

A Delay-Diffusion Model of Marine Plankton Ecosystem Exhibiting Cyclic Nature of Blooms

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Abstract. The cyclic nature of blooms is a very interesting characteristic of marine plankton ecosystem. Release of toxins by some phytoplanktons has an important role on planktonic interactions and hence on regulating the blooms. A mathematical model describing the phytoplankton-zooplankton interaction with these characteristics is studied. The time needed for liberation of toxins by phytoplanktons is considered. To account for the spatial heterogeneity of an aquatic environment, diffusivity of different plankton populations is also incorporated into the system. Stability and bifurcation behaviour of different steady states are analysed.

Key words: phytoplankton, zooplankton, bloom, toxin, time delay, diffusion, Hopf-bifurcation

1. Introduction

Phytoplanktons are one of the most important components of the marine ecosystem. They not only form a basis for all aquatic food chains but also perform a very useful service by providing oxygen for human life and by absorbing carbon-dioxide [1]. A remarkable feature of plankton population is the occurrence of rapid population explosion and almost equally rapid decline, separated by periods of almost stationary high or low population levels. In marine plankton ecology, this phenomenon is commonly known as *bloom*. Planktonic blooms may be divided into two categories, namely, *spring blooms* and *red tides*. Both involve surges of phytoplankton populations. The process by which these outbursts are controlled are not clearly understood. Some of the mechanisms proposed include availability of trace elements [2, 3], the vertical stability of water column and salinity.

Spring blooms occur seasonally as a result of changes in temperature or nutrient availability which is almost certainly induced by seasonal changes in thermocline depth and strength and consequent mixing. Red tides, on the other hand, are localized outbreaks and occur in coastal waters, in estuaries and fronts [4]. Though they are not directly correlated with a particular seasonal variation their occurrence does coincide with regions and times of high water temperature.

Phytoplankton species which take part in bloom phenomenon are mainly of two types:

- (i) The diatoms and the blue-green algae which are very small plankton species capable of a very fast rate of reproduction.
- (ii) The dinoflagellates (e.g. *Chattonila*, *Gymnodinium* and *Protogonyaulax*) which are relatively large species with a significantly slower rate of reproduction. They are however able to swim actively to adjust their depth in water column and hence can take full advantage of the available light and nutrient conditions. These two groups adopt fundamentally different approaches to exploit their environment.

Blooms of blue-green algae are connected with a varied number of health problems ranging from skin irritation to liver damage depending upon the time and duration of exposure. Dinoflagellates secrete large quantities of neurotoxins during red tides. These toxins can cause substantial mortality to fish. The poison absorbed by shell fish can cause paralysis and death in sea birds, marine mammals and humans [5]. The toxins produced by the dinoflagellate *Alexandrium tamarense* and the diatom *Pseudo-nitzschia australis* [6] are liberated into the sea water before they are eaten up and may well affect the zooplankton when they are in water. The fact that a considerable number of phytoplanktons (e.g. *Pseudo-nitzschia* sp, *Gambierdiscus toxicus*, *Prorocentrum* sp, *Coolia monotis*, *Amphidinium carterae*, *Alexandrium* sp, *Pfiesteria piscicida*, *Prymnesium patelliferum* etc.) are able to produce toxins had been established by the work of many researchers [7–10].

The toxins produced by different phytoplankton species has a significant role in shaping the dynamical behaviour of marine plankton ecosystems [11–13]. The zooplankton population tries to avoid the areas where the concentration of phytoplankton is very large. The reason may be either dense concentration or the effect of toxic substances released by phytoplanktons. Buskey and Stockwell [14] have shown in their field study that macro- and meso-zooplankton populations are reduced during the blooms of *Chrysophyte Aureococcus anophagefferens*. A number of field studies [8, 15, 16] as well as laboratory experiments [17–19] have established that toxicity has a great impact on phytoplankton-zooplankton interaction dynamics.

Biological delay systems of one type or another have been considered by a number of authors [20–23]. These systems governed by integro-differential equations exhibit much more rich dynamics than ordinary differential systems. In the phytoplankton-zooplankton system considered above, the liberation of toxic substances by the phytoplankton species is not an instantaneous process but is mediated by some time delay. This phenomenon is supported by the observation that zooplankton mortality due to the toxic phytoplankton bloom occurs after some time lapse. The field study conducted by Chattopadhyay *et al.* [24] suggests that the abundance of *Paracalanus* (zooplankton) population reduces after some time lapse of the bloom of toxic phytoplankton *Noctiluea Scintillans*. This time delay can

be interpreted biologically as the time required for the phytoplankton species to mature before they can produce toxic substances.

In a marine environment, due to the presence of strong water current, the nutrient and the different plankton populations will move in the horizontal as well as vertical directions. Thus diffusion plays an important role in modeling aquatic ecosystem. Steele [25] and Sjoberge [26] studied the effect of diffusion on a planktonic model within the marine pelagic environment. Okubo [27] also studied a number of such ecosystems in different contexts.

The present study deals with a mathematical model describing the interaction of phytoplankton and zooplankton population in an aquatic environment. The toxic substances released by some phytoplankton species and their adverse effect on zooplankton is considered. In order to account for the time needed by the phytoplankton to mature after which they can release toxins, a distributed time delay is incorporated into the system. Moreover, due to the presence of water current in a marine environment, the effect of diffusion on different populations is considered. The stability and bifurcation behaviour of different steady states are analyzed. Expressions for some of the bifurcating solutions are also constructed.

