa + 2)\eta(Y)\eta(Z)(X - \eta(X)\xi + h'X).
 \end{aligned} \tag{3.11}$$

With the aid of (3.1), it has been proved by Dileo and Pastore in [23, Lemma 4.1] that on M^{2n+1} there holds

$$\begin{aligned}
 (\nabla_X h')Y &= g((\kappa + 1)X - h'X, Y)\xi + \eta(Y)((\kappa + 1)X - h'X) \\
 & \quad - 2(\kappa + 1)\eta(X)\eta(Y)\xi.
 \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11) yields

$$\begin{aligned}
 & (\nabla_X \mathcal{L}_V \nabla)(Y, Z) \\
 &= -2(\kappa + 2)(g(Y + h'Y, Z) - \eta(Y)\eta(Z))(X + h'X) \\
 &\quad - 2(\kappa + 2)g(\kappa X - 2h'X + (1 - 2\kappa)\xi, Y)\eta(Z)\xi \\
 &\quad - 2(\kappa + 2)g(\kappa X - 2h'X + (1 - 2\kappa)\xi, Z)\eta(Y)\xi.
 \end{aligned} \tag{3.13}$$

Notice that the following formula was given by Yano in [31, pp. 23]

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z) \tag{3.14}$$

for any vector fields X, Y, Z . Putting (3.13) into the above relation we complete the proof. \square

Following Blair [24, pp.72] and Tanno [32] we give

Definition 3.1. On an almost contact metric manifold (M, ϕ, ξ, η, g) a vector field X is said to be infinitesimal contact transformation if $\mathcal{L}_V \eta = f\eta$ for some function f . In particular, X is said to be strict infinitesimal contact transformation if $\mathcal{L}_X \eta = 0$.

Our main result in this paper is the following.

Theorem 3.1. *If a triple (g, V, λ) on a non-Kenmotsu $(\kappa, \mu)'$ -almost Kenmotsu manifold is a \ast -Ricci soliton, then the soliton constant $\lambda = 0$, and either the manifold is locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ and the potential vector field is Killing, or the potential vector field is strict infinitesimal contact transformation.*

Proof. Let $\{e_i : i = 0, 1, 2, \dots, 2n\}$ be an orthonormal basis for tangent space at each point of the manifold. Taking the inner product of (3.13) with X , setting $X = e_i$ in (3.13) and summing over $i \in \{0, 1, 2, \dots, 2n\}$ we obtain

$$\sum_{i=0}^{2n} g((\nabla_{e_i} \mathcal{L}_V \nabla)(Y, Z), e_i) = -4n(\kappa + 2)g(Y - \eta(Y)\xi + h'Y, Z), \tag{3.15}$$

where we have employed $\text{tr}h' = 0$ and $h'^2 = (\kappa + 1)\phi^2$. Similarly, taking the inner product of (3.13) with Y , replacing Y by e_i in (3.13) and summing over $i \in \{0, 1, 2, \dots, 2n\}$ we obtain

$$\sum_{i=0}^{2n} g((\nabla_X \mathcal{L}_V \nabla)(e_i, Z), e_i) = 0. \tag{3.16}$$

From (3.15) and (3.16), contracting X in (3.5) or (3.14) we have

$$(\mathcal{L}_V \text{Ric})(Y, Z) = -4n(\kappa + 2)g(Y - \eta(Y)\xi + h'Y, Z). \tag{3.17}$$

From (3.2), the Ricci tensor can be written as

$$\text{Ric}(Y, Z) = -2ng(Y, Z) + 2n(\kappa + 1)\eta(Y)\eta(Z) + 2ng(h'Y, Z).$$

Taking the Lie derivative of this relation along the potential vector field V we obtain

$$\begin{aligned}
 & (\mathcal{L}_V \text{Ric})(Y, Z) \\
 &= -4n(\kappa + 2 - \lambda)g(Y, Z) + 4n(\kappa + 2 - 2\lambda(\kappa + 1))\eta(Y)\eta(Z) \\
 &\quad + 2n(\kappa + 1)(\eta(Y)g(\mathcal{L}_V \xi, Z) + \eta(Z)g(\mathcal{L}_V \xi, Y)) \\
 &\quad - 2ng((\mathcal{L}_V h')Y, Z) - 4n(\kappa + 2 - \lambda)g(h'Y, Z)
 \end{aligned} \tag{3.18}$$

