A higher-order nonlinear evolution equation for broader bandwidth gravity waves in deep water

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From Zakharov's integral equation a nonlinear evolution equation for broader bandwidth gravity waves in deep water is obtained, which is one order higher than the corresponding equation derived by Trulsen and Dysthe ["A modified nonlinear Schrödinger equation for broader bandwidth gravity waves on deep water," Wave Motion **24**, 281 (1996)]. The instability regions in the perturbed wave-number space for a uniform Stokes wave obtained from this equation are in surprisingly good agreement with the exact results obtained by McLean *et al.* ["Three dimensional instability of finite amplitude water waves," Phys. Rev. Lett. **46**, 817 (1981)].
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I. INTRODUCTION

A narrow bandwidth weakly nonlinear surface gravity wave packet in deep water, where the order of smallness of wave steepness and bandwidth are same, is described by a nonlinear Schrödinger equation. Such type of equations for surface gravity wave packet in deep water were derived by Zakharov,[1] Hasimoto and Ono,[2] and Davey.[3] This nonlinear Schrödinger equation is the lowest order nonlinear evolution equation and the order of this equation is ϵ^3, where ϵ is the order of smallness of wave steepness. A modification of this nonlinear Schrödinger equation to fourth-order accuracy called the modified nonlinear Schrödinger (MNLS) equation was derived by Dysthe[4] for a surface gravity wave packet in deep water. Though the stability analysis of a uniform Stokes wave obtained from the MNLS equation gives better results compared to the same obtained from NLS equation, limitations in bandwidth limit the applicability of NLS and MNLS equations to ocean waves. In view of this Trulsen and Dysthe[5] derived an equation relaxing this bandwidth constraint. In this equation the order of bandwidth has been taken as $\epsilon^{1/2}$ while keeping the same accuracy in nonlinearity. This equation remains correct up to order $\epsilon^{7/2}$ terms. The instability regions of a uniform Stokes wave in the perturbed wave-number space obtained from this equation are in better agreement with the exact results of McLean *et al.*[6] Recently Trulsen *et al.*[7] modelled the evolution of weakly nonlinear waves in deep water by the cubic nonlinear Schrödinger equation replacing dispersive terms by full linear dispersive terms given in an integral form. The stability analysis for Stokes wave made from this equation agrees well with the exact results of McLean *et al.*[6]

In the present paper we derive an equation which is one order higher than the equation derived by Trulsen and Dysthe.[5] Consequently the equation remains correct up to ϵ^4 terms, and it contains dispersive and nonlinear terms correct up to sixth and second orders, respectively, in spectral width. The equation has been derived from Zakharov's integral equation[1,7] following Stiassnie,[9] who first showed that the MNLS equation obtained by Dysthe[4] can be derived from Zakharov's integral equation and the equation thus derived does not remain coupled with the velocity potential for wave-induced slow motion. It is to be noted here that the coupled evolution equations of Dysthe[4] is probably more convenient for numerical simulations. From the higher-order nonlinear evolution equation an instability condition is obtained for a uniform Stokes wave. With the help of this instability condition instability regions in the perturbed wave-number space are shown in figures for some different values of wave steepness. The instability regions obtained are in surprisingly good agreement with the regions obtained by McLean *et al.*[6]

II. DERIVATION OF THE HIGHER ORDER EVOLUTION EQUATION

The Zakharov's integral equation is

$$i\frac{\partial B(\boldsymbol{k},t)}{\partial t} = \int\int\int_{-\infty}^{\infty} T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)B^*(\boldsymbol{k}_1,t)B(\boldsymbol{k}_2,t)B(\boldsymbol{k}_3,t) \times \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\exp\{i[\omega(\boldsymbol{k})+\omega(\boldsymbol{k}_1) - \omega(\boldsymbol{k}_2)-\omega(\boldsymbol{k}_3)]t\}d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3, \tag{2.1}$$

where $B(\boldsymbol{k},t)$ is related to the free-surface elevation $\zeta(\boldsymbol{x},t)$ by

$$\zeta(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{|\boldsymbol{k}|}{2\omega(\boldsymbol{k})}\right]^{1/2}\{B(\boldsymbol{k},t)\exp\{i[\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t]\} + \text{c.c.}\}d\boldsymbol{k}. \tag{2.2}$$

Here c.c. denotes complex conjugate, $\boldsymbol{k}=(k,l)$ is the wave vector, $\boldsymbol{x}=(x,y)$ is the horizontal spatial vector, and ω is the linearized wave frequency related to $\boldsymbol{k}$ through the linear dispersion relation, $\omega(\boldsymbol{k})=[g(|\boldsymbol{k}|)]^{1/2}$, g being the acceleration due to gravity. $T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ is a scalar function given in the Appendix of Crawford *et al.*[8]

In order to derive an equation which is one order higher than the evolution equation derived by Trulsen and Dysthe,[5]

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the wave energy is supposed to be centered around $\boldsymbol{k}=\boldsymbol{k}_0=(k_0,0)$ and the wave vectors $\boldsymbol{k},\boldsymbol{k}_j, j=1,2,3$ are written as

$$\boldsymbol{k}=\boldsymbol{k}_0+\boldsymbol{\chi}, \quad \boldsymbol{k}_j=\boldsymbol{k}_0+\boldsymbol{\chi}_j, \quad \boldsymbol{\chi}=(\chi,\lambda), \quad \boldsymbol{\chi}_j=(\chi_j,\lambda_j),$$

and it is assumed that the spectral widths $|\boldsymbol{\chi}|/k_0$, $|\boldsymbol{\chi}_j|/k_0=O(\epsilon^{1/2})$, where ϵ is the order of smallness of wave steepness.

