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A fourth-order evolution equation for deep water surface gravity waves in the presence of wind blowing over water

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The stability of a train of nonlinear surface gravity waves in deep water in the presence of wind blowing over water is considered. An evolution equation is derived for the wave envelope that is correct to fourth order in the wave steepness. The importance of the fourth-order term in the evolution equation was pointed out by Dysthe [Proc. R. Soc. London Ser. A **369**, 105 (1979)]. From this evolution equation the expressions for the maximum growth rate of the instability and the frequency at marginal stability are derived, and graphs are plotted for those two expressions against the wave steepness.

I. INTRODUCTION

In recent years there has been considerable interest in the stability of finite amplitude surface gravity waves in deep water. Much of this interest has been focused on the instability of a uniform wave train to modulational perturbations.

For small but finite amplitude, the most successful approach of this study is through the application of the lowest-order nonlinear evolution equation. Zakharov's[1] study is along this line, allowing for finite amplitude wave trains to be subjected to modulational perturbations in two horizontal directions both along and perpendicular to the direction of the wave train. Davey and Stewartson[2] made an extension of this to water of finite depth. Further extensions of this were made by Djordjevic and Redekopp[3] to include capillarity and by Das[4] to include density stratification.

For small amplitude, $ka < 0.1$, say, the predictions from the nonlinear Schrödinger equation, when compared with Longuet-Higgin's[5,6] exact results, are fairly accurate. Here k is the wavenumber and a is the amplitude of the wave.

But for $ka > 0.15$ the predictions from the nonlinear Schrödinger equation do not agree with the exact results of Longuet-Higgins.[5,6] Dysthe[7] has shown that a surprising improvement on these results relating to stability of a finite amplitude wave can be attained by extending the perturbation analysis one step further, i.e., adding the order ϵ^4 term in the nonlinear Schrödinger equation.

From this fourth-order evolution equation Janssen[8] has elaborated on the Dysthe approach by investigating the effect of wave-induced flow on the long-time behavior of Benjamin–Feir instability and has also applied this equation to the homogeneous random field of gravity waves and obtained the nonlinear energy transfer function recently found by Dungey and Hui.[9] Stiassnie[10] has shown that Zakharov's integral equation yields the modified or fourth-order nonlinear Schrödinger equation for the particular case of narrow spectrum. Hogan[11] has considered the stability of a train of nonlinear capillary–gravity waves on the surface of an ideal fluid of infinite depth. He derived from the Zakharov equation under the assumption of a narrow band of waves, and including the full form of the interaction coefficient for capillary–gravity waves, an evolution equation for the wave envelope that is correct to fourth order in the wave steepness.

The second-order corrections to the first-order stability properties are shown to depend on the interaction between the mean flow and the frequency dispersion term of the wave envelope. Brinch-Nielsen and Jonsson[12] have also derived the fourth-order evolution equation for a three-dimensional Stokes wave on arbitrary water depth. In deep water the equations reduce to those of Dysthe,[7] and on finite depth the third-order terms agree with these of Benney and Roskes,[13] Davey and Stewartson,[2] and Hasimoto and Ono.[14]

In the present paper we extend the analysis of Dysthe[7] for the case of wind blowing over water. Therefore this paper considers the influence of wind on Benjamin–Feir instability. Here we derive a nonlinear evolution equation in the next highest order for the case of wind blowing over water. This evolution equation remains valid when the wind velocity is less than a critical velocity. This critical velocity is defined by the fact that a wave becomes linearly unstable if the wind velocity exceeds this critical velocity. In the linearly stable region there are two modes that we designate as positive and negative modes. Starting from the derived evolution equation we make a stability analysis of a uniform wave train as given by Dysthe.[7] The maximum growth rate γ_M of instability has been plotted against dimensionless wave steepness α_0 for some different values of dimensionless wind velocity V. It is found that γ_M increases up to certain values of α_0 and then decreases with the increase of α_0. The growth rate is found to be appreciably much higher for wind velocity approaching its critical value. The perturbed frequency Ω at marginal stability has also been plotted against wave steepness α_0 for some different values of dimensionless wind velocity.

Exact results based on full water wave equations, for perturbations both along and perpendicular to the direction of propagation of the wave, were also obtained by McLean *et al.*[15]; these results include the result of Longuet-Higgins[5,6] as a special case. As in the latter's analysis, the wavenumber of perturbation along the direction of propagation of the wave is restricted to rational numbers, while that perpendicular to the direction of propagation of the wave is zero. Yuen[16] has presented results that are extensions of those by McLean *et al.*[15] to the case of interfacial waves with current jump.

Some of Yuen's[16] results were confirmed experimentally by Bliven *et al.*,[17] and recently Yuen's[16] analysis has been

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extended by Pullin and Grimshaw[18] to the case of basic current shear.