for any vector fields Y, Z , where we have applied (3.6). Comparing (3.18) with (3.17) gives

$$\begin{aligned}
 g((\mathcal{L}_V h')Y, Z) &= 2\lambda g(Y, Z) + 2\lambda g(h'Y, Z) - 4\lambda(\kappa + 1)\eta(Y)\eta(Z) \\
 &\quad + (\kappa + 1)(\eta(Y)g(\mathcal{L}_V \xi, Z) + \eta(Z)g(\mathcal{L}_V \xi, Y)).
 \end{aligned}$$

Note that by setting $X = Y = \xi$ in (3.6) we obtain $\eta(\nabla_\xi V) + \lambda = 0$. Applying this equation and replacing $Y = Z = \xi$ in the previous relation we obtain $\lambda = 0$ because of $\kappa < -1$ and hence

$$(\mathcal{L}_V h')Y = (\kappa + 1)(\eta(Y)\mathcal{L}_V \xi + g(\mathcal{L}_V \xi, Y)\xi) \tag{3.19}$$

for any vector field Y . With the aid of $\lambda = 0$, replacing Y in (3.19) by ξ we get

$$h'\mathcal{L}_V \xi = -(\kappa + 1)\mathcal{L}_V \xi.$$

The action of the operator h' on the above relation gives $h'\mathcal{L}_V \xi = \mathcal{L}_V \xi$ because of (3.1), $\lambda = 0$ and $\kappa < -1$. Comparing this equation with the previous one gives either $\mathcal{L}_V \xi = 0$ or $\kappa = -2$.

It has been proved by Dileo and Pastore in [23, pp. 56] that if $\kappa = -2$, then a $(\kappa, \mu)'$ -almost Kenmotsu manifold is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. In this case, from (3.6) we see that the potential vector field V is Killing. Note that the almost Kenmotsu structure on the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ was constructed in [23]. If $\kappa \neq -2$, then we get $\mathcal{L}_V \xi = 0$, and with the aid of $\lambda = 0$ and $(\mathcal{L}_V g)(\xi, X) = 0$ we have

$$(\mathcal{L}_V \eta)X = \mathcal{L}_V \eta(X) - \eta(\nabla_V X) + \eta(\nabla_X V) = 0$$

for any vector field X . Then, V is strict infinitesimal contact transformation. This completes the proof. □

Remark 3.1. In [15], it was proved that if a 3-dimensional cosymplectic manifold admits a Ricci soliton, then either the manifold is locally flat or the potential vector field is an infinitesimal contact transformation. In view of this result and our Theorem 3.1, it is interesting to investigate the existence and properties of infinitesimal contact transformations on almost Kenmotsu and cosymplectic manifolds in future.

Remark 3.2. Ghosh and Patra in [20, Theorem 16] proved that a non-Sasakian (κ, μ) -contact manifold of dimension > 3 admitting a non-trivial *-Ricci soliton is locally isometric to $\mathbb{R}^{n+1} \times \mathbb{S}^n(4)$. Our Theorem 3.1 is in analogy with Ghosh-Patra’s result in almost Kenmotsu geometry.

4 Examples

Before closing this paper, we present two concrete examples of $(\kappa, \mu)'$ -almost Kenmotsu manifolds of dimension three admitting a *-Ricci soliton.

It was shown in [22, Theorem 3.2] that any 3-dimensional non-unimodular Lie group admits a left invariant almost Kenmotsu structure satisfying the (κ, μ, ν) -nullity condition. In particular, $(\kappa, \mu)'$ -almost Kenmotsu manifolds of dimension three were completely classified in [23, Theorem 5.1].