2. Description of the Model

The model, we present, describes the bloom phenomenon as a spatially heterogeneous system where the population evolve with space and time. The growth of the phytoplankton is assumed to follow a logistic law [28] with intrinsic growth rate γ and carrying capacity k . The predation functional response are assumed to follow the *law of mass action* which is the most fundamental form of predational response.

In most biological models, the diffusion considered is a Fickian diffusion where the population flux is proportional to the concentration gradient. This approach to diffusion assumes that the individuals of the population are performing an unbiased random walk. These assumptions translate into a partial differential equation for the population, essentially just the corresponding ordinary differential equation model with a Laplacian term added to the right hand side to represent the diffusion [28]. When diffusion is introduced into a mathematical model involving delays, the normal practice, following the above philosophy, is to add a diffusion term to the corresponding delay-differential equation model where the delay is discrete in nature. But of late, it has been recognized that there are some modeling difficulties with this simple approach. The difficulty arose from the fact that the phenomenon of diffusion and time delays, though associated with space and time respectively, are not independent of each other. So when time delays are introduced in a model involving diffusion to account for the population at a previous time, it must also be kept in mind that the individuals of the population have not been at the same point in space at previous times. Thus to account for the drift of individuals to their present position from all possible positions at previous times, the delay term has to involve a weighted spatial averaging over the whole of the infinite domain which

can be mathematically realized by considering the delay to be of distributed nature. With these considerations, our model equations are

$$\begin{aligned}\frac{\partial P}{\partial t} &= \gamma P \left(1 - \frac{P}{k}\right) - \alpha P Z + D \Delta P \\ \frac{\partial Z}{\partial t} &= \beta P Z - \mu Z - \theta Z F ** P + \Delta Z\end{aligned}\quad (1)$$

where $(x, t) \in \mathfrak{R}^n \times (0, \infty)$ and

$$F ** P(\mathbf{x}, t) = \int_{\mathfrak{R}^n} \int_{-\infty}^t F(\mathbf{x} - \mathbf{y}, t - s) P(\mathbf{y}, s) ds d\mathbf{y} \quad (2)$$

Here P and Z represent the population densities of phytoplankton and zooplankton respectively. D is the constant diffusion coefficient of the phytoplankton and it is assumed that the diffusion of phytoplankton is much less than that of its predator zooplankton. The following assumptions are made on the kernel $F(\mathbf{x}, t)$.

1. $F \in L^1(\mathfrak{R}^n \times (0, \infty))$ and $tF \in L^1(\mathfrak{R}^n \times (0, \infty))$. This assumption implies that the convolution $F ** P$ is spatio-temporal. The assumption of integrability on $tF(\mathbf{x}, t)$ is needed for technical reasons.
2. F satisfies the condition $F ** 1 = 1$ that is

$$\int_{\mathfrak{R}^n} \int_0^\infty F(\mathbf{x}, t) dt d\mathbf{x} = 1$$

which implies that the uniform steady state solutions of the system are the same as that of the system without delay.

3. The kernel F defined by $F = F(r, t)$ where $r = |\mathbf{x}|$ quantifies the effect that $P(\mathbf{y}, s)$ has on $P(\mathbf{x}, t)$, ($s \leq t$). It is assumed that the nonlocal effect depends only on the distance and not on the direction of \mathbf{y} from \mathbf{x} .
4. $F \geq 0$ as F is a weighting function.

3. Stability Analysis

In this section, we perform the linear stability analysis of the system (1). The equilibrium points of the system are:

1. Trivial equilibrium point: $E_T \equiv (0, 0)$
2. Axial equilibrium point: $E_A \equiv (k, 0)$
3. Interior equilibrium point: $E_I \equiv (P^*, Z^*)$ where

$$P^* = \frac{\mu}{\beta - \theta} \quad \text{and} \quad Z^* = \frac{\gamma}{\alpha} \left(1 - \frac{P^*}{k}\right)$$

Now for the existence of the steady-state of coexistence E_I , we should have

$$\beta - \theta > \frac{\mu}{k} \quad (3)$$

First we analyze the stability to general perturbation of the trivial equilibrium point E_T . For this we have to consider the system

$$\begin{aligned}\frac{\partial P}{\partial t} &= \gamma P + D \Delta P \\ \frac{\partial Z}{\partial t} &= -\mu Z + \Delta Z\end{aligned}\quad (4)$$

The characteristic equation corresponding to system (4) shows that if $q^2 < \gamma/D$ or $q < \sqrt{\gamma/D}$, the system will be linearly stable.

Next we investigate the stability of the axial equilibrium $(k, 0)$. For this we substitute $P = k + \tilde{P}$ and $Z = \tilde{Z}$. With these substitutions we obtain from (1)

$$\begin{aligned}\frac{\partial \tilde{P}}{\partial t} &= -\gamma \tilde{P} - \alpha k \tilde{Z} + D \Delta \tilde{P} \\ \frac{\partial \tilde{Z}}{\partial t} &= (\beta k - \mu) \tilde{Z} - \theta \tilde{Z} (F ** \tilde{P}) + \Delta \tilde{Z}\end{aligned}\quad (5)$$

