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A Korteweg–de Vries equation modified by viscosity for waves in a channel of uniform but arbitrary cross section

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A KdV equation modified by viscosity is derived for weakly nonlinear long waves propagating in a channel of uniform but arbitrary cross section. The case of high Reynolds number is considered, and the method of matched asymptotic expansion is employed. The equation derived here is found to be similar to the corresponding equation for a two-dimensional layer of liquid derived by previous authors. The only difference is that the dispersive, nonlinear, and viscous terms are multiplied by constants dependent on the cross section geometry of the channel.

I. INTRODUCTION

The effect of viscosity on weakly nonlinear long waves, in which there is a balance between nonlinear and dispersive effects, has been considered by some authors. Starting from equations derived by Chester,[1] in connection with oscillation of a liquid in a tank near resonant frequency, Miles[2] derived a KdV equation modified by viscosity for weakly nonlinear long waves on the free surface of a two-dimensional layer of fluid having a finite depth. Kakutani and Matsuuchi[3] derived a similar equation using the method of matched asymptotic expansion. Koop and Butler[4] have shown how some of the existing internal wave theories may be extended to include higher-order nonlinear effects of viscous effects. They have derived a KdV equation modified by viscosity for internal wave propagating in a density stratified fluid confined between two rigid horizontal boundaries. They have also derived an expression for the amplitude decay caused by viscosity of internal solitary wave in a two-layer fluid confined between two rigid horizontal planes. This amplitude decay for the case of free upper boundary had been considered by Leon, Segur, Hammack.[5]

In this paper we derive an equation describing a weakly nonlinear long wave propagating along a channel of arbitrary but uniform cross section including the effect of viscosity. Equations describing propagation of such waves along a channel of uniform but arbitrary cross section without considering the effect of viscosity were derived by Pregrine.[6] Using a two-layer model, Grimshaw[7] derived a KdV equation that describes propagation of internal weakly nonlinear long waves in a channel of arbitrary cross section without considering the effect of viscosity.

We consider the case $R^{-1} \ll kh \ll 1$, which is equivalent to the equation $\delta \ll h \ll k^{-1}$. Here R is the Reynolds number, k is the wavenumber, h is the mean depth, and δ is the thickness of the boundary layer. Obviously this condition corresponds to the case of high Reynold's numbers and long wavelength. In this case, as has been considered by previous authors, the effect of viscosity is dominant in two boundary layers. One of these two layers is adjacent to the free surface, and the other is adjacent to the channel surface. It will be found subsequently that there is no influence of the boundary layer adjacent to the free surface on the KdV equation modified by viscosity. The flow field is divided into two regions. One is the region outside the boundary layers, which we call the outer region; the other is the thin boundary layer adjacent to the free surface and the channel surface, which we call the inner region. The solutions in the two regions are matched by the well-known matched asymptotic method.[8,9]

The equation derived here is found to be similar to the corresponding equation for a two-dimensional layer of fluid derived by Miles[2] and Kakutani and Matsuuchi.[3] The only difference is that the dispersive, nonlinear, and viscous terms are multiplied by some constants dependent on the geometry of the cross section of the channel. To determine this constant for the dispersive term, it becomes necessary to solve a two-dimensional Neumann-type boundary value problem, which is solved here for rectangular and isosceles triangular cross section of the channel. From the equation for a channel of rectangular cross section, we recover the equation derived by Kakutani and Matsuuchi[3] by appropriate passage to the limit.

II. BASIC EQUATIONS

The undisturbed free surface is taken as the $z = 0$ plane, the x axis is directed along the channel, and the z axis is vertical and intersects the lowest portion of the channel surface. Figure 1 shows the vertical section of the channel, which is perpendicular to the x axis.

Let S be the cross-sectional area of the channel filled with water at equilibrium, l be the width of equilibrium free surface, $z = f(y)$ be the equation of the curve of intersection of the channel surface with the $x = 0$ plane, g be the acceleration caused by gravity, and ρ be the constant density of water. We introduce dimensionless variables as follows:

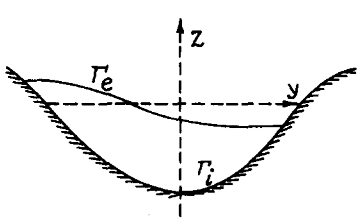

FIG. 1. Channel cross section.