Let G be a Lie group whose Lie algebra \mathfrak{g} is given by

$$[e_1, e_2] = (1 + \theta)e_2, [e_2, e_3] = 0, [e_3, e_1] = (\theta - 1)e_3,$$

where θ is a positive constant. Let g be the metric defined on G by $g(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$. We denote by $\xi = -e_1$ and denote by η the dual 1-form of ξ with respect to g . We define a $(1, 1)$ -type tensor field ϕ by

$$\phi(\xi) = 0, \phi(e_2) = e_3, \phi(e_3) = -e_2.$$

We denote e by e_2 and ϕe by e_3 , respectively. According to the Koszul formula, the Levi-Civita connection on G can be written as follows:

$$\nabla_\xi \xi = 0, \nabla_\xi e = 0, \nabla_\xi \phi e = 0, \tag{4.1}$$

$$\nabla_e \xi = (1 + \theta)e, \nabla_e e = -(\theta + 1)\xi, \nabla_e \phi e = 0, \tag{4.2}$$

$$\nabla_{\phi e} \xi = (1 - \theta)\phi e, \nabla_{\phi e} e = 0, \nabla_{\phi e} \phi e = (\theta - 1)\xi, \tag{4.3}$$

One can check that on G there exists a left invariant almost Kenmotsu structure (ϕ, ξ, η, g) satisfying the $(-1 - \theta^2, -2)'$ -nullity condition. In particular, Lie group G is locally isometric to the product $\mathbb{H}^2(-4) \times \mathbb{R}$ (see also [23]) when $\theta = 1$. Next, we show that there exist two kinds of \ast -Ricci solitons on G .

Case I: $\theta = 1$. We suppose that $V := \alpha\xi + \beta e + \gamma\phi e$ is a Killing vector field defined on G , i.e., $g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0$ for any vector fields X, Y , where α, β and γ are three smooth functions varying along ξ, e and ϕe . Locally, we observe that V is a Killing vector field if and only if the following six equations hold:

$$\begin{aligned} g(\nabla_\xi V, e) + g(\nabla_e V, \xi) &= 0, \\ g(\nabla_\xi V, \phi e) + g(\nabla_{\phi e} V, \xi) &= 0, \\ g(\nabla_e V, \phi e) + g(\nabla_{\phi e} V, e) &= 0, \\ g(\nabla_\xi V, \xi) = g(\nabla_e V, e) = g(\nabla_{\phi e} V, \phi e) &= 0. \end{aligned}$$

The above six equations can be rewritten as the following

$$\xi(\beta) + e(\alpha) = 2\beta, \quad \xi(\gamma) + \phi e(\alpha) = 0, \quad e(\gamma) + \phi e(\beta) = 0, \tag{4.4}$$

$$\xi(\alpha) = 0, \quad 2\alpha + e(\beta) = 0, \quad \phi e(\gamma) = 0. \tag{4.5}$$

One can find many solutions of the system of partial differential equations (4.4)-(4.5). For instance, if we set $\alpha = 0$ and $\gamma = 1$, from (4.4)-(4.5) we obtain

$$\xi(\beta) = 2\beta, \quad e(\beta) = 0, \quad \phi e(\beta) = 0. \tag{4.6}$$

On the $(-2, -2)'$ -almost Kenmotsu 3-manifold G mentioned above, $\beta e + \phi e$ is a Killing vector field if β satisfies (4.6). Next we give some local expressions of these solutions.

Example 4.1. Let $M^3 := \{(x, y, z) \in \mathbb{R}^3\}$ be an open submanifold of the Euclidean space \mathbb{R}^3 . On M^3 , we consider three linear independently vector fields

$$e_1 = \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

One can check that

$$[e_1, e_2] = 2e_2, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

On M^3 we define a $(1, 1)$ -type tensor field ϕ by $\phi(e_1) = 0, \phi e_2 = e_3$ and $\phi e_3 = -e_2$, and we define a Riemannian metric g such that $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$. We denote by $\xi = -e_1$ and η its dual 1-form with respect to the metric g . One can check that $(M^3, \phi, \xi, \eta, g)$ is a $(-2, -2)'$ -almost Kenmotsu 3-manifold. On M^3 , from (4.6) we have $\beta = ce^{-2x}$, where c is a non-zero constant. Thus, $ce^{-2x} \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ is a Killing vector field and hence $(ce^{-2x} \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, 0, g)$ is a \ast -Ricci soliton on M^3 .