As the convolution term in (5) involves values of P at all times previous to t , for a properly posed problem in $\mathfrak{R}^n \times (0, \infty)$ we need to provide initial data for all $t \leq 0$. We take $P = k$ for $t < 0$ (so that $\tilde{P} = 0$ for all $t < 0$) and $P(\mathbf{x}, 0) = P_0(\mathbf{x})$, $Z(\mathbf{x}, 0) = Z_0(\mathbf{x})$. With these assumptions we can apply Laplace Convolution Theorem on (5). Taking Laplace transformation of (5) we get,

$$\begin{aligned}s \bar{\tilde{P}}(\mathbf{x}, s) - \bar{\tilde{P}}_0(\mathbf{x}) &= \gamma \bar{\tilde{P}} - \alpha k \bar{\tilde{Z}} + D \Delta \bar{\tilde{P}} \\ s \bar{\tilde{Z}}(\mathbf{x}, s) - \bar{\tilde{Z}}_0(\mathbf{x}) &= (\beta k - \mu) \bar{\tilde{Z}} - \theta \int_{\mathfrak{R}^n} \bar{F}(\mathbf{x} - \mathbf{y}, s) \bar{\tilde{P}}(\mathbf{y}, s) dy + \Delta \bar{\tilde{Z}}\end{aligned}\quad (6)$$

where bar denotes Laplace transform and the transform variable is given by s . In order to test linear stability to plane perturbations, let us take the initial conditions as

$$\begin{aligned}P_0(\mathbf{x}) &= 1 + \varepsilon P' e^{i\mathbf{q}\cdot\mathbf{x}} \\ Z_0(\mathbf{x}) &= \varepsilon Z' e^{i\mathbf{q}\cdot\mathbf{x}}\end{aligned}\quad (7)$$

where \mathbf{q} is the wave vector. Also,

$$\begin{aligned}\bar{\tilde{P}}_0(\mathbf{x}) &= \varepsilon P' e^{i\mathbf{q}\cdot\mathbf{x}} \\ \bar{\tilde{Z}}_0(\mathbf{x}) &= \varepsilon Z' e^{i\mathbf{q}\cdot\mathbf{x}}\end{aligned}\quad (8)$$

Considering the solution of the transformed system in the form

$$\begin{bmatrix} \bar{P}(\mathbf{x}, s) \\ \bar{Z}(\mathbf{x}, s) \end{bmatrix} = \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} e^{i\mathbf{q}\cdot\mathbf{x}} \quad (9)$$

we get

$$\begin{bmatrix} s + \mu + Dq^2 & \alpha k \\ \theta \bar{H}(s, q^2) & s + (\mu - k\beta) + q^2 \end{bmatrix} \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} = \varepsilon \begin{bmatrix} P' \\ Z' \end{bmatrix} \quad (10)$$

where $q^2 = \mathbf{q} \cdot \mathbf{q}$ and

$$\bar{H}(s, q^2) = \int_{\mathbb{R}^n} \int_0^\infty F(\mathbf{x}, t) e^{-st} e^{-i\mathbf{q}\cdot\mathbf{x}} dt d\mathbf{x} \quad (11)$$

The singularities of (\bar{P}, \bar{Z}) are determined by the roots of

$$s^2 + s[\gamma + \mu - k\beta + (D + 1)q^2] + (\gamma + Dq^2)(\mu - k\beta + q^2) - \alpha k\theta \bar{H} = 0 \quad (12)$$

If all the roots of (12) have negative real parts then the steady-state will be stable. Next we state the following properties of $\bar{H}(s, q^2)$.

1. For real s , $\bar{H}(s, q^2)$ is a real-valued function of q^2 and s .
2. $\bar{H}(0, 0) = 1$.
3. If $\text{Re}(s) \geq 0$, then $|\bar{H}(s, q^2)| \leq 1$.
4. When s is real and non-negative then $\bar{H}(s, q^2) \leq 1$.
5. For real, non-negative s and for $(s, q^2) \neq (0, 0)$, $\bar{H}(s, q^2) < 1$.
6. For $(\omega, q^2) \neq (0, 0)$, $\bar{H}(i\omega, q^2) \neq 1$.

The proofs of these properties are available in standard literature. Coming back to stability analysis, we note that Eq. (12) should have one positive root if

$$\mu + q^2 < k\beta < \mu + (D + 1)q^2 + \gamma \quad (13)$$

In this case instability of the equilibrium point may occur and the equilibrium point will be an unstable saddle.

Let us now study the most interesting equilibrium point, namely, the equilibrium point of co-existence $E_I(P^*, Z^*)$. In order to perform the stability analysis of this equilibrium point we set,

$$\begin{aligned} P &= P^* + \hat{P} \\ Z &= Z^* + \hat{Z} \end{aligned} \quad (14)$$

The corresponding linearised system will be

$$\begin{aligned}\frac{\partial \hat{P}}{\partial t} &= \frac{\gamma P^*}{k} \hat{P} - \alpha P^* \hat{Z} + D \Delta \hat{Z} \\ \frac{\partial \hat{Z}}{\partial t} &= \beta Z^* \hat{P} + (\beta P^* - \mu) \hat{Z} - \theta \int_{\mathbb{R}^n} \int_{-\infty}^t F(\mathbf{x} - \mathbf{y}, t - s) \hat{P} \, ds d\mathbf{y} + \Delta \hat{P}\end{aligned}\quad (15)$$

Taking Laplace transform and applying the technique as in the case of the axial equilibrium E_A , we get the corresponding equation as

$$\begin{aligned}f(s, \theta, q^2) &\equiv s^2 + s \left(\mu - \beta P^* + q^2 + \frac{\gamma P^*}{k} + Dq^2 \right) \\ &\quad + \left(\frac{P^* \gamma}{k} + Dq^2 \right) (\mu - \beta P^* + q^2) - \alpha P^* \theta \bar{H} + \alpha \beta P^* Z^* = 0\end{aligned}\quad (16)$$

We rewrite Eq. (16) as $f(s, \theta, q^2) \equiv s^2 + Ms + N$, the roots of which will determine the stability criteria of the equilibrium point.