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$$(x,y,z) = (x',y',z')/h, \quad t = t'(g/h)^{1/2},$$
$$\mathbf{u} = (u,v,w) = (u',v',w')/c_0, \quad p = p'/(h\rho g),$$
$$A = A'/h^2, \quad Q = Q'/(h^2 c_0), \quad \zeta = \zeta'/h,$$

where primed quantities are dimensional variables. Here $h = S/l$ is the mean depth and $c_0 = \sqrt{gh}$ is the linear phase velocity.

The Navier–Stokes equation and equation of continuity in these dimensionless variables are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + (0,0,1) = \frac{1}{R}\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

where $R = (c_0/h)/\nu$ is the Reynold's number, $c_0 = \sqrt{gh}$ the characteristic velocity, and ν the kinematic coefficient of viscosity.

The equation of continuity can also be written as

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \tag{3}$$

where $A(x,t)$ is the cross-sectional area of the channel filled with water and

$$Q(x,t) = \iint_A u\, ds \tag{4}$$

is the total flow along the channel at any instant.

The boundary conditions to be satisfied at the free surface are the following:

$$w - \frac{\partial \zeta}{\partial t} = u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} \quad \text{at } z = \zeta, \tag{5}$$

$$\begin{aligned} p - p_0 = \frac{2}{R}\bigg[&\left(\frac{\partial \zeta}{\partial x}\right)^2\frac{\partial u}{\partial x} + \left(\frac{\partial \zeta}{\partial y}\right)^2\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\ &- \frac{\partial \zeta}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \frac{\partial \zeta}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ &+ \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\bigg] \\ &\times\left[1 + \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]^{-1}, \quad \text{at } z = \zeta, \end{aligned} \tag{6}$$

where $z = \zeta(x,y,t)$ is the equation of the disturbed free surface and p_0 is the constant atmospheric pressure. Equation (6) corresponds to the continuity of normal stress at the free surface.

As we consider the existence of a boundary layer adjacent to the channel surface, the velocity should satisfy the no-slip condition at the channel surface, i.e.,

$$\mathbf{u} = 0 \tag{7}$$

at the channel surface.

The equation obtained by integrating the x component of Eq. (1) over the cross-sectional area $A(x,t)$ is

$$\begin{aligned} &\iint_A \left(\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} - \frac{1}{R}\frac{\partial^2 u}{\partial x^2}\right) dS \\ &\quad + \int_{\Gamma_e} \frac{u[\partial \zeta/\partial t + u(\partial \zeta/\partial x)]}{[1 + (\partial \zeta/\partial y)^2]^{1/2}}\, dl \\ &\quad = -\frac{1}{R}\int_{\Gamma_i}\frac{\partial u}{\partial n}\, dl - \frac{1}{R}\int_{\Gamma_e}\frac{\partial u}{\partial n}\, dl, \end{aligned} \tag{8}$$

where $\Gamma_e(x,t)$ and $\Gamma_i(x,t)$ are curves of intersection of the vertical plane through the point $(x,0,0)$ with the free surface and wet portion of the channel surface, respectively, i.e., the area $A(x,t)$ is bounded by the curves Γ_e and Γ_i, and ∂n is an element of inward drawn normal to the channel surface or free surface. (See Fig. 1.) In writing (8) we have used Eq. (2) and boundary conditions (5) and (7).

We make the following usual coordinate stretching to derive an equation for weakly nonlinear long waves propagating along the channel:

$$\xi = \epsilon^{1/2}(x - t), \quad \tau = \epsilon^{3/2}t, \tag{9}$$

where ϵ is a measure of smallness of k^2, i.e., a measure of weakness of dispersion.

In terms of ξ and τ, Eqs. (8), (1)–(3), and boundary conditions (5) and (6) assume the following form:

$$\begin{aligned} &\iint_A \left(-\epsilon^{1/2}\frac{\partial u}{\partial \xi} + \epsilon^{3/2}\frac{\partial u}{\partial \tau} + 2\epsilon^{1/2}u\frac{\partial u}{\partial \xi} + \epsilon^{1/2}\frac{\partial p}{\partial \xi} - \frac{\epsilon^{7/2}}{R}\frac{\partial^2 u}{\partial \xi^2}\right) dS \\ &\quad + \int_{\Gamma_e}\frac{u[-\epsilon^{1/2}(\partial \zeta/\partial \xi) + \epsilon^{3/2}(\partial \zeta/\partial \tau) + \epsilon^{1/2}u(\partial \zeta/\partial \xi)]}{[1 + (\partial \zeta/\partial y)^2]^{1/2}}\, dl = -\frac{\epsilon^{5/2}}{R}\int_{\Gamma_i}\frac{\partial u}{\partial n}\, dl - \frac{\epsilon^{5/2}}{R}\int_{\Gamma_e}\frac{\partial u}{\partial n}\, dl, \end{aligned} \tag{10}$$