Now introducing a new variable $A(\boldsymbol{\chi},t)$ defined by

$$A(\boldsymbol{\chi},t)=B(\boldsymbol{k},t)\exp\{-i[\omega(\boldsymbol{k})-\omega(\boldsymbol{k}_0)]t\}, \tag{2.3}$$

where $\boldsymbol{k}=\boldsymbol{k}_0+\boldsymbol{\chi}$, the evolution equation (2.1) can be written as

$$i\frac{\partial A(\boldsymbol{\chi},t)}{\partial t}-[\omega(\boldsymbol{k})-\omega(\boldsymbol{k}_0)]A(\boldsymbol{\chi},t)=\int\int\int_{-\infty}^{\infty}T(\boldsymbol{k}_0+\boldsymbol{\chi},\boldsymbol{k}_0+\boldsymbol{\chi}_1,\boldsymbol{k}_0+\boldsymbol{\chi}_2,\boldsymbol{k}_0+\boldsymbol{\chi}_3)A^*(\boldsymbol{\chi}_1,t)A(\boldsymbol{\chi}_2,t)\times A(\boldsymbol{\chi}_3,t)\delta(\boldsymbol{\chi}+\boldsymbol{\chi}_1-\boldsymbol{\chi}_2-\boldsymbol{\chi}_3)d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 d\boldsymbol{\chi}_3. \tag{2.4}$$

Also $\zeta(\boldsymbol{x},t)$ in this new variable becomes

$$\zeta(\boldsymbol{x},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{|\boldsymbol{k}_0+\boldsymbol{\chi}|}{4g}\right)^{1/4}\{A(\boldsymbol{\chi},t)\exp i[(\boldsymbol{k}_0+\boldsymbol{\chi})\cdot\boldsymbol{x}-\omega(\boldsymbol{k}_0)t]+\text{c.c.}\}d\boldsymbol{\chi}. \tag{2.5}$$

Taylor expanding the factor $|\boldsymbol{k}_0+\boldsymbol{\chi}|^{1/4}$ and keeping terms up to third order in spectral width, $\zeta(\boldsymbol{x},t)$ can finally be expressed as

$$\zeta(\boldsymbol{x},t)=\frac{1}{2}\big[a(\boldsymbol{x},t)\exp\{i[\boldsymbol{k}_0\cdot\boldsymbol{x}-\omega(\boldsymbol{k}_0)t]\}+a^*(\boldsymbol{x},t)\exp\{-i[\boldsymbol{k}_0\cdot\boldsymbol{x}-\omega(\boldsymbol{k}_0)t]\}\big], \tag{2.6}$$

where

$$a(\boldsymbol{x},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}a(\boldsymbol{\chi},t)e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}}d\boldsymbol{\chi}, \tag{2.7}$$

$$a(\boldsymbol{\chi},t)=\left[\frac{2\omega(\boldsymbol{k}_0)}{g}\right]^{1/2}\left(1+\frac{\chi}{4k_0}-\frac{3\chi^2}{32k_0^2}+\frac{\lambda^2}{8k_0^2}\right)A(\boldsymbol{\chi},t). \tag{2.8}$$

Multiplying (2.4) by $(1/2\pi)\{[2\omega(\boldsymbol{k}_0)/g]\}^{1/2}[1+(\chi/4k_0)-(3\chi^2/32k_0^2)+(\lambda^2/8k_0^2)]e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}}$ we integrate with respect to $\boldsymbol{\chi}$, in which we replace $A(\boldsymbol{\chi},t)$ by $a(\boldsymbol{\chi},t)$ by the use of the relation (2.8). Performing integration with respect to $\boldsymbol{\chi}$ in the first term on the left-hand side and in the term on the right-hand side, we get

$$\begin{aligned}
&i\frac{\partial a(\boldsymbol{x},t)}{\partial t}-\frac{1}{2\pi}\int_{-\infty}^{\infty}[\omega(\boldsymbol{k}_0+\boldsymbol{\chi})-\omega(\boldsymbol{k}_0)]a(\boldsymbol{\chi},t)e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}}d\boldsymbol{\chi}\\
&=\left[\frac{g}{2\omega(\boldsymbol{k}_0)}\right]\frac{1}{2\pi}\int\int\int_{-\infty}^{\infty}T(\boldsymbol{k}_0+\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1,\boldsymbol{k}_0+\boldsymbol{\chi}_1,\boldsymbol{k}_0+\boldsymbol{\chi}_2,\boldsymbol{k}_0+\boldsymbol{\chi}_3)\\
&\times\left[1+\frac{1}{4k_0}(\chi_2+\chi_3-\chi_1)-\frac{3}{32k_0^2}(\chi_2+\chi_3-\chi_1)^2+\frac{1}{8k_0^2}(\lambda_2+\lambda_3-\lambda_1)^2\right]\\
&\times\left[1+\frac{1}{4k_0}\chi_1-\frac{3}{32k_0^2}\chi_1^2+\frac{1}{8k_0^2}\lambda_1^2\right]^{-1}\cdot\left[1+\frac{1}{4k_0}\chi_2-\frac{3}{32k_0^2}\chi_2^2+\frac{1}{8k_0^2}\lambda_2^2\right]^{-1}\\
&\times\left[1+\frac{1}{4k_0}\chi_3-\frac{3}{32k_0^2}\chi_3^2+\frac{1}{8k_0^2}\lambda_3^2\right]^{-1}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)a(\boldsymbol{\chi}_3,t)\\
&\times\exp[i(\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 d\boldsymbol{\chi}_3.
\end{aligned} \tag{2.9}$$