Suppose, $q^2 \geq 0$ be fixed. Let us first show that all the roots of Eq. (16) have negative real parts. Consider the region $\text{Re}(z) \geq 0$, $|z| = R$ and let ∂R be the boundary of the region. We divide the region into two parts, namely, $\partial^1(R)$, which is the circular part $\text{Re}(z) > 0$ and $\partial^2(R)$, defined by

$$\partial^2(R) = \{z = iy : -R \leq y \leq R\}$$

Then by construction, $\partial(R) = \partial^1(R) + \partial^2(R)$. Thus for $f(z, \theta, q^2) \neq 0$ with $\text{Re}(z) = 0$, the number of roots of $f(z, \theta, q^2)$ inside the region $\partial(R)$ will be given by the argument principle as

$$\nu(R) = \frac{1}{2\pi i} \int_{\partial R} \frac{f'(z, \theta, q^2)}{f(z, \theta, q^2)} dz \equiv J_1(R) + J_2(R)\quad (17)$$

where

$$\begin{aligned}J_m(R) &= \frac{1}{2\pi i} \int_{\partial^m(R)} \frac{f'(z, \theta, q^2)}{f(z, \theta, q^2)} dz \\ m &= 1, 2\end{aligned}\quad (18)$$

The number of roots with $\text{Re}(z) \geq 0$ is

$$\nu(\infty) = \lim_{R \rightarrow \infty} \nu(R)\quad (19)$$

Next we introduce a function

$$h(z) = \frac{2 - \alpha P^* \theta \bar{H}}{z} \quad (20)$$

and denote by T , the strength of the response delay described by the kernel F as

$$T = \int_{\mathfrak{R}^n} \int_0^\infty t F(\mathbf{x}, t) dt d\mathbf{x} \quad (21)$$

Now as

$$\bar{H}'(z, q^2) = - \int_{\mathfrak{R}^n} \int_0^\infty t F(\mathbf{x}, t) e^{-zt} e^{-i\mathbf{q} \cdot \mathbf{x}} dt d\mathbf{x}$$

it follows that

$$|\bar{H}'(z, q^2)| \leq T \quad (22)$$

when $\operatorname{Re}(z) \geq 0$. Again from

$$|\bar{H}'(\operatorname{Re}^{i\theta}, q^2)| \leq \int_{\mathfrak{R}^n} \int_0^\infty t F(\mathbf{x}, t) e^{-(R \cos \theta)t} dt d\mathbf{x}$$

we find

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\bar{H}'(\operatorname{Re}^{i\theta}, q^2)| d\theta &\leq 2 \int_{\mathfrak{R}^n} \int_0^\infty t F(\mathbf{x}, t) \int_{-\frac{\pi}{2}}^0 e^{-(R \cos \theta)t} d\theta dt d\mathbf{x} \\ &= 2 \int_{\mathfrak{R}^n} \int_0^\infty t F(\mathbf{x}, t) \int_0^{\frac{\pi}{2}} e^{-R \sin \theta t} d\theta dt d\mathbf{x} \\ &\leq 2 \int_{\mathfrak{R}^n} \int_0^\infty t F(\mathbf{x}, t) \int_0^{\frac{\pi}{2}} e^{-R 2t\theta/\pi} d\theta dt d\mathbf{x} \\ &= \frac{\pi}{R} \int_{\mathfrak{R}^n} \int_0^\infty t F(\mathbf{x}, t) (1 - e^{-Rt}) dt d\mathbf{x} \\ &\leq \frac{\pi}{R} \end{aligned} \quad (23)$$

where we have used Jordan's inequality in step three above. Now,

$$\begin{aligned} |J_1(R) - 1| &= \left| \frac{1}{2\pi i} \int_{\partial^1(R)} \frac{f'(z, \theta, q^2)}{f(z, \theta, q^2)} dz - \frac{1}{2\pi i} \int_{\partial^1(R)} \frac{2}{z} dz \right| \\ &= \frac{1}{2\pi} \left| \int_{\partial^1(R)} \left(\frac{f'(z, \theta, q^2)}{f(z, \theta, q^2)} - h(z) \right) dz - \alpha P^* \theta \int_{\partial^1(R)} \frac{\bar{H}'(z, q^2)}{z} dz \right| \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \left| \frac{\alpha P^* Z^* \theta \bar{H}'(z^2 + (M-1)z + N) - Mz - 2N}{z(z^2 + Mz + N)} \right| \\
 &\leq \frac{\alpha P^* \theta \frac{\pi}{R} [R^2 + (M-1)R + N] + MR + 2N}{R(R^2 + MR + N)}
 \end{aligned}$$

which tends to zero as $R \rightarrow \infty$. This shows that

$$J_1(\infty) = 1 \quad (24)$$

Again,

$$\begin{aligned}
 J_2(R) &= \frac{1}{2\pi i} \int_R^{-R} \frac{f'(iy, \theta, q^2)}{f(iy, \theta, q^2)} i dy \\
 &= \frac{1}{2\pi i} [\log f(-iR, \theta, q^2) - \log f(iR, \theta, q^2)]
 \end{aligned}$$