The present paper deals with the special case of the results obtained by Yuen.[16] Since this considers the influence of wind on the Benjamin–Feir instability, it was restricted to small wavenumbers of perturbations and to small wave steepness. As the present paper is restricted to the air–water interface and Yuen's paper does not give computations for this case (γ = ratio of the densities of air to water = 0.001 29), there is no possibility of comparing our results with that of Yuen.[16] But to check our results we have made computations for $\gamma = 0.1$ for two sets of values of (α_0, V) and have found results that are in very good agreement with the computations of Yuen.[16] The case $\gamma = 0$ has been considered by McLean *et al.*[15] and Longuet-Higgins[5,6] and is a particular case of Yuen's[16] paper. As previously mentioned, Dysthe has shown that the fourth-order nonlinear evolution equation gives results that agree with the exact results of Longuet-Higgins[5,6] for wave steepness greater than 0.15. Thus our result for the case $\gamma = 0$, which coincides with the results of Dysthe,[7] is in good agreement with the results of Longuet-Higgins and also with those of McLean *et al.*[15] and Yuen[16] for small wavenumbers and for small wave steepness.

II. BASIC EQUATIONS

The common horizontal interface between water and air in the undisturbed state is taken as the $z = 0$ plane. In the undisturbed state air flows over water with a velocity U in a direction that is taken as the x axis. We take $z = \zeta(x, y, t)$ as the equation of the common interface at any time t in the perturbed state. Let ρ and ρ' be the densities of water and air, respectively. We introduce the dimensionless quantities φ^*, φ'^*, ζ^*, (x^*, y^*, z^*), t^*, V^*, and γ^*, which are, respectively, the perturbed velocity potential in water, perturbed velocity potential in air, surface elevation of the water–air interface, space coordinates, time, air flow velocity, and the ratio of the densities of air to water. These dimensionless quantities are related to the corresponding dimensional quantities by the following relations:

$$\varphi^* = \sqrt{k_0^3/g}\,\varphi, \quad \varphi'^* = \sqrt{k_0^3/g}\,\varphi', \quad \zeta^* = k_0\zeta,$$
$$x^* = k_0 x, \quad y^* = k_0 y, \quad z^* = k_0 z, \tag{1}$$
$$t^* = \omega t, \quad V^* = \sqrt{k_0/g}\,U, \quad \gamma^* = \rho'/\rho,$$

where k_0 is some characteristic wavenumber. In the future all these quantities will be written in their dimensionless form but with their asterisks deleted.

The perturbed velocity potentials φ and φ', from which perturbed velocities $\mathbf{u}$ and $\mathbf{u}'$ of water and air, respectively, can be obtained from the relations $\mathbf{u} = \nabla\varphi$ and $\mathbf{u}' = \nabla\varphi'$, satisfy the following Laplace equations,

$$\nabla^2\varphi = 0, \quad \nabla^2\varphi' = 0, \tag{2}$$

in $-\infty < z < \zeta$ and $\zeta < z < \infty$, respectively.

The kinematic boundary conditions to be satisfied at the interface are the following:

$$\frac{\partial\varphi}{\partial z} - \frac{\partial\zeta}{\partial t} = (\nabla_h\varphi)\cdot(\nabla_h\zeta), \quad \text{when } z = \zeta, \tag{3}$$

$$\frac{\partial\varphi'}{\partial z} - \frac{\partial\zeta}{\partial t} - V\frac{\partial\zeta}{\partial x} = (\nabla_h\varphi')\cdot(\nabla_h\zeta), \quad \text{when } z = \zeta, \tag{4}$$

where

$$\nabla_h \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right).$$

The condition of continuity of pressure at the interface gives

$$\frac{\partial\varphi}{\partial t} - \gamma\frac{\partial\varphi'}{\partial t} + (1-\gamma)\zeta - \gamma V\frac{\partial\varphi'}{\partial x} = -\frac{1}{2}(\nabla\varphi)^2 + \frac{\gamma}{2}(\nabla\varphi')^2, \quad \text{when } z = \zeta. \tag{5}$$

Finally φ and φ' should satisfy the following conditions at infinity:

$$\varphi \to 0 \text{ as } z \to -\infty, \quad \varphi' \to 0 \text{ as } z \to \infty. \tag{6}$$

We look for solutions of the above equations in the form

$$G = G_0 + \sum_{n=1}^{\infty} (G_n \exp in\Psi + G_n^* \exp - in\Psi), \tag{7}$$

where G stands for φ, φ', ζ; $\Psi = kx - \omega t$; $\varphi_0, \varphi'_0, \varphi_n, \varphi'_n, \varphi_n^*, \varphi_n'^*$ are functions of z, $x_1 = \epsilon x$, $y_1 = \epsilon y$, $t_1 = \epsilon t$; and $\zeta_0, \zeta_n, \zeta_n^*$ are functions of x_1, y_1, t_1. Here $*$ denotes complex conjugate, ϵ is a slowness parameter, and ω, k satisfy the following linear dispersion relation with $l = 0$:

$$\lambda(\omega,k,l) \equiv (1+\gamma)\omega^2 - 2\gamma\omega k V + \gamma k^2 V^2 - (1-\gamma)(k^2 + l^2)^{1/2} = 0. \tag{8}$$