$$-\epsilon^{1/2}\frac{\partial u}{\partial \xi} + \epsilon^{3/2}\frac{\partial u}{\partial \tau} + \epsilon^{1/2}u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \epsilon^{1/2}\frac{\partial p}{\partial \xi} = \frac{\epsilon^{7/2}}{R}\frac{\partial^2 u}{\partial \xi^2} + \frac{\epsilon^{5/2}}{R}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{11}$$

$$-\epsilon^{1/2}\frac{\partial v}{\partial \xi} + \epsilon^{3/2}\frac{\partial v}{\partial \tau} + \epsilon^{1/2}u\frac{\partial v}{\partial \xi} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial p}{\partial y} = \frac{\epsilon^{7/2}}{R}\frac{\partial^2 v}{\partial \xi^2} + \frac{\epsilon^{5/2}}{R}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{12}$$

$$-\epsilon^{1/2}\frac{\partial w}{\partial \xi} + \epsilon^{3/2}\frac{\partial w}{\partial \tau} + \epsilon^{1/2}u\frac{\partial w}{\partial \xi} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} + 1 = \frac{\epsilon^{7/2}}{R}\frac{\partial^2 w}{\partial \xi^2} + \frac{\epsilon^{5/2}}{R}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{13}$$

$$\epsilon^{1/2}\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{14}$$

$$\epsilon^{3/2}\frac{\partial A}{\partial \tau} - \epsilon^{1/2}\frac{\partial A}{\partial \xi} + \epsilon^{1/2}\frac{\partial Q}{\partial \xi} = 0, \tag{15}$$

$$(w)_0+\epsilon^{1/2}\frac{\partial\zeta}{\partial\xi}-\epsilon^{3/2}\frac{\partial\zeta}{\partial\tau}=-\zeta\left(\frac{\partial w}{\partial z}\right)_0+\epsilon^{1/2}(u)_0\frac{\partial\zeta}{\partial\xi}+(v)_0\frac{\partial\zeta}{\partial y}+\cdots, \tag{16}$$

$$\left[(p)_0+\zeta\left(\frac{\partial p}{\partial z}\right)_0+\cdots\right]-p_0=\frac{2\epsilon^{5/2}}{R}\left\{\left(\frac{\partial w}{\partial z}\right)_0+\zeta\left(\frac{\partial^2 w}{\partial z^2}\right)_0-\frac{\partial\zeta}{\partial y}\left[\left(\frac{\partial v}{\partial z}\right)_0+\left(\frac{\partial w}{\partial y}\right)_0\right]-\frac{\partial\zeta}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)_0+\left(\frac{\partial u}{\partial z}\right)_0\right]+\cdots\right\}, \tag{17}$$

where $(\quad)_0$ implies the value of the quantity inside the parentheses at $z=0$. In the last two equations only terms up to the second degree have been written. The smallness of $1/R$ has been assumed to be of the order $\epsilon^{5/2}$ following Kakutani and Matsuuchi.[3]

III. DERIVATION OF THE MODIFIED KdV EQUATION

We first write equations in both the inner and outer region in the lowest order and solve them.

A. Equations in the lowest order

1. Outer expansion

We seek expansion for the quantities in the outer region in the following form, in which quantities in the outer region have been designated by the subscript e.

$$\begin{bmatrix}\zeta_e\\ u_e\\ p_e\\ Q_e\\ A_e\end{bmatrix}=\begin{bmatrix}0\\ 0\\ p_0-z\\ 0\\ S/h^2\end{bmatrix}+\epsilon\begin{bmatrix}\zeta_e^{(1)}\\ u_e^{(1)}\\ p_e^{(1)}\\ Q_e^{(1)}\\ A_e^{(1)}\end{bmatrix}+\epsilon^2\begin{bmatrix}\zeta_e^{(2)}\\ u_e^{(2)}\\ p_e^{(2)}\\ Q_e^{(2)}\\ A_e^{(2)}\end{bmatrix}+\cdots, \tag{18}$$

$$\begin{bmatrix}v_e\\ w_e\end{bmatrix}=\epsilon^{3/2}\begin{bmatrix}v_e^{(1)}\\ w_e^{(1)}\end{bmatrix}+\epsilon^{5/2}\begin{bmatrix}v_e^{(2)}\\ w_e^{(2)}\end{bmatrix}+\cdots. \tag{19}$$

This expansion in (19) was first used by Peters.[10]

Substituting the expansion (18) and (19) into Eqs. (10)–(15) and boundary conditions (7), (16), and (17) and then equating various powers of ϵ, we get a sequence of equations for the quantities like $\zeta_e^{(n)}$ $(n=1,2,\ldots)$, etc. We now proceed to solve these sequence of equations.