But, $\overline{f(iR, \theta, q^2)} = f(-iR, \theta, q^2)$ and hence using the principal branch $|\arg(z)| < \pi$ of the logarithm we deduce

$$J_2(R) = -\frac{1}{\pi} \arg f(iR, \theta, q^2) \quad (25)$$

Our next task is to find the limit of $\arg f(iR, \theta, q^2)$ as $R \rightarrow \infty$. For this we take,

$$\bar{H}(iR, q^2) = C(R) - iS(R) \quad (26)$$

where

$$\begin{aligned}
 C(R) &= \int_{\mathbb{R}^n} \int_0^\infty F(\mathbf{x}, t) \cos(Rt + \mathbf{q} \cdot \mathbf{x}) dt d\mathbf{x} \\
 S(R) &= \int_{\mathbb{R}^n} \int_0^\infty F(\mathbf{x}, t) \sin(Rt + \mathbf{q} \cdot \mathbf{x}) dt d\mathbf{x}
 \end{aligned}$$

Then $|C(R)| \leq 1$ and $|S(R)| \leq 1$ for all $R \geq 0$.

Now,

$$\operatorname{Ref}(iR, \theta, q^2) = -R^2 + M_1 N_1 - \alpha P^* \theta C(R) - \alpha \beta P^* Z^*$$

where $M_1 = \mu - \beta P^* + q^2$ and $N_1 = P^* \gamma / k + Dq^2$.

Therefore for large R ,

$$\operatorname{Ref}(iR, \theta, q^2) \rightarrow -R^2 \quad (27)$$

Also, $\text{Imf}(iR, \theta, q^2) = R(M_1 + N_1) + \alpha P^* \theta S(R)$ and hence

$$|\text{Imf}(iR, \theta, q^2)| \leq R \quad (28)$$

which is a constant. Now,

$$\begin{aligned} f(0, \theta, q^2) &= \left(\frac{\gamma P^*}{k} + Dq^2 \right) (\mu - \beta P^* + q^2) - \alpha P^* Z^* \theta C(0) + \alpha \beta P^* Z^* \\ &\geq \left(\frac{\gamma P^*}{k} + Dq^2 \right) (\mu - \beta P^* + q^2) - \alpha P^* Z^* \theta + \alpha \beta P^* Z^* \end{aligned} \quad (29)$$

Therefore, $f(0, \theta, q^2) > 0$ implies

$$\begin{aligned} k\mu\gamma\theta^2 + \theta\{-\mu - q^2(D+1) - 2k\mu\gamma\beta + \mu^2\gamma\} + \mu\gamma \\ + k\beta Dq^2 + q^2\beta + k\mu\gamma\beta^2 - \gamma\beta\mu^2 > 0 \end{aligned} \quad (30)$$

which is true for all positive values of $\theta > \theta_{\max}$ as $k\beta > \mu$ where θ_{\max} is the greatest root of $f(0, \theta, q^2) = 0$. This together with (27) and (28) (that is the asymptotic behaviour of $f(iR, \theta, q^2)$) implies that $\arg f(iR, \theta, q^2)$ tends to an odd multiple of π as $R \rightarrow \infty$. Let

$$\arg f(iR, \theta, q^2) \rightarrow (1 - 2m)\pi, \quad m \in Z$$

Then $J_2(\infty) = 2m - 1$ and consequently from (17), (19) and (24) we get, $\nu(\infty) = 2m$ which indicates that $m \geq 0$.

We intend to show in particular, that $m = 0$. Now, $m = 0$ corresponds to $\arg f(i\infty, \theta, q^2) = \pi$. A sufficient condition for ensuring $\arg f(i\infty, \theta, q^2) = \pi$ is that $\text{Imf}(iR, \theta, q^2) > 0$ when $\text{Ref}(iR, \theta, q^2) = 0$. Now, $\text{Ref}(iR, \theta, q^2) = 0$ implies,

$$-R^2 + M_1 N_1 - \alpha P^* \theta C(R) - \alpha \beta P^* Z^* = 0 \quad (31)$$

When (30) is satisfied, then since $f(0, \theta, q^2) > 0$ therefore at such points $R > 0$. Hence, when $\text{Ref}(iR, \theta, q^2) = 0$ we have,

$$\begin{aligned} \text{Imf}(iR, \theta, q^2) &= R(M_1 + N_1) + \alpha P^* \theta S(R) \\ &\geq R(M_1 + N_1) + \alpha P^* \theta R \\ &= \frac{1}{k(\beta - \theta)} [k\beta q^2(D+1) + \mu\gamma - k\theta\{\mu(1 + \alpha) + q^2(D+1)\}] \end{aligned}$$

Therefore, $\text{Imf}(iR, \theta, q^2) > 0$ implies

$$\theta < \frac{1}{k} \left[\frac{kq^2\beta(D+1) + \mu\gamma}{\mu(\alpha+1) + q^2(D+1)} \right] \equiv \theta_0 \quad (\text{say}) \quad (32)$$

The above analysis leads to the following

Theorem 3.1. *For a given wave number q^2 , the interior equilibrium point $E_1(P^*, Z^*)$ will be linearly stable if $\theta_{\max} < \theta < \theta_0$.*

From the above analysis, it is clear that toxication plays an important role in shaping the dynamical behaviour of the system (1). If the toxication factor θ is less than a critical value θ_0 , then the system is locally asymptotically stable. Thus there is no exchange of stability when $\theta < \theta_0$. Our next task is to analyze the system (1) for $\theta > \theta_0$.