We now suppose that the first harmonic linear wave, whose nonlinear evolution equation we are going to study, has its wavenumber equal to the characteristic wavenumber k_0. Thus we have $k = 1$ and the linear dispersion relation determining ω becomes

$$(1+\gamma)\omega^2 - 2\gamma\omega V + \gamma V^2 - (1-\gamma) = 0. \tag{9}$$

This equation gives two values of ω,

$$\omega_\pm = (\gamma V \pm \sqrt{1 - \gamma^2 - \gamma V^2})/(1+\gamma), \tag{10}$$

which corresponds to two modes, and we designate these two modes as positive and negative modes. The positive mode moves in the positive direction of the x axis with a frequency $(\sqrt{1-\gamma^2-\gamma V^2} + \gamma V)/(1+\gamma)$, while the negative mode moves in the negative direction of the x axis with a frequency $(\sqrt{1-\gamma^2-\gamma V^2} - \gamma V)/(1+\gamma)$. If V is replaced by $-V$ the frequency of the positive mode becomes equal to the frequency of the negative mode. So the results for the negative mode can be obtained from those for the positive mode by replacing V by $-V$. Therefore we have made a nonlinear analysis for the positive mode, and then we have obtained the results for the negative mode by replacing V by $-V$.

From the expression (10) for $\omega_\pm$, we find that for linear stability V should satisfy the following condition:

$$|V| < \sqrt{(1-\gamma^2)/\gamma}. \tag{11}$$

Thus our present analysis will remain valid as long as the dimensionless flow velocity of the wind becomes less than the critical value $\sqrt{(1-\gamma^2)/\gamma}$. For air flowing over water $\gamma = 0.001\,29$, and this critical value becomes 27.84.

III. DERIVATION OF EVOLUTION EQUATION

On substituting the expansions (7) in Eq. (2) and then equating coefficients of exp $in\Psi$ ($n = 0,1,2$), we obtain the following equations for $n = 0,1,2$:

$$\frac{d^2\varphi_n}{dz^2} - \Delta_n^2\varphi_n = 0, \quad \frac{d^2\varphi'_n}{dz^2} - \Delta_n^2\varphi'_n = 0. \tag{12}$$

Here Δ_n is the operator given by

$$\Delta_n = \left[\left(n - i\epsilon\frac{\partial}{\partial x_1}\right)^2 - \epsilon^2\frac{\partial^2}{\partial y_1^2}\right]^{1/2}, \quad n = 0,1,2. \tag{13}$$

The solution of these equations satisfying boundary conditions (6) can be put in the form

$$\varphi_n = \exp(\Delta_n z)A_n, \quad \varphi'_n = \exp(-\Delta_n z)A'_n \tag{14}$$

for $n = 1,2$, and

$$\bar\varphi_0 = \exp(\epsilon\bar k z)\bar A_0, \quad \bar\varphi'_0 = \exp(-\epsilon\bar k z)\bar A'_0. \tag{15}$$

In the above A_n, A'_n ($n = 1,2$) are functions of x_1, y_1, t_1 and $\bar\varphi_0, \bar\varphi'_0$ are Fourier transforms of φ_0, φ'_0, respectively, defined by

$$(\bar\varphi_0,\bar\varphi'_0) = \iiint_{-\infty}^{\infty} (\varphi_0,\varphi'_0)\exp[i(-\bar k_x x_1 - \bar k_y y_1 + \bar\omega t_1)]dx_1\,dy_1\,dt_1, \tag{16}$$

where $\bar k^2 = \bar k_x^2 + \bar k_y^2$ and $\bar A_0, \bar A'_0$ are functions of $\bar k_x$, $\bar k_y$, and $\bar\omega$.

On substituting the expansions (7) in the Taylor expanded form of Eqs. (3)–(5) about $z = 0$ and then equating coefficients of exp $in\Psi$ for $n = 0,1,2$ on both sides, we obtain the following three sets of equations in which we substitute the solutions (14) and (15) for $\varphi_1, \varphi_2, \bar\varphi_0, \varphi'_1, \varphi'_2, \bar\varphi'_0$. For the sake of convenience we take the Fourier transform of the set of equations corresponding to $n = 0$.

(i) For $n = 1$,

$$\Delta_1 A_1 + i\omega_1\zeta_1 = a_1, \tag{17}$$

$$-\Delta_1 A'_1 + i\omega_1\zeta_1 - iVk_1\zeta_1 = b_1, \tag{18}$$

$$-i\omega_1 A_1 + i\gamma\omega_1 A'_1 - i\gamma V k_1 A'_1 + (1-\gamma)\zeta_1 = c_1; \tag{19}$$

(ii) for $n = 2$,

$$\Delta_2 A_2 + i\omega_2\zeta_2 = a_2, \tag{20}$$

$$-\Delta_2 A'_2 + i\omega_2\zeta_2 - iVk_2\zeta_2 = b_2, \tag{21}$$

$$-i\omega_2 A_2 + i\gamma\omega_2 A'_2 - i\gamma V k_2 A'_2 + (1-\gamma)\zeta_2 = c_2; \tag{22}$$

(iii) for $n = 0$,

$$\epsilon\bar k\bar A_0 + i\epsilon\bar\omega\bar\zeta_0 = F(a_0), \tag{23}$$

$$-\epsilon\bar k\bar A'_0 + i\epsilon\bar\omega\bar\zeta_0 - i\epsilon V\bar k_x\bar\zeta_0 = F(b_0), \tag{24}$$

$$-i\epsilon\bar\omega\bar A_0 + i\gamma\epsilon\bar\omega\bar A'_0 - i\epsilon\gamma V\bar k_x\bar A'_0 + (1-\gamma)\bar\zeta_0 = F(c_0), \tag{25}$$

where $\omega_j = j\omega + i\epsilon(\partial/\partial t_1)$, $k_j = j - i\epsilon(\partial/\partial x_1)$, $j = 1,2$, a_n, b_n, c_n ($n = 0,1,2$) are contributions from nonlinear terms, and $F(\;)$ implies a Fourier transform of the quantity inside parentheses.