Equations (12) and (13) give the following two equations in the lowest order, $O(\epsilon)$:

$$\frac{\partial p_e^{(1)}}{\partial y}=0,\quad \frac{\partial p_e^{(1)}}{\partial z}=0.$$

These two equations imply that

$$p_e^{(1)}=P(\xi,\tau),\quad \text{independent of } y,z. \tag{20}$$

In the lowest order, $O(\epsilon)$, Eq. (17) gives

$$\zeta_e^{(1)}=(p_e^{(1)})_0=P(\xi,\tau). \tag{21}$$

Equation (11) gives in the lowest order, $O(\epsilon^{3/2})$,

$$-\frac{\partial u_e^{(1)}}{\partial\xi}+\frac{\partial p_e^{(1)}}{\partial\xi}=0. \tag{22}$$

By the use of (20) and (21) this gives

$$\frac{\partial u_e^{(1)}}{\partial\xi}=\frac{\partial\zeta_e^{(1)}}{\partial\xi}. \tag{23}$$

Equation (15) gives in the lowest order, $O(\epsilon^{3/2})$, the same equation as (23).

Equations (14) and (16) give the following two equations in the lowest order, $O(\epsilon^{3/2})$:

$$\frac{\partial v_e^{(1)}}{\partial y}+\frac{\partial w_e^{(1)}}{\partial z}=-\frac{\partial u_e^{(1)}}{\partial\xi}, \tag{24}$$

$$(w_e^{(1)})_0=-\frac{\partial\zeta_e^{(1)}}{\partial\xi}. \tag{25}$$

2. Inner expansion

a. Boundary layer adjacent to the channel surface. In order to write equations in the boundary layer in the neighborhood of any point P on the channel surface, we make the following usual coordinate stretching:

$$Z=z_1/\epsilon,\quad Y=y_1, \tag{26}$$

where y_1,z_1 are Cartesian coordinates with respect to the tangent and normal to the curve Γ_i at P as coordinate axes. By the usual stretching in the boundary layer approximation we write

$$v_{y_1}=v_i,\quad v_{z_1}=\epsilon w_i, \tag{27}$$

where v_{y_1},v_{z_1} are components of $\mathbf{u}$ along Py_1 and Pz_1 axes.

With these transformations and choosing Px, Py_1, Pz_1 as coordinate axes, Eqs. (11)–(13) assume the following form:

$$-\epsilon^{1/2}\frac{\partial u_i}{\partial\xi}+\epsilon^{3/2}\frac{\partial u_i}{\partial\tau}+\epsilon^{1/2}u_i\frac{\partial u_i}{\partial\xi}+v_i\frac{\partial u_i}{\partial Y}+w_i\frac{\partial u_i}{\partial Z}+\epsilon^{1/2}\frac{\partial p_i}{\partial\xi}=\frac{\epsilon^{5/2}}{R}\left(\epsilon\frac{\partial^2 u_i}{\partial\xi^2}+\frac{\partial^2 u_i}{\partial Y^2}+\frac{1}{\epsilon^2}\frac{\partial^2 u_i}{\partial Z^2}\right), \tag{28}$$

$$-\epsilon^{1/2}\frac{\partial v_i}{\partial\xi}+\epsilon^{3/2}\frac{\partial v_i}{\partial\tau}+\epsilon^{1/2}u_i\frac{\partial v_i}{\partial\xi}+v_i\frac{\partial v_i}{\partial Y}+w_i\frac{\partial v_i}{\partial Z}+\frac{\partial p_i}{\partial Y}+\sin\alpha=\frac{\epsilon^{5/2}}{R}\left(\epsilon\frac{\partial^2 v_i}{\partial\xi^2}+\frac{\partial^2 v_i}{\partial Y^2}+\frac{1}{\epsilon^2}\frac{\partial^2 v_i}{\partial Z^2}\right), \tag{29}$$