Now expanding the product of the four factors appearing under the integral sign on the right-hand side of the above equation keeping terms up to second order in spectral width, (2.9) can be written as

$$\begin{aligned}
&i\frac{\partial a(\boldsymbol{x},t)}{\partial t}-\frac{1}{2\pi}\int_{-\infty}^{\infty}[\omega(\boldsymbol{k}_0+\boldsymbol{\chi})-\omega(\boldsymbol{k}_0)]a(\boldsymbol{\chi},t)e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}}d\boldsymbol{\chi}\\
&=\left[\frac{g}{2\omega(\boldsymbol{k}_0)}\right]\frac{1}{2\pi}\int\int\int_{-\infty}^{\infty}T(\boldsymbol{k}_0+\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1,\boldsymbol{k}_0+\boldsymbol{\chi}_1,\boldsymbol{k}_0+\boldsymbol{\chi}_2,\boldsymbol{k}_0+\boldsymbol{\chi}_3)\\
&\times\left[1-\frac{1}{2k_0}\chi_1+\frac{1}{8k_0^2}(\chi_1^2-2\chi_2\chi_3+2\chi_3\chi_1+2\chi_1\chi_2+2\lambda_2\lambda_3-2\lambda_3\lambda_1-2\lambda_1\lambda_2)\right]\\
&\times a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)a(\boldsymbol{\chi}_3,t)\exp\{i(\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}\}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 d\boldsymbol{\chi}_3.
\end{aligned} \tag{2.10}$$

Next expanding T in Taylor's series and keeping terms up to second order in spectral width, we get

$$T=\frac{k_0^3}{4\pi^2}\left\{1+\frac{3}{2k_0}(\chi_2+\chi_3)-\frac{1}{8k_0^2}(3\chi_1^2-5\chi_2^2-5\chi_3^2-11\chi_2\chi_3-3\chi_3\chi_1-3\chi_1\chi_2)-\frac{1}{2k_0^2}(\lambda_1^2-2\lambda_2\lambda_3-\lambda_3\lambda_1-\lambda_1\lambda_2)\right.$$
$$-\frac{1}{2k_0}\left[\frac{(\chi_3-\chi_1)^2}{|\boldsymbol{\chi}_3-\boldsymbol{\chi}_1|}+\frac{(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}\right]-\frac{1}{2k_0^2}(\chi_2+\chi_3)\left[\frac{(\chi_3-\chi_1)^2}{|\boldsymbol{\chi}_3-\boldsymbol{\chi}_1|}+\frac{(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}\right]-\frac{1}{2k_0^2}(\lambda_2+\lambda_3)\left[\frac{(\chi_3-\chi_1)(\lambda_3-\lambda_1)}{|\boldsymbol{\chi}_3-\boldsymbol{\chi}_1|}\right.$$
$$\left.\left.+\frac{(\chi_1-\chi_2)(\lambda_1-\lambda_2)}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}\right]-\frac{1}{8k_0^2}\left[\frac{(\chi_3-\chi_1)^4}{|\boldsymbol{\chi}_3-\boldsymbol{\chi}_1|^2}+\frac{(\chi_1-\chi_2)^4}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|^2}\right]\right\}. \tag{2.11}$$

Taylor expanding $\omega(\boldsymbol{k}_0+\boldsymbol{\chi})-\omega(\boldsymbol{k}_0)$ appearing on the left-hand side of (2.10) in powers of $\boldsymbol{\chi}$ and keeping terms up to sixth order in spectral width we get

$$\omega(\boldsymbol{k}_0+\boldsymbol{\chi})-\omega(\boldsymbol{k}_0)$$
$$=\sqrt{gk_0}\left[\frac{1}{2k_0}\chi-\frac{1}{8k_0^2}(\chi^2-2\lambda^2)+\frac{1}{16k_0^3}(\chi^3-6\chi\lambda^2)-\frac{1}{128k_0^4}(5\chi^4-60\chi^2\lambda^2+12\lambda^4)\right.$$
$$\left.+\frac{1}{256k_0^5}(7\chi^5-140\chi^3\lambda^2+84\chi\lambda^4)-\frac{1}{1024k_0^6}(21\chi^6-630\chi^4\lambda^2+756\chi^2\lambda^4-56\lambda^6)\right]. \tag{2.12}$$

By the use of the expansions (2.11) and (2.12) the evolution equation (2.10) can be written as follows where terms up to $O(\epsilon^4)$ have been retained, assuming that the order of wave steepness is ϵ and the same of spectral width is $\epsilon^{1/2}$:

$$i\frac{\partial a(\boldsymbol{x},t)}{\partial t}-\frac{\omega_0}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2k_0}\chi-\frac{1}{8k_0^2}(\chi^2-2\lambda^2)+\frac{1}{16k_0^3}(\chi^3-6\chi\lambda^2)-\frac{1}{128k_0^4}(5\chi^4-60\chi^2\lambda^2+12\lambda^4)+\frac{1}{256k_0^5}(7\chi^5-140\chi^3\lambda^2\right.$$
$$\left.+84\chi\lambda^4)-\frac{1}{1024k_0^6}(21\chi^6-630\chi^4\lambda^2+756\chi^2\lambda^4-56\lambda^6)\right]a(\boldsymbol{\chi},t)e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}}d\boldsymbol{\chi}=\frac{gk_0^3}{16\pi^3\omega_0}\int\int\int_{-\infty}^{\infty}\left\{1-\frac{1}{2k_0}(\chi_1-3\chi_2-3\chi_3)\right.$$
$$-\frac{1}{8k_0^2}(2\chi_1^2-5\chi_2^2-5\chi_3^2-9\chi_2\chi_3+\chi_3\chi_1+\chi_1\chi_2)-\frac{1}{4k_0^2}(2\lambda_1^2-5\lambda_2\lambda_3-\lambda_3\lambda_1-\lambda_1\lambda_2)-\frac{1}{2k_0}\left[\frac{(\chi_3-\chi_1)^2}{|\boldsymbol{\chi}_3-\boldsymbol{\chi}_1|}+\frac{(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}\right]$$
$$+\frac{1}{4k_0^2}(\chi_1-2\chi_2-2\chi_3)\left[\frac{(\chi_3-\chi_1)^2}{|\boldsymbol{\chi}_3-\boldsymbol{\chi}_1|}+\frac{(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}\right]-\frac{1}{2k_0^2}(\lambda_2+\lambda_3)\left[\frac{(\chi_3-\chi_1)(\lambda_3-\lambda_1)}{|\boldsymbol{\chi}_3-\boldsymbol{\chi}_1|}+\frac{(\chi_1-\chi_2)(\lambda_1-\lambda_2)}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}\right]$$
$$\left.-\frac{1}{8k_0^2}\left[\frac{(\chi_3-\chi_1)^4}{|\boldsymbol{\chi}_3-\boldsymbol{\chi}_1|^2}+\frac{(\chi_1-\chi_2)^4}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|^2}\right]\right\}e^{i(\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)a(\boldsymbol{\chi}_3,t)d\boldsymbol{\chi}_1d\boldsymbol{\chi}_2d\boldsymbol{\chi}_3. \tag{2.13}$$