To solve the above three sets of equations we make the following perturbation expansion for the quantities A_n, A'_n, and ζ_n ($n = 0,1,2$):

$$E_1 = \sum_{n=1}^{\infty}\epsilon^n E_{1n}, \quad E_m = \sum_{n=2}^{\infty}\epsilon^n E_{mn} \quad (m = 0,2), \tag{26}$$

where E_j stands for A_j, A'_j, ζ_j ($j = 0,1,2$).

On substituting the expansions (26) in the above three sets of equations and then equating coefficients of various powers of ϵ on both sides, we obtain a sequence of equations.

From the first-order (i.e., lowest order) and second-order equations of (17) and (18) we obtain solutions for A_{11}, A'_{11} and A_{12}, A'_{12}, respectively. Next, from the first-order and second-order equations of (20)–(22), we obtain solutions for $A_{22}, A'_{22}, \zeta_{22}$ and $A_{23}, A'_{23}, \zeta_{23}$, respectively. Finally, from the first-order equations of (23)–(25) we obtain solutions for $A_{02}, A'_{02}, \zeta_{02}$ and from the second-order equation of (25) we obtain a solution for ζ_{03}. All these solutions are given in the Appendix.

Equation (19), which has not been used in obtaining the above perturbation solutions, can be put in the following convenient form after eliminating A_1, A'_1 by the use of Eqs. (17) and (18):

$$\lambda(\omega_1,k_1,l_1)\zeta_1 = -\Delta_1 c_1 - i\omega_1 a_1 - i\gamma(\omega_1 - k_1 V)b_1, \tag{27}$$

where

$$\omega_1 = \omega + i\epsilon\frac{\partial}{\partial t_1}, \quad k_1 = 1 - i\epsilon\frac{\partial}{\partial x_1}, \quad l_1 = -i\epsilon\frac{\partial}{\partial y_1}. \tag{28}$$

We keep terms up to order ϵ^4 in Eq. (27), then substitute solutions for various perturbed quantities appearing on its right-hand side, and finally use the transformations

$$\xi = x_1 - c_g t_1, \quad \eta = y_1, \quad \tau = \epsilon t_1. \tag{29}$$

Thus we obtain the following fourth-order nonlinear evolution equation:

$$\begin{aligned} &i\lambda_\omega\frac{\partial\zeta_{11}}{\partial\tau} + \frac{\lambda_\omega}{2}\frac{dc_g}{dk}\frac{\partial^2\zeta_{11}}{\partial\xi^2} + \frac{1-\gamma}{2}\frac{\partial^2\zeta_{11}}{\partial\eta^2} \\ &+ \epsilon\left(i\lambda_\omega\frac{\partial\zeta_{12}}{\partial\tau} + \frac{\lambda_\omega}{2}\frac{dc_g}{dk}\frac{\partial^2\zeta_{12}}{\partial\xi^2} + \frac{1-\gamma}{2}\frac{\partial^2\zeta_{12}}{\partial\eta^2} + 2(1+\gamma)c_g\frac{\partial^2\zeta_{11}}{\partial\xi\,\partial\tau} + \frac{i(1-\gamma)}{2}\frac{\partial^3\zeta_{11}}{\partial\xi\,\partial\eta^2}\right) \\ &= \delta_1\zeta_{11}^2\zeta_{11}^* + \epsilon\left[\delta_1\zeta_{11}^2\zeta_{12}^* + 2\delta_1\zeta_{11}\zeta_{11}^*\zeta_{12} + i(\delta_3 - \delta_2 c_g)\zeta_{11}\zeta_{11}^*\frac{\partial\zeta_{11}}{\partial\xi} + i\delta_4\zeta_{11}^2\frac{\partial\zeta_{11}^*}{\partial\xi}\right. \\ &\left. + \delta_5\zeta_{11}\frac{\partial}{\partial\xi}F^{-1}\left(\frac{1}{\bar k}F\frac{\partial}{\partial\xi}(\zeta_{11}\zeta_{11}^*)\right)\right], \end{aligned} \tag{30}$$

where $F^{-1}(\ )$ is the inverse of the Fourier transform of the quantity inside the parentheses and

$$\delta_1 = 2\left(\omega^2 + \gamma(\omega - V)^2 + \frac{[\omega^2 - \gamma(\omega - V)^2]^2}{1-\gamma}\right),$$

$$\delta_2 = 4\left(\omega + \gamma(\omega - V) + \frac{2[\omega - \gamma(\omega - V)][\omega^2 - \gamma(\omega - V)^2]}{1-\gamma} - \frac{2[(1+\gamma)\omega - \gamma V][\omega^2 - \gamma(\omega - V)^2]^2}{(1-\gamma)^2}\right),$$