$$-\epsilon^{3/2}\frac{\partial w_i}{\partial\xi}+\epsilon^{5/2}\frac{\partial w_i}{\partial\tau}+\epsilon^{3/2}u_i\frac{\partial w_i}{\partial\xi}+\epsilon v_i\frac{\partial w_i}{\partial Y}+\epsilon w_i\frac{\partial w_i}{\partial Z}+\frac{1}{\epsilon}\frac{\partial p_i}{\partial Z}+\cos\alpha=\frac{\epsilon^{5/2}}{R}\left(\epsilon^2\frac{\partial^2 w_i}{\partial\xi^2}+\epsilon\frac{\partial^2 w_i}{\partial Y^2}+\frac{1}{\epsilon}\frac{\partial^2 w_i}{\partial Z^2}\right), \tag{30}$$

where α is the angle made by the normal to the channel surface at P with the vertical. The quantities in the inner region have been designated by a subscript i.

We now make the following perturbation expansion for the quantities inside the boundary layer:

$$\begin{bmatrix} u_i \\ p_i \end{bmatrix} = \begin{bmatrix} 0 \\ p_0 + h_P - Y\sin\alpha - \epsilon Z\cos\alpha \end{bmatrix} + \epsilon \begin{bmatrix} u_i^{(1)} \\ p_i^{(1)} \end{bmatrix} + \epsilon^2 \begin{bmatrix} u_i^{(2)} \\ p_i^{(2)} \end{bmatrix} + \dots, \tag{31}$$

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \epsilon^{3/2} \begin{bmatrix} v_i^{(1)} \\ w_i^{(1)} \end{bmatrix} + \epsilon^{5/2} \begin{bmatrix} v_i^{(2)} \\ w_i^{(2)} \end{bmatrix} + \dots, \tag{32}$$

where h_P is the depth of P below the undisturbed free surface.

Substituting these expansions in (28)–(30), we get the following equations in the lowest order, respectively, in $O(\epsilon^{3/2})$, $O(\epsilon)$ and $O(1)$:

$$-\frac{\partial u_i^{(1)}}{\partial \xi} + \frac{\partial p_i^{(1)}}{\partial \xi} = \frac{1}{R}\frac{\partial^2 u_i^{(1)}}{\partial Z^2}, \tag{33}$$

$$\frac{\partial p_i^{(1)}}{\partial Y} = 0, \tag{34}$$

$$\frac{\partial p_i^{(1)}}{\partial Z} = 0. \tag{35}$$

The last two equations imply that $p_i^{(1)}$ is independent of Y,Z. Therefore $p_i^{(1)} = p_i^{(1)}(\xi,\tau)$. Since $p_i^{(1)}$ and $p_e^{(1)}$ should satisfy the following matching conditions,

$$\lim_{Z\to\infty} p_i^{(1)} = \lim_{(y,z)\to P} p_e^{(1)},$$

we have

$$p_i^{(1)} = p_e^{(1)}(\xi,\tau). \tag{36}$$

Therefore, by the use of (22) we get

$$\frac{\partial p_i^{(1)}}{\partial \xi} = \frac{\partial u_e^{(1)}}{\partial \xi}. \tag{37}$$

Substituting this expansion for $\partial p_i^{(1)}/\partial \xi$ in (33) we get

$$\frac{1}{R}\frac{\partial^2 u_i^{(1)}}{\partial Z^2} = -\frac{\partial u_i^{(1)}}{\partial \xi} + \frac{\partial u_e^{(1)}}{\partial \xi}. \tag{38}$$

The boundary conditions on $u_i^{(1)}$ are

$$u_i^{(1)} = 0 \quad \text{at } Z = 0, \qquad u_i^{(1)} \to u_e^{(1)} \quad \text{as } Z \to \infty. \tag{39}$$

It should be kept in mind that $u_e^{(1)}$ is independent of Y,Z.

If we set

$$u_i^{(1)} = F + u_e^{(1)}, \tag{40}$$

then F satisfies the following equation and boundary conditions

$$\frac{\partial^2 F}{\partial Z^2} = -R\frac{\partial F}{\partial \xi}, \tag{41}$$

$$F \to 0 \quad \text{as } Z \to \infty, \qquad F = -u_e^{(1)} \quad \text{at } Z = 0. \tag{42}$$

The quantity that we need in the present analysis is the viscous stress,

$$T = \frac{1}{R}\left(\frac{\partial u_i^{(1)}}{\partial Z}\right)_{Z=0} = \frac{1}{R}\left(\frac{\partial F}{\partial Z}\right)_{Z=0}, \tag{43}$$

experienced by the channel surface.