Now in order to derive the desired nonlinear evolution equation we are to evaluate the Fourier inversion integrals appearing in the above equation. The evaluation of the inversion integrals in the linear terms can be obtained by the use of the formula for the derivative of Fourier transform and the inversion integral in nonlinear terms, which are of first order in spectral width and have been evaluated by Stiassnie.[9] We are only to evaluate the inversion integrals in the nonlinear terms which are of second order in spectral width. The evaluation of some of these inversion integrals is given in the Appendix. The rest are either similar to these or can be obtained by the formula for the derivative of the Fourier transform. Thus evaluating the Fourier inversion integrals in (2.13) and then introducing the dimensionless quantities in primes,

$$a'=k_0a,\quad x'=k_0x,\quad y'=k_0y,\quad t'=\omega_0t, \tag{2.14}$$

and finally dropping primes on the dimensionless quantities, we finally arrive at the following higher-order nonlinear evolution equation for broader bandwidth surface gravity wave packet, which is correct to $O(\epsilon^4)$ terms:

$$i\frac{\partial a}{\partial t}+\frac{i}{2}\frac{\partial a}{\partial x}-\frac{1}{8}\left(\frac{\partial^2 a}{\partial x^2}-2\frac{\partial^2 a}{\partial y^2}\right)-\frac{i}{16}\left(\frac{\partial^3 a}{\partial x^3}-6\frac{\partial^3 a}{\partial x\,\partial y^2}\right)+\frac{1}{128}\left(5\frac{\partial^4 a}{\partial x^4}-60\frac{\partial^4 a}{\partial x^2\,\partial y^2}+12\frac{\partial^4 a}{\partial y^4}\right)+\frac{i}{256}\left(7\frac{\partial^5 a}{\partial x^5}-140\frac{\partial^5 a}{\partial x^3\,\partial y^2}\right.$$
$$\left.+84\frac{\partial^5 a}{\partial x\,\partial y^4}\right)-\frac{1}{1024}\left(21\frac{\partial^6 a}{\partial x^6}-630\frac{\partial^6 a}{\partial x^4\,\partial y^2}+756\frac{\partial^6 a}{\partial x^2\,\partial y^4}-56\frac{\partial^6 a}{\partial y^6}\right)=\frac{1}{2}a^2a^*-\frac{3}{2}iaa^*\frac{\partial a}{\partial x}-\frac{i}{4}a^2\frac{\partial a^*}{\partial x}+\frac{1}{2}aH\frac{\partial}{\partial x}(aa^*)$$
$$-\frac{5}{8}aa^*\frac{\partial^2 a}{\partial x^2}-\frac{9}{16}a^*\left(\frac{\partial a}{\partial x}\right)^2-\frac{5}{8}a^*\left(\frac{\partial a}{\partial y}\right)^2-\frac{1}{8}a\frac{\partial a}{\partial x}\frac{\partial a^*}{\partial x}+\frac{1}{4}a\frac{\partial a}{\partial y}\frac{\partial a^*}{\partial y}+\frac{1}{8}a^2\frac{\partial^2 a^*}{\partial x^2}+\frac{1}{4}a^2\frac{\partial^2 a^*}{\partial y^2}-\frac{i}{2}\frac{\partial a}{\partial x}H\frac{\partial}{\partial x}(aa^*)$$
$$-\frac{i}{2}\frac{\partial a}{\partial y}H\frac{\partial}{\partial y}(aa^*)-\frac{i}{4}a\frac{\partial}{\partial x}H\left(a\frac{\partial a^*}{\partial x}\right)-\frac{i}{2}a\frac{\partial}{\partial x}H\left(a^*\frac{\partial a}{\partial x}\right)-\frac{i}{2}a\frac{\partial}{\partial y}H\left(a^*\frac{\partial a}{\partial y}\right)-\frac{1}{8}a\frac{\partial^2}{\partial x^2}P\frac{\partial}{\partial x}(aa^*), \tag{2.15}$$

where H, the Hilbert transform operator, and P, an operator similar to this, are given by

$$H\Psi(\boldsymbol{x})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\xi-x)}{|\boldsymbol{\xi}-\boldsymbol{x}|^3}\Psi(\boldsymbol{\xi})d\boldsymbol{\xi},$$

$$P\Psi(\boldsymbol{x})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\xi-x)}{|\boldsymbol{\xi}-\boldsymbol{x}|^2}\Psi(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{2.16}$$