$$\delta_3 = -2\left(\omega^2 + \gamma(\omega - V)^2 + \frac{[(1-\gamma) + 4\gamma V(\omega - V)][\omega^2 - \gamma(\omega - V)^2]^2}{(1-\gamma)^2} + 2[\omega^2 + \gamma(\omega - V)(\omega - 2V)]\right.$$
$$\left. + \frac{[\omega^2 - \gamma(\omega - V)^2][\omega^2 - \gamma(\omega - V)(\omega - 3V)]}{1-\gamma} + \frac{2[\omega^2 - \gamma(\omega - V)(\omega - 2V)][\omega^2 - \gamma(\omega - V)^2]}{(1-\gamma)}\right), \quad (31)$$

$$\delta_4 = -2\left(\omega^2 + \gamma(\omega - V)^2 + \frac{[\omega^2 - \gamma(\omega - V)^2]^2}{1-\gamma}\right),$$

$$\delta_5 = 4[\omega^2 + \gamma(\omega - V)^2],$$

$$c_g = \frac{d\omega}{dk}, \quad \frac{dc_g}{dk} = \frac{d^2\omega}{dk^2}, \quad \lambda_\omega = \frac{\partial\lambda}{\partial\omega}.$$

The derivatives $d\omega/dk$, $d^2\omega/dk^2$, and $\partial\lambda/\partial\omega$ are to be evaluated from (8) at $k=1$ and $l=0$. Writing

$$\zeta = \zeta_{11} + \epsilon\zeta_{12} \quad (32)$$

and using the relation

$$\frac{\partial\zeta_{11}}{\partial\tau} = \frac{i}{2}\frac{dc_g}{dk}\frac{\partial^2\zeta_{11}}{\partial\xi^2} + \frac{i(1-\gamma)}{2\lambda_\omega}\frac{\partial^2\zeta_{11}}{\partial\eta^2} - \frac{i\delta_1}{\lambda_\omega}\zeta_{11}^2\zeta_{11}^* + O(\epsilon), \quad (33)$$

which can be obtained from (30), the evolution equation (30) can finally be put in the following form:

$$2i\frac{\partial\zeta}{\partial\tau} - \beta_1\frac{\partial^2\zeta}{\partial\xi^2} + \beta_2\frac{\partial^2\zeta}{\partial\eta^2} + \beta_3\frac{\partial^3\zeta}{\partial\xi^3} + \beta_4\frac{\partial^3\zeta}{\partial\xi\,\partial\eta^2}$$
$$= \Lambda_1\zeta^2\zeta^* + i\Lambda_2\zeta\zeta^*\frac{\partial\zeta}{\partial\xi} + i\Lambda_3\zeta^2\frac{\partial\zeta^*}{\partial\xi} + \Lambda_4\zeta H\frac{\partial}{\partial\xi}(\zeta\zeta^*), \quad (34)$$

where

$$\beta_1 = -\frac{dc_g}{dk}, \quad \beta_2 = \frac{1-\gamma}{\lambda_\omega},$$

$$\beta_3 = \frac{2i(1+\gamma)}{\lambda_\omega}c_g\frac{dc_g}{dk}, \quad \beta_4 = \frac{i(1-\gamma)}{\lambda_\omega}\left(1 + \frac{2c_g}{\lambda_\omega}\right),$$

$$\Lambda_1 = 2\delta_1/\lambda_\omega, \quad (35)$$

$$\Lambda_2 = 2[\delta_3 - \delta_2 c_g + 4(1+\gamma)c_g\delta_1/\lambda_\omega]/\lambda_\omega,$$

$$\Lambda_3 = 2[\delta_4 + 2(1+\gamma)c_g\delta_1/\lambda_\omega]/\lambda_\omega,$$

$$\Lambda_4 = 2\delta_5/\lambda_\omega,$$

and where the operator H is defined by

$$H\Psi = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{(\xi' - \xi)\Psi(\xi',\eta')}{[(\xi'-\xi)^2 + (\eta'-\eta)^2]^{3/2}}\,d\xi'\,d\eta'. \quad (36)$$

If $\Psi(\xi,\eta)$ is independent of η, $H\Psi$ becomes the following after performing integration over η':

$$H\Psi = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\Psi(\xi')}{\xi'-\xi}\,d\xi',$$

which is the Hilbert transform of Ψ.

If we set $\gamma = V = 0$, then Eq. (34) reduces to an equation equivalent to Eq. (2) of Janssen.[8] Equation (34), with ω set equal to ω_+ given by (10), is the evolution equation for the positive mode.

IV. STABILITY OF A FINITE AMPLITUDE WAVE TRAIN

The solution for the uniform wave train of Eq. (34) is

$$\zeta = (\alpha_0/2)\exp(i\Delta\omega\tau), \quad (37)$$

where α_0 is a real constant and the nonlinear frequency shift $\Delta\omega$ is given by

$$\Delta\omega = -\tfrac{1}{8}\alpha_0^2\Lambda_1. \quad (38)$$

To study modulational stability of this wave train we introduce the following perturbation in the uniform solution:

$$\zeta = (\alpha_0/2)(1+\zeta')\exp i(\theta' + \Delta\omega\tau), \quad (39)$$

where ζ', θ' are small perturbations of amplitude and phase, respectively.