Case II: $\theta \neq 1$. Following the second case of Theorem 3.1, if there exists a \ast -Ricci soliton (V, λ, g) on G , then it becomes (3.6) with $\kappa = -\theta^2 - 1 \neq -2$, and we also have $\lambda = 0$ and $\mathcal{L}_V \xi = 0$. Also, in this case, the potential vector field V is strict infinitesimal contact transformation. Suppose that V is the vector field satisfying the above conditions.

Replacing Y by ξ in (3.6), with the aid of $\lambda = 0$ and $\mathcal{L}_V \xi = 0$, we see that $\eta(V)$ is a constant, say α . Then, we may write $V = \alpha\xi + \beta e + \gamma\phi e$ with β, γ smooth functions. In this case, with the aid of (4.1)-(4.3), $\mathcal{L}_V \xi = 0$ is equivalent to

$$\xi(\beta) = \beta(1 + \theta), \quad \xi(\gamma) = \gamma(1 - \theta). \tag{4.7}$$

Putting $X = Y = e$ in (3.6), using $\lambda = 0$ and (4.1)-(4.3), we have

$$e(\beta) = -\alpha(\theta + 1) - \theta^2 + 1. \tag{4.8}$$

Similarly, putting $X = Y = \phi e$ in (3.6), using $\lambda = 0$ and (4.1)-(4.3), we have

$$\phi e(\gamma) = \alpha(\theta - 1) - \theta^2 + 1. \quad (4.9)$$

Finally, putting $X = e$ and $Y = \phi e$ in (3.6), using $\lambda = 0$ and (4.1)-(4.3), we have

$$\phi e(\beta) + e(\gamma) = 0. \quad (4.10)$$

For simplicity, we set $\gamma = 0$, then $V = \alpha\xi + \beta e + \gamma\phi e$ on G with $\theta = 2$ is a *-Ricci soliton if and only if

$$\xi(\beta) = 3\beta, \quad e(\beta) = -12, \quad \phi e(\beta) = 0. \quad (4.11)$$

Example 4.2. Let $M^3 := \{(x, y, z) \in \mathbb{R}^3\}$ be an open submanifold of the Euclidean space \mathbb{R}^3 . On M^3 , we consider three linear independently vector fields

$$e_1 = \frac{\partial}{\partial x} - 3y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

One can check that

$$[e_1, e_2] = 3e_2, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = e_3.$$

On M^3 we define a $(1, 1)$ -type tensor field ϕ by $\phi(e_1) = 0$, $\phi e_2 = e_3$ and $\phi e_3 = -e_2$, and we define a Riemannian metric g such that $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$. We denote by $\xi = -e_1$ and η its dual 1-form with respect to the metric g . One can check that $(M^3, \phi, \xi, \eta, g)$ is a $(-5, -2)$ '-almost Kenmotsu 3-manifold. On M^3 , from (4.11) we have $\beta = ce^{-3x} - 12y$, where c is a non-zero constant. Thus, $(3 \frac{\partial}{\partial x} + (ce^{-3x} - 12y) \frac{\partial}{\partial y}, 0, g)$ is a *-Ricci soliton on M^3 .