III. STABILITY OF A UNIFORM STOKES WAVE

A uniform Stokes wave solution of (2.15) is

$$a=a_0\exp(-i\Delta\omega t)\equiv\zeta^{(0)}, \tag{3.1}$$

where a_0 is a real constant and

$$\Delta\omega=\frac{1}{2}a_0^2. \tag{3.2}$$

To study modulational instability of the uniform wave train solution (3.1), we introduce the following perturbation in the uniform wave train solution (3.1):

$$a=\zeta^{(0)}\left[1+\hat{a}(\boldsymbol{x},t)\right], \tag{3.3}$$

where $\hat{a}$ is complex. Substituting (3.3) in the evolution equation (2.15) and then linearizing with respect to $\hat{a}$, we get the following equation:

$$i\frac{\partial \hat{a}}{\partial t}+\frac{i}{2}\frac{\partial \hat{a}}{\partial x}-\frac{1}{8}\left(\frac{\partial^2 \hat{a}}{\partial x^2}-2\frac{\partial^2 \hat{a}}{\partial y^2}\right)-\frac{i}{16}\left(\frac{\partial^3 \hat{a}}{\partial x^3}-6\frac{\partial^3 \hat{a}}{\partial x\,\partial y^2}\right)+\frac{1}{128}\left(5\frac{\partial^4 \hat{a}}{\partial x^4}-60\frac{\partial^4 \hat{a}}{\partial x^2\,\partial y^2}+12\frac{\partial^4 \hat{a}}{\partial y^4}\right)+\frac{i}{256}\left(7\frac{\partial^5 \hat{a}}{\partial x^5}-140\frac{\partial^5 \hat{a}}{\partial x^3\,\partial y^2}\right.$$
$$\left.+84\frac{\partial^5 \hat{a}}{\partial x\,\partial y^4}\right)-\frac{1}{1024}\left(21\frac{\partial^6 \hat{a}}{\partial x^6}-630\frac{\partial^6 \hat{a}}{\partial x^4\,\partial y^2}+756\frac{\partial^6 \hat{a}}{\partial x^2\,\partial y^4}-56\frac{\partial^6 \hat{a}}{\partial y^6}\right)=\frac{1}{2}a_0^2(\hat{a}+\hat{a}^*)-\frac{3}{2}ia_0^2\frac{\partial \hat{a}}{\partial x}-\frac{i}{4}a_0^2\frac{\partial \hat{a}^*}{\partial x}-\frac{5}{8}a_0^2\frac{\partial^2 \hat{a}}{\partial x^2}$$
$$+\frac{1}{8}a_0^2\frac{\partial^2 \hat{a}^*}{\partial x^2}+\frac{1}{4}a_0^2\frac{\partial^2 \hat{a}^*}{\partial y^2}+\frac{a_0^2}{2}H\left(\frac{\partial \hat{a}}{\partial x}+\frac{\partial \hat{a}^*}{\partial x}\right)-\frac{i}{4}a_0^2\frac{\partial}{\partial x}H\left(\frac{\partial \hat{a}^*}{\partial x}\right)-\frac{i}{2}a_0^2\frac{\partial}{\partial x}H\left(\frac{\partial \hat{a}}{\partial x}\right)-\frac{i}{2}a_0^2\frac{\partial}{\partial y}H\left(\frac{\partial \hat{a}}{\partial y}\right)-\frac{1}{8}a_0^2\frac{\partial^2}{\partial x^2}P\left(\frac{\partial \hat{a}}{\partial x}+\frac{\partial \hat{a}^*}{\partial x}\right). \tag{3.4}$$

Setting $\hat{a}=a_r+ia_i$, where a_r and a_i are real, and then separating into real and imaginary parts we get two coupled linear equations for a_r and a_i. These two equations produce the following nonlinear dispersion relation if it is assumed that the space-time dependence of a_r and a_i is of the form $\exp[i(\boldsymbol{p}\cdot\boldsymbol{x}-\Omega t)]$, where $\boldsymbol{p}=p\hat{x}+q\hat{y}$:

$$\left(L-\frac{5}{4}pa_0^2+\frac{1}{4}\frac{p^3}{|\boldsymbol{p}|}a_0^2\right)\left(L-\frac{7}{4}pa_0^2+\frac{3}{4}\frac{p^3}{|\boldsymbol{p}|}a_0^2\right)$$
$$-\left(M-\frac{3}{4}p^2a_0^2-\frac{1}{4}q^2a_0^2\right)\left(M-a_0^2-\frac{1}{2}p^2a_0^2+\frac{1}{4}q^2a_0^2\right.$$
$$\left.+\frac{p^2}{|\boldsymbol{p}|}a_0^2+\frac{1}{4}\frac{p^4}{|\boldsymbol{p}|^2}a_0^2\right)=0, \tag{3.5}$$

where

$$L=\Omega-\frac{1}{2}p-\frac{1}{16}(p^3-6pq^2)-\frac{1}{256}(7p^5-140p^3q^2$$
$$+84pq^4)+\frac{1}{2}\frac{pq^2}{|\boldsymbol{p}|}a_0^2,$$

$$M=\frac{1}{8}(p^2-2q^2)+\frac{1}{128}(5p^4-60p^2q^2+12q^4)$$
$$+\frac{1}{1024}(21p^6-630p^4q^2+756p^2q^4-56q^6). \tag{3.6}$$

The dispersion relation (3.5) can be written as

$$(\Omega+N)^2=R(R+S)-U(U-S)+V^2, \tag{3.7}$$

where N,R,S,U, and V are given by

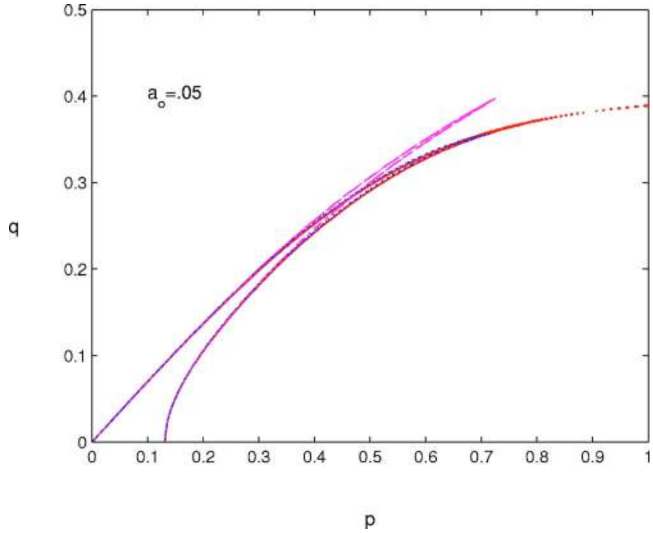

FIG. 1. Stable-unstable regions in perturbed wave-number space; $a_0=0.05$. The dash-dot lines are for results from the evolution equation derived by Trulsen and Dysthe (Ref. 5) and the solid lines are the results from the present evolution equation (2.15).