Inserting (39) into (34) and linearizing we obtain the equations

$$P_1\zeta' + P_2\theta' - (\Lambda_2 + \Lambda_3)\frac{\alpha_0^2}{4}\frac{\partial\zeta'}{\partial\xi} = 0 \quad (40)$$

and

$$P_1\theta' - P_2\zeta' + (\Lambda_3 - \Lambda_2)\frac{\alpha_0^2}{4}\frac{\partial\theta'}{\partial\xi} + \frac{\Lambda_1}{2}\alpha_0^2\zeta'$$
$$+ \frac{\Lambda_4}{4\pi}\alpha_0^2\iint_{-\infty}^{\infty}\frac{(\xi'-\xi)}{[(\xi'-\xi)^2 + (\eta'-\eta)^2]^{3/2}}\times\frac{\partial\zeta'}{\partial\xi'}\,d\xi'\,d\eta' = 0, \quad (41)$$

where P_1 and P_2 are two operators given by

$$P_1 = 2\frac{\partial}{\partial\tau} + \beta_3\frac{\partial^3}{\partial\xi^3} + \beta_4\frac{\partial^3}{\partial\xi\,\partial\eta^2}, \quad (42)$$

$$P_2 = \beta_2\frac{\partial^2}{\partial\eta^2} - \beta_1\frac{\partial^2}{\partial\xi^2}. \quad (43)$$

If we now suppose the time dependence of ζ' and θ' to be of the form $\exp(-i\Omega'\tau)$, then Eqs. (40), (41), and (43) remain the same as before but P_1 now stands for

$$P_1 = -2i\Omega' + \beta_3 \frac{\partial^3}{\partial \xi^3} + \beta_4 \frac{\partial^3}{\partial \xi\, \partial \eta^2}. \qquad (44)$$

We now take the Fourier transform of Eqs. (40) and (41) with respect to ξ,η defined by

$$[\bar{\zeta}',\bar{\theta}'] = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} [\zeta'(\xi,\eta),\theta'(\xi,\eta)] \times \exp[-i(\lambda\xi + \mu\eta)]\, d\xi\, d\eta, \qquad (45)$$

and obtain the following two equations:

$$\{\overline{P}_1 + [(\Lambda_2 + \Lambda_3)/8]\alpha_0^2\lambda\}\bar{\zeta}' + i\overline{P}_2\bar{\theta}' = 0, \qquad (46)$$

$$\left[\overline{P}_1 + \left(\frac{(\Lambda_2 - \Lambda_3)}{8}\right)\alpha_0^2\lambda\right]\bar{\theta}' - i\left[\overline{P}_2 - \frac{\alpha_0^2}{4}\Lambda_1\left(1 - \frac{\Lambda_4\lambda^2}{\Lambda_1\sqrt{\lambda^2+\mu^2}}\right)\right]\bar{\zeta}' = 0, \qquad (47)$$

where

$$\overline{P}_1 = \Omega - c_g\lambda + (\beta_3/2)\lambda^3 + (\beta_4/2)\lambda\mu^2, \qquad (48)$$

$$\overline{P}_2 = (\beta_1/2)\lambda^2 - (\beta_2/2)\mu^2, \qquad (49)$$

where

$$\Omega = \Omega' + c_g\lambda. \qquad (50)$$

For a nontrivial solution of (46) and (47) to exist the determinant of these two equations must be equal to zero, and this gives the nonlinear dispersion relation

$$[\overline{P}_1 + \tfrac{1}{8}(\Lambda_2 + \Lambda_3)\alpha_0^2\lambda\,][\overline{P}_1 + \tfrac{1}{8}(\Lambda_2 - \Lambda_3)\alpha_0^2\lambda\,] = \overline{P}_2\left[\overline{P}_2 - \frac{\alpha_0^2}{4}\Lambda_1\left(1 - \frac{\Lambda_4\lambda^2}{\Lambda_1\sqrt{\lambda^2+\mu^2}}\right)\right]. \qquad (51)$$

We restrict our future analysis to the case of a one-dimensional perturbation, i.e., $\mu = 0$.

From (51), for $\mu = 0$,

$$\overline{P}_1 = -\frac{\Lambda_2}{8}\alpha_0^2\lambda \pm \sqrt{\overline{P}_2\left[\overline{P}_2 - \frac{\alpha_0^2\Lambda_1}{4}\left(1 - \frac{\Lambda_4}{\Lambda_1}|\lambda|\right)\right]}. \qquad (52)$$

From (48), (49), and (52)

$$\Omega = c_g\lambda - \frac{\beta_3}{2}\lambda^3 - \frac{\Lambda_2}{8}\alpha_0^2\lambda \pm \sqrt{\frac{\beta_1}{2}\lambda^2\left[\frac{\beta_1}{2}\lambda^2 - \frac{\alpha_0^2}{4}\Lambda_1\left(1 - \frac{\Lambda_4}{\Lambda_1}|\lambda|\right)\right]} = \Omega_1 \pm \sqrt{\Omega_2}, \qquad (53)$$

where

$$\Omega_1 = c_g\lambda - (\beta_3/2)\lambda^3 - (\Lambda_2/8)\alpha_0^2\lambda \qquad (54)$$

and

$$\Omega_2 = \frac{\beta_1}{2}\lambda^2\left(\frac{\beta_1}{2}\lambda^2 - \frac{\alpha_0^2}{4}\Lambda_1 + \frac{\alpha_0^2\Lambda_4}{4}|\lambda|\right). \qquad (55)$$