4. Unstability and Bifurcation Behavior

In this section, we will investigate the unstability and bifurcation behavior of the system.

Theorem 4.1. *If $\theta > \theta_0$, the interior equilibrium point $E_1(P^*, Z^*)$ of the system (1) will become unstable.*

Proof: In the previous section, we have found out that the total number of roots with positive real part is $2m$ where $m \in \mathbb{Z}$. Now, $m \geq 2$ would correspond to a situation where, as R ranges from 0 to infinity, the graph of $f(iR, \theta, q^2)$ winds around the origin before permanently entering the left half of the complex plane. $m = 1$ means that the graph of $f(iR, \theta, q^2)$ passes underneath the origin, that is, $\text{Imf}(iR, \theta, q^2) < 0$ when $\text{Ref}(iR, \theta, q^2) = 0$. But, $\text{Imf}(iR, \theta, q^2) < 0$ implies $0 > \theta_0$. This condition together with (30) gives the criteria for unstability of the system as $\theta > \theta_0$, which completes the proof. \square

The following table gives the abbreviations of different parameters associated with the model system (1)–(2) together with their observational values [29].

Taking observational values from Table I and using (32), we find that $\theta_0 \simeq 0.034/\text{day}$. Therefore, from Theorems 3.1 and 4.1, it is clear that when the toxin production rate is less than 0.034/day, the system will remain asymptotically stable

Table I. Description of parameters and values

Parameters	Description	Values (Per day)
γ	Growth rate of phytoplankton	0.3
k	Environmental carrying capacity	108 μgN
α	Grazing efficiency of zooplankton population	0.7
β	Growth efficiency of zooplankton population	0.035
μ	Natural death rate of zooplankton	0.012
D	Diffusion coefficient of phytoplankton	0.03

and as it increases and crosses the value 0.034/day, the system will become unstable. Hence, an increase in the production of toxic substances by the phytoplanktons may destabilize an otherwise stable planktonic ecosystem which ultimately give rise to the bloom phenomenon.

Theorem 4.2. *The system (1) will undergo a Hopf-bifurcation near the interior equilibrium point E_I for $\theta > \theta_0$ when the following parameter conditions are satisfied*

$$\max \left[\frac{q^2}{2\beta - \theta}, \frac{\gamma - Dq^2}{k(\beta - \theta)} \right] < \mu < \min \left[\frac{k}{2}(\beta - \theta), \frac{q^2(D + 1)}{k\theta - \gamma} \right]; \quad \beta > \frac{\gamma}{k} \quad (33)$$

Proof: From Theorems 3.1 and 4.1 above, it can be said that as θ is increased to a value sufficiently large, the real part of at least two roots of (16) changes from negative to positive. So these roots must have crossed the imaginary axis as a pair of complex conjugates which signifies that the system undergoes a bifurcation as θ is increased.

We consider the roots of (16) in the form

$$\lambda = u + iv \quad (34)$$

Substituting (34) in (16), separating real and imaginary parts and differentiating partially we find

$$\begin{aligned} \frac{\partial u}{\partial \theta} = & \left[\frac{\mu\gamma}{k(\beta - \theta)^2} \left\{ \frac{3\mu\beta}{\beta - \theta} - \mu - q^2 + \theta \bar{H}_R \left(k - \frac{2\mu}{\beta - \theta} \right) \right\} \right. \\ & \left. + \frac{\mu\beta}{(\beta - \theta)^2} \left\{ \frac{P^*\gamma}{k} + Dq^2 - \gamma \right\} + \alpha P^* Z^* \bar{H}_R \right] \frac{A}{A^2 + v_0^2} \\ & + 2v_0 \left[\alpha \bar{H}_I \left\{ P^* Z^* + \theta \left(\frac{\mu\gamma}{\alpha(\beta - \theta)^2} - \frac{2\mu^2\gamma}{k\alpha(\beta - \theta)^2} \right) \right\} \right. \\ & \left. - 2v_0^2 \left(\frac{\gamma}{k} - \beta \right) \frac{\mu}{(\beta - \theta)^2} \right] \frac{1}{A^2 + v_0^2} \end{aligned} \quad (35)$$

where

$$A = \mu - \beta P^* + \frac{\gamma P^*}{k} + q^2(D + 1) - \alpha P^* \theta \bar{H}_s$$

$v = v_0$ when $u = 0$. \bar{H}_R denotes the real part of \bar{H} and \bar{H}_I denotes the imaginary part of \bar{H} . From the parameter restrictions assumed in the theorem, it follows

that

$$\frac{\partial u}{\partial \theta} > 0$$

that is the transversibility condition is satisfied. Hence the theorem. \square

It has already been shown that for the existence of the interior equilibrium point E_1 , the condition $\beta - \theta > 0$ should be satisfied. The above analysis demonstrates that a destabilization of the system takes place for $\theta > \theta_0$. Combining these two, we obtain a range for the toxication parameter θ , namely, $\theta_0 < \theta < \beta$ for which planktonic bloom occurs. Numerical calculations using values from Table I shows this range of θ as $0.034 < \theta < 0.035$.

From the analysis of Sections 3 and 4 it is clear that $\theta = \theta_0$ is a bifurcation point of the system, below which the system is stable and above which it becomes unstable. Phytoplankton bloom is the manifestation of the instability of the planktonic population. Again, if the death rate of zooplankton satisfies condition (33) then the bifurcation is of Hopf-type which ecologically signifies the periodic behaviour of the bloom phenomenon.