Solving (41) and (42) by the use of Fourier transform with respect to ξ and substituting this solution in (43) we get

$$T = (2R)^{-1/2}\int_{-\infty}^{\infty} |k|^{1/2}[1 - i\,\mathrm{sgn}(k)]\bar{u}_e^{(1)} e^{ik\xi}\,dk, \tag{44}$$

where $\bar{u}_e$ is the Fourier transform of $u_e^{(1)}$ with respect to ξ given by

$$\bar{u}_e = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_e^{(1)} e^{-ik\xi}\,d\xi.$$

Applying the convolution theorem, (44) can be written as

$$T = -\frac{1}{2}(\pi R)^{-1/2}\int_{-\infty}^{\infty} \frac{\partial u_e^{(1)}}{\partial \xi'}\frac{1 - \mathrm{sgn}(\xi - \xi')}{|\xi - \xi'|^{1/2}}\,d\xi'. \tag{45}$$

This expression shows that T is independent of the position of the point P on the channel surface.

b. Boundary layer adjacent to the free surface. The equation, which corresponds to the vanishing of tangential stress along the tangent to the free surface parallel to the xz plane, is

$$(1 - \zeta_x^2)\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - \zeta_y\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - 2\zeta_x\left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right) - \zeta_x\zeta_y\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = 0$$

$$\text{at } z = \zeta. \tag{46}$$

In order to write equations in the boundary layer adjacent to the free surface in the neighborhood of any point P $\{x_0,y_0,\zeta(x_0,y_0,t)\}$ on the free surface, we make the following coordinate stretching:

$$X = x_1, \quad Y = y_1, \quad Z = z_1/\epsilon, \tag{47}$$

where x_1,y_1,z_1 are the coordinates of any point with respect to PT_1, PT_2, PN as coordinate axes. Here PN is the normal to the free surface at P having direction cosines $(l,m,n) = (-\zeta_x, -\zeta_y, 1)/(1 + \zeta_x^2 + \zeta_y^2)^{1/2}$; PT_1 is the tangent to the free surface at P parallel to the xz plane having direction cosines $(-1, 0, -\zeta_x)/(1 + \zeta_x^2)^{1/2}$; PT_2 is the tangent to the free surface at P perpendicular to both PN and PT_1.

Neglecting order ϵ terms we find that in the lowest order, PT_1, PT_2, PN are directed along the three-coordinate axes $0x$, $0y$, $0z$. The coordinate stretching (47) then becomes the following in the lowest order:

$$X = x - x_0, \quad Y = y - y_0, \quad Z = z/\epsilon. \tag{48}$$

If u_1,v_1,w_1 are the components of velocity along the coordinate axes PT_1, PT_2, PN, then in the lowest order we have

$$u_1 = u, \quad v_1 = v, \quad w_1 = w. \tag{49}$$

By usual stretching in the boundary layer approximation we write

$$u_1 = u = u_i, \quad v_1 = v = v_i, \quad w_1 = w = \epsilon w_i. \tag{50}$$

For u_i,v_i,w_i, we make the following usual perturbation expansion:

$$u_i = \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \dots, \tag{51}$$

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \epsilon^{3/2}\begin{bmatrix} v_i^{(1)} \\ w_i^{(1)} \end{bmatrix} + \epsilon^{5/2}\begin{bmatrix} v_i^{(2)} \\ w_i^{(2)} \end{bmatrix} + \dots .$$

By the use of these perturbation expansions, Eq. (46) gives the following in the lowest order:

$$\frac{\partial u_i^{(1)}}{\partial Z}=0 \quad \text{at } P. \tag{52}$$

In the integral, $\int_{\Gamma_e}(\partial u/\partial n)dl$, appearing in Eq. (10), the integrand at P becomes the following in the lowest order in view of the above:

$$\left(\frac{\partial u}{\partial n}\right)_{\text{at } P}=\left(\frac{\partial u_i^{(1)}}{\partial Z}\right)_{\text{at } P},$$

which is zero according to (52). Therefore we have

$$\int_{\Gamma_e}\frac{\partial u}{\partial n}\,dl=0 \tag{53}$$

in the lowest order, so we need not proceed further with the analysis in this boundary layer.