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References

- [1] Hamilton R.S., Three-manifolds with positive Ricci curvature, *J. Differ. Geom.*, 1982, 17, 255–306
- [2] Hamilton R.S., The Ricci flow on surfaces, In: *Mathematics and General Relativity*, *Contemp. Math.* American Mathematical Society., 1988, 71, 237–262
- [3] Sharma R., Certain results on K -contact and (κ, μ) -contact manifolds, *J. Geom.*, 2008, 89, 138–147
- [4] Cho J.T., Sharma R., Contact geometry and Ricci solitons, *Int. J. Geom. Methods Modern Phys.*, 2010, 7, 951–960
- [5] Ghosh A., Ricci solitons and contact metric manifolds, *Glasgow Math. J.*, 2013, 55, 123–130
- [6] Ghosh A., Sharma R., K -contact metrics as Ricci solitons, *Beitr. Algebra Geom.*, 2012, 53, 25–30
- [7] Sharma R., Ghosh A., Sasakian 3-manifolds as a Ricci soliton represents the Heisenberg group, *Int. J. Geom. Methods Modern Phys.*, 2011, 8, 149–154
- [8] Ghosh A., Kenmotsu 3-metric as a Ricci soliton, *Chaos, Solitons & Fractals*, 2011, 44, 647–650
- [9] Ghosh A., An η -Einstein Kenmotsu metric as a Ricci soliton, *Publ. Math. Debrecen*, 2013, 82, 591–598
- [10] Wang Y., Liu X., Ricci solitons on three-dimensional η -Einstein almost Kenmotsu manifolds, *Taiwanese J. Math.*, 2015, 19, 91–100
- [11] Wang Y., Gradient Ricci almost solitons on two classes of almost Kenmotsu manifolds, *J. Korean Math. Soc.*, 2016, 53, 1101–1114
- [12] Wang Y., Ricci solitons on almost Kenmotsu 3-manifolds, *Open Math.*, 2017, 15, 1236–1243
- [13] Cho J.T., Almost contact 3-manifolds and Ricci solitons, *Int. J. Geom. Methods Modern Phys.*, 2013, 10, 1220022
- [14] Wang Y., A generalization of the Goldberg conjecture for coKähler manifolds, *Mediterr. J. Math.*, 2016, 13, 2679–2690
- [15] Wang Y., Ricci solitons on 3-dimensional cosymplectic manifolds, *Math. Slovaca.*, 2017, 67, 979–984
- [16] Wang Y., Ricci solitons on almost coKähler manifolds, *Canad. Math. Bull.*, 2019, 62, DOI: <https://doi.org/10.4153/S0008439518000632>
- [17] Tachibana S., On almost-analytic vectors in almost Kählerian manifolds, *Tôhoku Math. J.*, 1959, 11, 247–265
- [18] Kaimakamis G., Panagiotidou K., *-Ricci solitons of real hypersurfaces in non-flat complex space forms, *J. Geom. Phys.*, 2014, 86, 408–413

- [19] Hamada T., Real hypersurfaces of complex space forms in terms of Ricci \ast -tensor, *Tokyo J. Math.*, 2002, 25, 473–483
- [20] Ghosh A., Patra D.S., \ast -Ricci soliton within the frame-work of Sasakian and (κ, μ) -contact manifold, *Int. J. Geom. Methods Modern Phys.*, 2018, 15, 1850120
- [21] Majhi P., De U.C., Suh Y.J., \ast -Ricci solitons on Sasakian 3-manifolds, *Publ. Math. Debrecen*, 2018, 93, 241–252
- [22] Wang Y., Contact 3-manifolds and \ast -Ricci soliton, submitted
- [23] Dileo G., Pastore A.M., Almost Kenmotsu manifolds and nullity distributions, *J. Geom.*, 2009, 93, 46–61
- [24] Blair D.E., *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 2010, 203, Birkhäuser, New York
- [25] Janssens D., Vanhecke L., Almost contact structures and curvature tensors, *Kodai Math. J.*, 1981, 4, 1–27
- [26] Kenmotsu K., A class of almost contact Riemannian manifolds, *Tôhoku Math. J.*, 1972, 24, 93–103
- [27] Dileo G., Pastore A.M., Almost Kenmotsu manifolds and local symmetry, *Bull. Belg. Math. Soc. Simon Stevin*, 2007, 14, 343–354
- [28] Pastore A.M., Saltarelli V., Almost Kenmotsu manifolds with conformal Reeb foliation, *Bull. Belg. Math. Soc. Simon Stevin*, 2011, 18, 655–666
- [29] Wang Y., Wang W., Some results on $(\kappa, \mu)'$ -almost Kenmotsu manifolds, *Quaest. Math.*, 2018, 41, 469–481
- [30] Dileo G., A classification of certain almost α -Kenmotsu manifolds, *Kodai Math. J.*, 2011, 34, 426–445
- [31] Yano K., *Integral Formulas in Riemannian Geometry*, 1970, Marcel Dekker, New York
- [32] Tanno S., Some transformations on manifolds with almost contact and contact metric structures II, *Tohoku Math. J.*, 1963, 15, 322–331