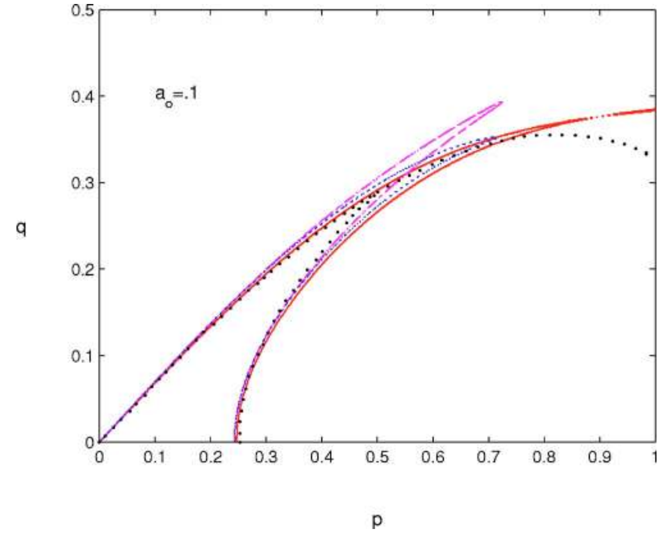

FIG. 2. Stable-unstable regions in perturbed wave-number space; $a_0=0.1$. The dotted lines are for the exact numerical results of McLean *et al.* (Ref. 6), dash-dot lines are for the results from the evolution equation derived by Trulsen and Dysthe (Ref. 5), dashed lines are for the results from the evolution equation of Trulsen *et al.* (Ref. 7), and the solid lines are for the results from the present evolution equation (2.15).

$$N=\frac{1}{2}p+\frac{1}{16}(p^3-6pq^2)+\frac{1}{256}(7p^5-140p^3q^2+84q^4)+\frac{1}{2}\left(3p-\frac{p^3}{|\boldsymbol{p}|}-\frac{pq^2}{|\boldsymbol{p}|}\right)a_0^2,$$

$$R=-\frac{1}{8}(p^2-2q^2)-\frac{1}{128}(5p^4-60p^2q^2+12q^4)-\frac{1}{1024}(21p^6-630p^4q^2+756p^2q^4-56q^6)+\frac{5}{8}p^2a_0^2,$$

$$S=\left(1-\frac{p^2}{|\boldsymbol{p}|}-\frac{1}{4}\frac{p^4}{|\boldsymbol{p}|^2}\right)a_0^2,$$

$$U=\frac{1}{8}(p^2+2q^2)a_0^2,$$

$$V=\frac{1}{4}\left(p+\frac{p^3}{|\boldsymbol{p}|}\right)a_0^2. \tag{3.8}$$

Equation (3.7) shows that there is instability, i.e., a uniform Stokes wave becomes unstable, if

$$R(R+S)-U(U-S)+V^2<0. \tag{3.9}$$

The instability regions in the p-q plane, where the above condition is satisfied, i.e., where a uniform Stokes wave becomes unstable are shown in Figs. 1, 2, 3, and 4, respectively, for the four different values 0.05, 0.1, 0.15, and 0.2 of wave steepness a_0. The regions within the curves and the line $q=0$ denote instability. In all these figures the regions of instability obtained by Trulsen and Dysthe[5] and Trulsen *et al.*[7] have been shown. The instability regions obtained by exact numerical calculations in the papers by McLean *et al.*[6] and McLean[10] have also been included in Figs. 2 and 4. These figures show that the results obtained from the higher-order evolution equation (2.15) for broader bandwidth waves are in very good agreement with the exact numerical results of McLean *et al.*[6] and McLean.[10] As higher-order dispersive terms and higher-order derivatives in the nonlinear terms in the evolution equation (2.15) have been neglected, the dispersion relation (3.7) and consequently the instability condition (3.9) remains valid only for $p,q<1$. For this reason these figures for $p>=0.9$ cannot be expected to give good results inconsistent with the exact results of McLean *et al.*[6]

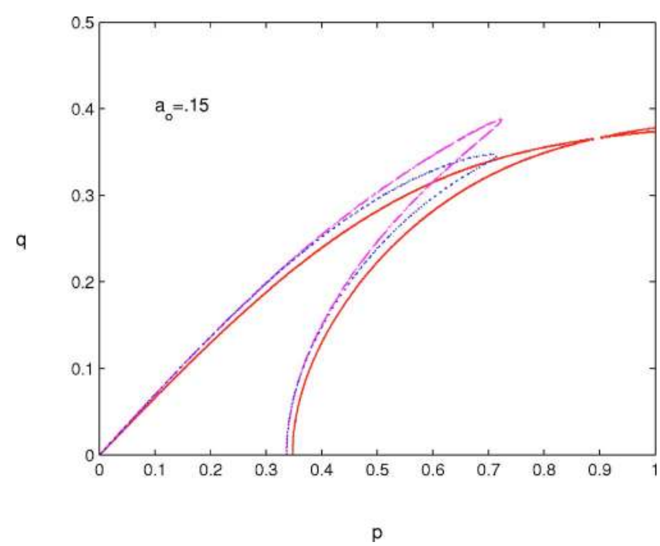

FIG. 3. Stable-unstable regions in perturbed wave-number space; $a_0=0.15$. The dash-dot lines are for the results from the evolution equation derived by Trulsen and Dysthe (Ref. 5), and the solid lines are for the results from the present evolution equation (2.15).