5. Construction of Bifurcating Solutions

From the analysis of the previous section, we see that as θ crosses a certain value, namely, θ_0 , two roots of the eigenvalue equation (16) will cross the imaginary axis as a pair of complex conjugates. The solutions to the linearised system will be proportional to $e^{i\mathbf{q}\cdot\mathbf{x}} \cdot e^{i\omega t}$, which implies that the bifurcating solution is periodic both in space and time. We seek a solution of (1) in the form

$$\begin{aligned} P(\mathbf{x}, t) &= \hat{P}(\xi, \tau) \\ Z(\mathbf{x}, t) &= \hat{Z}(\xi, \tau) \end{aligned} \tag{36}$$

where $\xi = \mathbf{q} \cdot \mathbf{x}$ and $\tau = \omega t \cdot \hat{P}$ and \hat{Z} are periodic of period 2π in the arguments ξ and τ .

With (36), system (1) after dropping the hats reduces to

$$\begin{aligned} \omega \frac{\partial P}{\partial \tau} &= \gamma P \left(1 - \frac{P}{k} \right) - \alpha P Z + Dq^2 \frac{\partial^2 P}{\partial \xi^2} \\ \omega \frac{\partial Z}{\partial \tau} &= \beta P Z - \mu Z - \theta Z(F ** P) + q^2 \frac{\partial^2 Z}{\partial \xi^2} \end{aligned} \tag{37}$$

where $(\xi, \tau) \in (0, 2\pi) \times (0, 2\pi)$.

Since we are looking for solutions symmetric in ξ , the solution should satisfy the following boundary conditions

$$P_\xi(0, \tau) = P_\xi(\pi, \tau) = Z_\xi(0, \tau) = Z_\xi(\pi, \tau) = 0 \quad (38)$$

P and Z should also satisfy

$$\begin{aligned} P(\xi, 0) &= P(\xi, 2\pi), & Z(\xi, 0) &= Z(\xi, 2\pi) \\ P_\tau(\xi, 0) &= P_\tau(\xi, 2\pi), & Z_\tau(\xi, 0) &= Z_\tau(\xi, 2\pi) \end{aligned} \quad (39)$$

From (38) and (39) it is obvious that the system is considered in $(\xi, \tau) \in (0, \pi) \times (0, 2\pi)$ only. The extension of the solution outside this interval is obvious.

In order to construct the bifurcating solutions we use Poincaré–Lindstedt procedure. In this method, the solution of (37) with the conditions (38) and (39) will be of the form

$$\begin{pmatrix} P \\ Z \end{pmatrix} = \begin{pmatrix} P^* \\ Z^* \end{pmatrix} + \epsilon \begin{pmatrix} P_1 \\ Z_1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} P_2 \\ Z_2 \end{pmatrix} + \dots \quad (40)$$

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \dots \quad (41)$$

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots \quad (42)$$

Again, $F ** P$ involves ω and so we have to expand each $F ** P_i$ in powers of ϵ namely,

$$\begin{aligned} F ** P_i &= (F ** P_i)|_{\epsilon=0} + \epsilon(F ** P_i)_\epsilon|_{\epsilon=0} + \frac{1}{2}\epsilon^2(F ** P_i)_{\epsilon\epsilon}|_{\epsilon=0} + \dots \\ &\equiv (F ** P_i)^0 + \epsilon(F ** P_i)_\epsilon^0 + \frac{1}{2}\epsilon^2(F ** P_i)_{\epsilon\epsilon}^0 + \dots \end{aligned} \quad (43)$$

Let us now define the linear operator

$$M = \omega_0 \frac{d}{d\tau} - \begin{bmatrix} -\frac{\gamma P^*}{k} + Dq^2 \frac{d^2}{d\xi^2} & -\alpha P^* \\ \beta Z^* - \theta_0 Z^*(F ** \cdot)^0 & \beta P^* - \mu - q^2 \frac{d^2}{d\xi^2} \end{bmatrix} \quad (44)$$

Then the first two perturbation equations are

$$M \begin{pmatrix} P_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (45)$$

$$M \begin{pmatrix} P_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} -\omega_1 P_{1\tau} - \frac{\gamma}{k} P_1^2 - \alpha P_1 Z_1 \\ -\omega_1 Z_{1\tau} + \beta P_1 Z_1 - \theta_0 Z_1 (F ** P_1)^0 \\ -\theta_1 Z^*(F ** P_1)^0 - \theta_0 Z^*(F ** P_1)_\epsilon^0 \end{pmatrix} \equiv \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \quad (46)$$

with $(\xi, \tau) \in (0, \pi) \times (0, 2\pi)$ and conditions given in (38) and (39). A simple calculation gives the formula

$$F ** e^{mi\tau} \cos n\xi = \bar{H}(mi\omega, n^2q^2)e^{mi\tau} \cos n\xi \quad (47)$$

where $m, n = 0, 1, 2, 3, \dots$. Considering the solution in the form $P_{m,n}(\xi, \tau) = e^{mi\tau} \cos n\xi$ we have from (47)

$$(F ** P_{m,n})^0 = \bar{H}(mi\omega_0, n^2q^2)P_{m,n} \quad (48)$$

The nontrivial solution of (45) is of the form

$$\begin{pmatrix} P_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ 1 \end{pmatrix} e^{i\tau \cos \xi} + \tilde{C} \quad (49)$$

together with

$$\begin{aligned} f(i\omega_0, \theta_0, q^2) &\equiv -\omega_0^2 + i\omega_0 A + \left(\frac{P^* \gamma}{k} + Dq^2 \right) (\mu - \beta P^* + q^2) \\ &\quad + \alpha P^* Z^* \{ \beta - \theta_0 \bar{H}(i\omega_0, q^2) \} = 0 \end{aligned} \quad (50)$$