If we set $\gamma = V = 0$, then the dispersion relation (51) reduces to

$$\left(\overline{P}_1 - \frac{7}{4}\alpha_0^2\lambda\right)\left(\overline{P}_1 - \frac{5}{4}\alpha_0^2\lambda\right) = \overline{P}_2\left[\overline{P}_2 - \alpha_0^2\left(1 - \frac{\lambda^2}{\sqrt{\lambda^2+\mu^2}}\right)\right], \qquad (56)$$

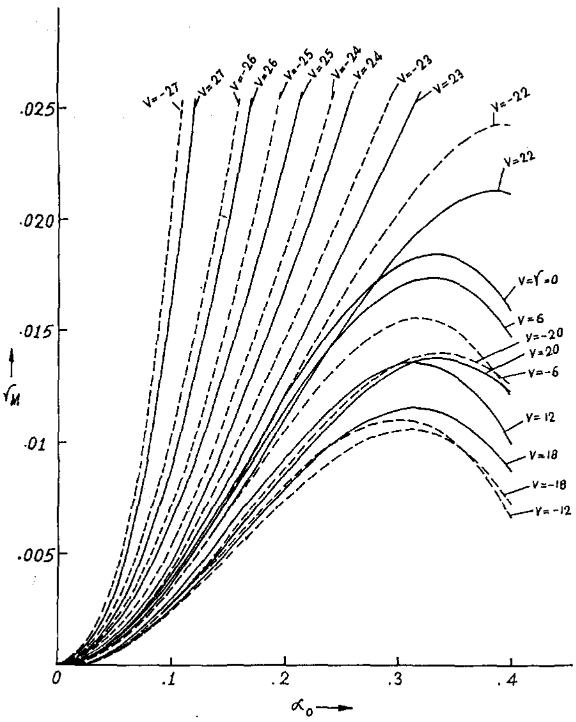

FIG. 1. Maximum growth rate γ_M as a function of nondimensional wave steepness α_0. Here $\gamma = 0.001\,29$ for all the graphs except the one with $\gamma = 0$ written on the graph. —: represents the positive mode; ---: negative mode.

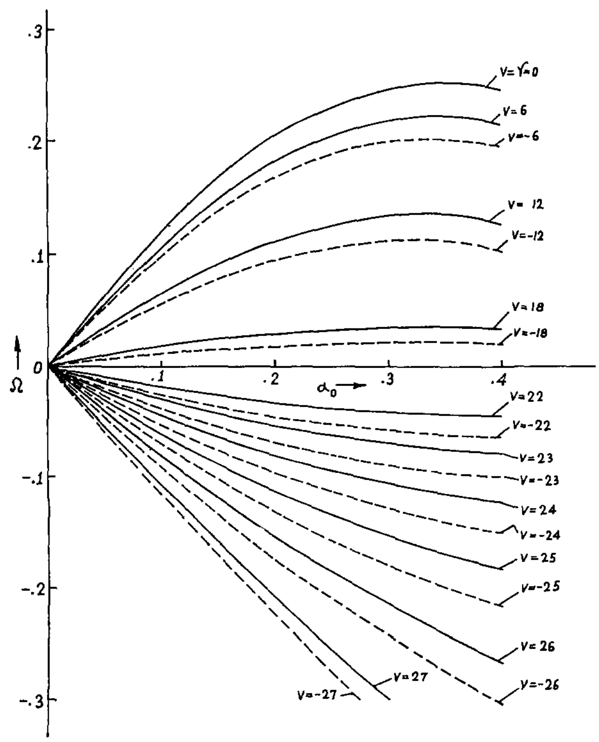

FIG. 2. Plot of perturbed frequency Ω at marginal stability against wave steepness α_0. Here $\gamma = 0.001\,29$ for all the graphs except for the one with $\gamma = 0$ written on the graph. —: represents the positive mode; ---: negative mode.

which is the same as Eq. (3.6) of Dysthe.[7]

Instability occurs when

$$\lambda^2 < \frac{\alpha_0^2 \Lambda_1 [1 - (\Lambda_4 |\lambda| / \Lambda_1)]}{2\beta_1} \quad (57)$$

and the maximum growth rate γ_M of instability is given by

$$\gamma_M = (\Lambda_1 \alpha_0^2 / 8)[1 - (\Lambda_4 \alpha_0 / \sqrt{4\Lambda_1 \beta_1})]. \quad (58)$$

At marginal stability Ω is real and is given by

$$\Omega = c_g \sqrt{\Lambda_1 / 2\beta_1}\, \alpha_0 (1 - \Lambda_4 \alpha_0 / \sqrt{8\Lambda_1 \beta_1}). \quad (59)$$

If we set $\gamma = V = 0$, then Eqs. (59) and (58) reduce to Eqs. (3.9) and (3.10) of Dysthe.[7]

In Fig. 1 the maximum growth rate γ_M of instability for the positive mode, which can be obtained from (58) by setting $\omega = \omega_+$, has been plotted against dimensionless wave steepness α_0 for some different values of dimensionless wind velocity V. By replacing V by $-V$ we make a similar plot for the negative mode, which is possible as stated at the end of Sec. II. From the figure it is seen that for both modes γ_M increases with the increase of α_0 up to a certain value of α_0, then its value decreases. The growth rate is found to be appreciably much higher for wind velocity approaching its critical value.