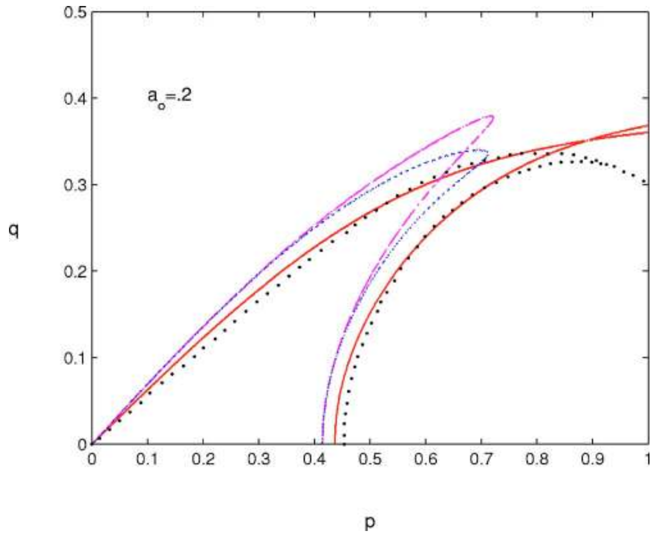

FIG. 4. Stable-unstable regions in perturbed wave-number space; $a_0=0.2$. The dotted lines are for the exact numerical results of McLean *et al.* (Ref. 6), the dash-dot lines are for the results from the evolution equation derived by Trulsen and Dysthe (Ref. 5), the dashed lines are for the results from the evolution equation of Trulsen *et al.* (Ref. 7), and the solid line are for the results from the present evolution equation (2.15).

IV. CONCLUSION

Starting from Zakharov's integral equation a nonlinear evolution equation for broader bandwidth gravity waves in deep water has been derived. This equation is one order higher than the equation derived by Trulsen and Dysthe.[5] The instability regions in the perturbed wave-number space, for a uniform Stokes wave obtained from the evolution equation (2.15) derived in the present investigation, are found to be in very good agreement with the exact numerical results of McLean *et al.*[6] and McLean.[10] Consequently though (2.15) derived here contains a few more nonlinear and dispersive terms than the corresponding equations of Trulsen and Dysthe[5] and Trulsen *et al.*,[7] it can possibly describe more correctly a Stokes wave on the surface of deep water.

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APPENDIX

$$I_1 = \frac{1}{8\pi^3}\int\int\int_{-\infty}^{\infty}\frac{(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)a(\boldsymbol{\chi}_3,t)\times e^{[i(\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 d\boldsymbol{\chi}_3.$$

After evaluating the integral we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|\boldsymbol{x}'|}e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}'}d\boldsymbol{x}' = \frac{1}{|\boldsymbol{\chi}|}. \tag{A1}$$

Performing the integration with respect to $\boldsymbol{\chi}_3$ and using the formula for the derivative of the Fourier transform, I_1 can be written as

$$\begin{aligned}
I_1 &= a(\boldsymbol{x},t)\frac{\partial}{\partial x}\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}\frac{i(\chi_1-\chi_2)}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 \\
&= a(\boldsymbol{x},t)\frac{\partial}{\partial x}\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} i(\chi_1-\chi_2)a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 \times \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|\boldsymbol{x}'|}e^{[i(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{x}']}d\boldsymbol{x}' \quad \text{[by (A1)]} \\
&= a(\boldsymbol{x},t)\frac{\partial}{\partial x}\frac{1}{8\pi^3}\int\int\int_{-\infty}^{\infty}\frac{i(\chi_1-\chi_2)}{|\boldsymbol{x}-\boldsymbol{\xi}|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)\times e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{\xi}]}d\boldsymbol{\xi}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2, \quad \boldsymbol{\xi}=\boldsymbol{x}-\boldsymbol{x}' \\
&= -a(\boldsymbol{x},t)\frac{\partial}{\partial x}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\boldsymbol{\xi}}{|\boldsymbol{x}-\boldsymbol{\xi}|}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\chi_1)a^*(\boldsymbol{\chi}_1,t)e^{[-i(\boldsymbol{\chi}_1\cdot\boldsymbol{\xi})]}d\boldsymbol{\chi}_1\right. \\
&\quad \left.\times\frac{1}{2\pi}\int_{-\infty}^{\infty}a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2\cdot\boldsymbol{\xi})]}d\boldsymbol{\chi}_2 + \frac{1}{2\pi}\int_{-\infty}^{\infty}a^*(\boldsymbol{\chi}_1,t)e^{[-i(\boldsymbol{\chi}_1\cdot\boldsymbol{\xi})]}d\boldsymbol{\chi}_1\times\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\chi_2)a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2\cdot\boldsymbol{\xi})]}d\boldsymbol{\chi}_2\right] \\
&= -a(\boldsymbol{x},t)\frac{\partial}{\partial x}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\boldsymbol{\xi}}{|\boldsymbol{x}-\boldsymbol{\xi}|}\frac{\partial}{\partial\xi}[a(\boldsymbol{\xi},t)a^*(\boldsymbol{\xi},t)] = -a(\boldsymbol{x},t)H\left\{\frac{\partial}{\partial x}[a(\boldsymbol{x},t)a^*(\boldsymbol{x},t)]\right\},
\end{aligned}$$

where H is the Hilbert transform operator defined by (2.16).

$$I_2 = \frac{1}{8\pi^3}\int\int\int_{-\infty}^{\infty}\frac{\chi_3(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)a(\boldsymbol{\chi}_3,t)\times e^{[i(\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 d\boldsymbol{\chi}_3.$$