B. Next higher order equations

In the next higher order, $O(\epsilon^2)$, Eqs. (12) and (13) give equations

$$-\frac{\partial v_e^{(1)}}{\partial \xi}+\frac{\partial p_e^{(2)}}{\partial y}=0,\quad -\frac{\partial w_e^{(1)}}{\partial \xi}+\frac{\partial p_e^{(2)}}{\partial z}=0. \tag{54}$$

Eliminating $p_e^{(2)}$ between these two equations we obtain

$$\frac{\partial}{\partial \xi}\left(\frac{\partial v_e^{(1)}}{\partial z}-\frac{\partial w_e^{(1)}}{\partial y}\right)=0, \tag{55}$$

which suggests that we can assume

$$v_e^{(1)}=\frac{\partial \phi^{(1)}}{\partial y},\quad w_e^{(1)}=\frac{\partial \phi^{(1)}}{\partial z}. \tag{56}$$

Introducing these in (24) and (25) and using the relation (23) we get

$$\frac{\partial^2\phi^{(1)}}{\partial y^2}+\frac{\partial^2\phi^{(1)}}{\partial z^2}=-\frac{\partial \zeta_e^{(1)}}{\partial \xi}, \tag{57}$$

$$\left(\frac{\partial \phi^{(1)}}{\partial z}\right)_0=-\frac{\partial \zeta_e^{(1)}}{\partial \xi}. \tag{58}$$

As $\mathbf{u}_e^{(1)}$ is the inviscid part of the outer velocity $\mathbf{u}_e$, it should satisfy the following inviscid boundary condition at the channel surface, which is consistent with the condition (7):

$$\mathbf{u}_e^{(1)}\cdot\mathbf{n}=0 \quad \text{or} \quad \frac{\partial \phi^{(1)}}{\partial n}=0 \tag{59}$$

on the channel surface.

Equations (57)–(59) suggest that we can take

$$\phi^{(1)}=-\frac{\partial \zeta_e^{(1)}}{\partial \xi}\psi(y,z)+x(\xi,\tau), \tag{60}$$

where ψ satisfies the Poisson's equation,

$$\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=1, \tag{61}$$

at every point in S, and Numann-type boundary conditions,

$$\left(\frac{\partial \psi}{\partial z}\right)_0=1,$$

$$\frac{\partial \psi}{\partial n}=0 \quad \text{on the channel surface.} \tag{62}$$

Expressing $v_e^{(1)}$ and $w_e^{(1)}$ in Eq. (49) in terms of $\phi^{(1)}$ as given by (56), we find that the expression $\partial\phi^{(1)}/\partial\xi+p_e^{(2)}$ is independent of y,z, so we take

$$-\frac{\partial \phi^{(1)}}{\partial \xi}+p_e^{(2)}=V(\xi,\tau).$$

Substituting here for $\phi^{(1)}$ given by (60) we get

$$p_e^{(2)}=-\frac{\partial^2\zeta_e^{(1)}}{\partial \xi^2}+\frac{\partial x}{\partial \xi}+V. \tag{63}$$

The two equations obtained by the operations,

$$\frac{\partial}{\partial y}[x-\text{comp. of }(1)]-\frac{\partial}{\partial x}[y-\text{comp. of }(1)],$$

$$\frac{\partial}{\partial z}[x-\text{comp. of }(1)]-\frac{\partial}{\partial x}[z-\text{comp. of }(1)]$$

give the following two equations in order $\epsilon^{5/2}$:

$$\frac{\partial}{\partial y}\left(\frac{\partial u_e^{(2)}}{\partial \xi}-\frac{\partial^2\phi^{(1)}}{\partial \xi^2}\right)=0,\quad \frac{\partial}{\partial z}\left(\frac{\partial u_e^{(2)}}{\partial \xi}-\frac{\partial^2\phi^{(1)}}{\partial \xi^2}\right)=0.$$

These two equations imply that the expression $\partial u_e^{(2)}/\partial\xi-\partial^2\phi^{(1)}/\partial\xi^2$ is independent of y and z. So we can assume

$$\frac{\partial u_e^{(2)}}{\partial \xi}=\frac{\partial^2\phi^{(1)}}{\partial \xi^2}+U(\xi,\tau). \tag{64}$$

Order ϵ^2 equation of (17) and Eq. (63) give

$$\zeta_e^{(2)}=-\frac{\partial^2\zeta_e^{(1)}}{\partial \xi^2}\psi(y,0)+\frac{\partial x}{\partial \xi}+V. \tag{65}$$