In Fig. 2 the perturbed frequency Ω at marginal stability for the positive mode given by (59) with $\omega = \omega_+$ has been plotted against wave steepness α_0 for some different values of wind velocity V. By replacing V by $-V$ we make a similar plot for the negative mode. In both the figures continuous graphs represent the positive mode, while the dotted graphs represent the negative mode.

Finally we check our results with that of Yuen[16] for $\gamma = 0.1$ and for two sets of values of (α_0, V).

For $\gamma = 0.1$, the value of Ω_2 will be negative if (i) $0 < \lambda < 0.362\,99$ when $\alpha_0 = 0.2$, $V = 1.573$; and (ii) $0 < \lambda < 0.147\,16$ when $\alpha_0 = 0.2$, $V = 2.83$. So for $\gamma = 0.1$, instability occurs when $0 < \lambda < 0.362\,99$ and $0 < \lambda < 0.147\,16$ for $\alpha_0 = 0.2$, $V = 1.573$ and $\alpha_0 = 0.2$, $V = 2.83$, respectively. These two instability regions correspond to the instability regions for $q = 0$ of Figs. 4(b) and 4(c), respectively, of Yuen.[16] Therefore our result in this case is in good agreement with the results of Yuen.[16]

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APPENDIX: THE SOLUTIONS TO EQS. (17)–(26)

$$A_{11} = -i\omega\zeta_{11}, \quad A_{12} = \omega \frac{\partial \zeta_{11}}{\partial x_1} + \frac{\partial \zeta_{11}}{\partial t_1} - i\omega\zeta_{12},$$

$$A'_{11} = i(\omega - V)\zeta_{11},$$

$$A'_{12} = -\omega \frac{\partial \zeta_{11}}{\partial x_1} - \frac{\partial \zeta_{11}}{\partial t_1} + i(\omega - V)\zeta_{12},$$

$$A_{22} = -i\omega\zeta_{22} + i\omega\zeta_{11}^2,$$

$$A'_{22} = i(\omega - V)\zeta_{22} + i(\omega - V)\zeta_{11}^2,$$

$$A_{23} = 2i\omega\zeta_{11}\zeta_{12} - \zeta_{11}\frac{\partial \zeta_{11}}{\partial t_1} - i\omega\zeta_{23} + \frac{1}{2}\frac{\partial \zeta_{22}}{\partial t_1} + \frac{\omega}{2}\frac{\partial \zeta_{22}}{\partial x_1},$$

$$A'_{23} = 2i(\omega - V)\zeta_{11}\zeta_{12} - \zeta_{11}\frac{\partial \zeta_{11}}{\partial t_1} - V\zeta_{11}\frac{\partial \zeta_{11}}{\partial x_1} + i(\omega - V)\zeta_{23} - \frac{1}{2}\frac{\partial \zeta_{22}}{\partial t_1} + \frac{\omega}{2}\frac{\partial \zeta_{22}}{\partial x_1}, \quad (A1)$$

$$A_{02} = 2\omega F^{-1}\left[\frac{1}{\bar{k}} F\left(\frac{\partial}{\partial x_1}(\zeta_{11}\zeta_{11}^*)\right)\right], \quad (A2)$$

$$A'_{02} = 2(\omega - V) F^{-1}\left[\frac{1}{\bar{k}} F\left(\frac{\partial}{\partial x_1}(\zeta_{11}\zeta_{11}^*)\right)\right], \quad (A3)$$

$$\zeta_{02} = 0, \quad (A4)$$

$$\zeta_{03} = F^{-1}\left(\frac{i\bar{\omega}}{(1-\gamma)}\bar{A}_{02} - \frac{i\gamma(\bar{\omega} - \bar{k}_x V)}{(1-\gamma)}\bar{A}'_{02}\right), \quad (A5)$$

$$\zeta_{22} = 2f^{-1}[\omega^2 - \gamma(\omega - V)^2]\zeta_{11}^2, \quad (A6)$$

$$\zeta_{23} = f^{-1}\Big(4[\omega^2 - \gamma(\omega - V)^2]\zeta_{11}\zeta_{12} + 4i[\omega - \gamma(\omega - V)]\zeta_{11}\frac{\partial \zeta_{11}}{\partial t_1} - 2i[\omega^2 - \gamma(\omega - V)(\omega - 3V)]\zeta_{11}\frac{\partial \zeta_{11}}{\partial x_1}\Big) + f^{-2}\Big(-16i\omega[\omega^2 - \gamma(\omega - V)^2]\zeta_{11}\frac{\partial \zeta_{11}}{\partial t_1} - 4i[\omega^2 - \gamma(\omega - V)^2]\zeta_{11}\frac{\partial \zeta_{11}}{\partial x_1}\Big), \quad (A7)$$

where

$$f = \lambda(2\omega, 2, 0). \quad (A8)$$

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