Performing the integration with respect to $\boldsymbol{\chi}_3$ and using the formula for the derivative of the Fourier transform, we can write

$$
\begin{aligned}
I_2 &= \frac{\partial a(\boldsymbol{x},t)}{\partial x}\frac{\partial}{\partial x}\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}\frac{(\chi_1-\chi_2)}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)\times e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2\\
&= \frac{\partial a(\boldsymbol{x},t)}{\partial x}\frac{\partial}{\partial x}\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}(\chi_1-\chi_2)a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|\boldsymbol{x}'|}e^{[i(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{x}']}d\boldsymbol{x}' \quad [\text{by (A1)}]\\
&= \frac{\partial a(\boldsymbol{x},t)}{\partial x}\frac{\partial}{\partial x}\frac{1}{8\pi^3}\int\int\int_{-\infty}^{\infty}\frac{(\chi_1-\chi_2)}{|\boldsymbol{x}-\boldsymbol{\xi}|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)\times e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{\xi}]}d\boldsymbol{\xi}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2, \quad \boldsymbol{\xi}=\boldsymbol{x}-\boldsymbol{x}'\\
&= i\frac{\partial a(\boldsymbol{x},t)}{\partial x}\frac{\partial}{\partial x}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\boldsymbol{\xi}}{|\boldsymbol{x}-\boldsymbol{\xi}|}\frac{\partial}{\partial\xi}[a(\boldsymbol{\xi},t)a^*(\boldsymbol{\xi},t)]
\end{aligned}
$$

(following the last steps of the evaluation of I_1)

$$
= i\frac{\partial a(\boldsymbol{x},t)}{\partial x}H\left\{\frac{\partial}{\partial x}[a(\boldsymbol{x},t)a^*(\boldsymbol{x},t)]\right\}.
$$

$$
I_3 = \frac{1}{8\pi^3}\int\int\int_{-\infty}^{\infty}\frac{\chi_2(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)a(\boldsymbol{\chi}_3,t)\times e^{[i(\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 d\boldsymbol{\chi}_3.
$$

As before performing the integration with respect to $\boldsymbol{\chi}_3$ and using the formula for the derivative of the Fourier transform, I_3 can be written as

$$
\begin{aligned}
I_3 &= -a(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}\frac{d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}\chi_2 a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}\\
&= -a(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}\chi_2 a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|\boldsymbol{x}'|}e^{[i(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{x}']}d\boldsymbol{x}' \quad [\text{by (A1)}]\\
&= -a(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{8\pi^3}\int\int\int_{-\infty}^{\infty}\frac{\chi_2}{|\boldsymbol{x}-\boldsymbol{\xi}|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)\times e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{\xi}]}d\boldsymbol{\xi}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2, \quad \boldsymbol{\xi}=\boldsymbol{x}-\boldsymbol{x}'\\
&= ia(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\boldsymbol{\xi}}{|\boldsymbol{x}-\boldsymbol{\xi}|}\times\frac{1}{2\pi}\int_{-\infty}^{\infty}a^*(\boldsymbol{\chi}_1,t)e^{[-i(\boldsymbol{\chi}_1\cdot\boldsymbol{\xi})]}d\boldsymbol{\chi}_1\times\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\chi_2)a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2\cdot\boldsymbol{\xi})]}d\boldsymbol{\chi}_2\\
&= ia(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\boldsymbol{\xi}}{|\boldsymbol{x}-\boldsymbol{\xi}|}a^*(\boldsymbol{\xi},t)\frac{\partial}{\partial\xi}[a(\boldsymbol{\xi},t)]\\
&= ia(\boldsymbol{x},t)\frac{\partial}{\partial x}H\left[a^*(\boldsymbol{x},t)\frac{\partial a(\boldsymbol{x},t)}{\partial x}\right],
\end{aligned}
$$

$$
I_4 = \frac{1}{8\pi^3}\int\int\int_{-\infty}^{\infty}\frac{(\chi_1-\chi_2)^4}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)a(\boldsymbol{\chi}_3,t)\times e^{[i(\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2 d\boldsymbol{\chi}_3.
$$

Evaluating the integral we get

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{x'}{|\boldsymbol{x}'|^2}e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}'}d\boldsymbol{x}' = \frac{i\chi}{|\boldsymbol{\chi}|^2}. \tag{A2}
$$

Performing integration with respect to $\boldsymbol{\chi}_3$ and using the formula for the derivative of Fourier transform, I_4 can be written as

$$I_4 = -a(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}\frac{(\chi_2-\chi_1)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|^2}a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)\times e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2$$

$$= a(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} i(\chi_1-\chi_2)a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}]}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{x'}{|\boldsymbol{x}'|}e^{[i(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{x}']}d\boldsymbol{x}' \quad [\text{by (A2)}]$$

$$= a(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{8\pi^3}\int\int\int_{-\infty}^{\infty}\frac{(x-\xi)}{|\boldsymbol{x}-\boldsymbol{\xi}|^2}i(\chi_1-\chi_2)a^*(\boldsymbol{\chi}_1,t)a(\boldsymbol{\chi}_2,t)\times e^{[i(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{\xi}]}d\boldsymbol{\xi}d\boldsymbol{\chi}_1 d\boldsymbol{\chi}_2, \quad \boldsymbol{\xi}=\boldsymbol{x}-\boldsymbol{x}'$$

$$= a(\boldsymbol{x},t)\frac{\partial^2}{\partial x^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\xi-x)}{|\boldsymbol{x}-\boldsymbol{\xi}|^2}\frac{\partial}{\partial\xi}(a(\boldsymbol{\xi},t)a^*(\boldsymbol{\xi},t))d\boldsymbol{\xi}$$

(following the last steps of the evaluaton of I_1)

$$= a(\boldsymbol{x})\frac{\partial^2}{\partial x^2}P\left\{\frac{\partial}{\partial x}[a(\boldsymbol{x},t)a^*(\boldsymbol{x},t)]\right\}.$$

Here the operator P is defined by (2.16).

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