It can be shown that the expressions for A and Q are

$$A=\frac{S}{h^2}+\epsilon\frac{l}{h}\zeta_e^{(1)}+\epsilon^2\int_{-l_1}^{l_2}dy\,\zeta_e^{(2)}(y,0)+\frac{\epsilon^2}{2}\zeta_e^{(1)2}\left(\frac{1}{f'(l_2)}-\frac{1}{f'(-l_1)}\right)+O(\epsilon^3), \tag{66}$$

$$Q=\epsilon\frac{S}{h^2}u_e^{(1)}+\epsilon^2\frac{l}{h}u_e^{(1)}\zeta_e^{(1)}+\epsilon^2\iint_{S/h^2}u_e^{(2)}\,ds+O(\epsilon^3), \tag{67}$$

where $-l_1$ and l_2 are the y coordinates of the points of intersection of the y axis with the channel surface.

By the use of these expressions we get the following equations in order $\epsilon^{5/2}$ from Eqs. (10) and (15):

$$\frac{\partial u_e^{(1)}}{\partial \tau}+u_e^{(1)}\frac{\partial \zeta_e^{(1)}}{\partial \xi}-U+\frac{\partial V}{\partial \xi}=-\frac{L}{l}T, \tag{68}$$

$$\frac{\partial \zeta_e^{(1)}}{\partial \tau}+\frac{h}{l}\frac{\partial^3\zeta_e^{(1)}}{\partial \xi^3}\left(\int_{-l_1}^{l_2}\psi(y,0)dy-\iint_{S/h^2}\psi\,dy\,dz\right)-\frac{h}{l}\left(\frac{1}{f'(l_2)}-\frac{1}{f'(-l_1)}-\frac{l}{h}\right)\zeta_e^{(1)}\frac{\partial \zeta_e^{(1)}}{\partial \xi}+u_e^{(1)}\frac{\partial \zeta_e^{(1)}}{\partial \xi}-\frac{\partial V}{\partial \xi}+U=0, \tag{69}$$

where L is the perimeter of the arc Γ_i in the unperturbed state.

Eliminating U or $\partial V/\partial\xi$ between (68) and (69), we get the following equation after setting $u_e^{(1)}=\zeta_e^{(1)}$ as can be obtained[3] from (23) and putting the expression for T given by (45):

$$\frac{\partial \zeta_e^{(1)}}{\partial \tau} + \alpha_1 \zeta_e^{(1)} \frac{\partial \zeta_e^{(1)}}{\partial \xi} + \alpha_2 \frac{\partial^3 \zeta_e^{(1)}}{\partial \xi^3} = \frac{\alpha_3}{4\sqrt{\pi R}} \int_{-\infty}^{\infty} \frac{\partial \zeta_e^{(1)}}{\partial \xi'} \frac{1 - \mathrm{sgn}(\xi - \xi')}{|\xi - \xi'|^{1/2}} d\xi', \tag{70}$$

where

$$\alpha_1 = \tfrac{3}{2} - (h/2l)(\tan\theta_1 + \tan\theta_2),$$
$$\alpha_2 = (h/2l)\left(\int_{-l_1}^{l_2} dy\, \psi(y,0) - \iint_{S/h^2} \psi\, dy\, dz\right), \tag{71}$$
$$\alpha_3 = L/l,$$

and θ_1 and θ_2 are angles made with the vertical by the tangents to the curve $z = f(y)$ at the points of its intersection with the y axis.

Equation (70) is the KdV equation modified by viscosity. For a channel of rectangular cross section, the solution of (61) and (62) is easily found to be

$$\psi = \tfrac{1}{2} z^2 + z. \tag{72}$$

So for a rectangular channel we get

$$\alpha_1 = \tfrac{3}{2}, \quad \alpha_2 = \tfrac{1}{6}, \quad \alpha_3 = (l + 2h)/l, \tag{73}$$

$\alpha_3 \to 1$ as $l \to \infty$. Proceeding to this limit, we recover the equation (4.33) obtained by Kakutani and Matsuuchi.[3]

For an isosceles triangular cross section of semivertical angle θ, the solution for ψ as given by Eqs. (61) and (62) is

$$\psi = \tfrac{1}{4}(z+2)^2 + \tfrac{1}{4} y^2. \tag{74}$$

With this value of ψ we get the following expressions for α_1, α_2, and α_3:

$$\alpha_1 = \tfrac{5}{4}, \quad \alpha_2 = \tfrac{1}{3}, \quad \alpha_3 = 1/\sin\theta. \tag{75}$$

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