\tilde{C} in (49) represents complex conjugate terms. Since $\begin{pmatrix} P_1 \\ Z_1 \end{pmatrix}$ is a solution of (45), we get

$$a_1 = \frac{\alpha P^*}{i\omega_0 + \frac{\gamma P^*}{k} + Dq^2} \quad (51)$$

$\begin{pmatrix} P_1 \\ Z_1 \end{pmatrix}$ is the unique solution of (45) provided that $f(mi\omega_0, n^2q^2) \neq 0$ for any pair of integers $(m, n) \neq (1, 1)$.

The adjoint M^* of the linear operator M will be of the form

$$M^* = -\omega_0 \frac{d}{d\tau} - \begin{bmatrix} -\frac{\gamma P^*}{k} + Dq^2 \frac{d^2}{d\xi^2} & \beta Z^* \\ -\alpha P^* & \beta P^* - \mu + q^2 \frac{d^2}{d\xi^2} \end{bmatrix} + \theta_0 Z^* M_1^* \quad (52)$$

where

$$M_1^* \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{mi\tau} \cos n\xi \right) = \begin{pmatrix} \overline{a_1 \bar{H}(mi\omega_0, n^2q^2) e^{mi\tau} \cos n\xi} \\ 0 \end{pmatrix} \quad (53)$$

The solution of the adjoint equation, namely,

$$M^* \begin{pmatrix} P_1^* \\ Z_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (54)$$

will be of the form

$$\begin{pmatrix} P_1^* \\ Z_1^* \end{pmatrix} = \begin{pmatrix} d_1 \\ 1 \end{pmatrix} e^{i\tau} \cos \xi + \tilde{C} \quad (55)$$

where \tilde{C} represents complex conjugate terms and

$$d_1 = \frac{\beta P^* - \mu - q^2 + i\omega_0}{\alpha P^*} \quad (56)$$

Now,

$$(F ** P_1)_\varepsilon^0 = a_1 i \omega_1 \bar{H}_s(i\omega_0, q^2) e^{i\tau} \cos \xi + \tilde{C} \quad (57)$$

The solvability criteria for (46) is given by the Fredholm Alternative in the form

$$\int_0^\pi \int_0^{2\pi} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} (\bar{d}_1 e^{-i\tau} \cos \xi \quad e^{-i\tau} \cos \xi) d\tau d\xi = 0 \quad (58)$$

Using the expressions for X_1 and Y_1 we get,

$$\begin{aligned} & \left[-i\omega_1(a_1 \bar{d}_1 + 1) \int_0^\pi \cos^2 \xi d\xi - \theta_1 Z^* \bar{H}(i\omega_0, q^2) \int_0^\pi \cos^2 \xi d\xi \right. \\ & \quad \left. - \theta_0 Z^* i a_1 \omega_1 \bar{H}_s(i\omega_0, q^2) \int_0^\pi \cos^2 \xi d\xi \right] 2\pi \\ & = \left[2 \frac{\gamma}{k} a_1^2 \bar{d}_1 \pi + (\alpha a_1 \bar{d}_1 + \beta a_1 + \theta_0 \bar{H}) \int_0^{2\pi} e^{i\tau} \right] \int_0^\pi \cos^3 \xi d\xi \end{aligned} \quad (59)$$

The right hand side of (59) is always zero. Therefore, (59) holds if each term in the left hand side is zero which is possible when $\omega_1 = 0$ and $\theta_1 = 0$. Consequently, $(F ** P_1)_\varepsilon^0 = 0$ and (46) becomes,

$$\begin{aligned} M \begin{pmatrix} P_2 \\ Z_2 \end{pmatrix} &= \cos^2 \xi \begin{pmatrix} -\frac{\gamma a_1^2}{k} e^{2i\tau} - \alpha a_1 e^{2i\tau} \\ \beta a_1 e^{2i\tau} - \theta_0 e^{2i\tau} \bar{H}(i\omega_0, q^2) a_1 \end{pmatrix} + \tilde{C} \\ &= \frac{1}{2} (1 + \cos 2\xi) e^{2i\tau} \begin{pmatrix} -\frac{\gamma a_1^2}{k} - \alpha a_1 \\ \beta a_1 - \theta_0 \bar{H}(i\omega_0, q^2) a_1 \end{pmatrix} \end{aligned} \quad (60)$$

where \tilde{C} represents complex conjugate terms. Equation (60) has solutions in the form

$$\begin{pmatrix} P_2 \\ Z_2 \end{pmatrix} = \left[\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \cos 2\xi \right] e^{2i\tau} \quad (61)$$

where A_i 's and B_i 's can be obtained by using simple algebra as

$$\begin{aligned}
 A_1 &= -\frac{\gamma a_1^2}{2k(2i\omega_0 + \frac{\gamma P^*}{k})} \\
 B_1 &= -\frac{\gamma a_1^2}{2k(2i\omega_0 + \frac{\gamma P^*}{k} + 4Dq^2)} \\
 A_2 &= -\frac{\theta_0 \bar{H}(i\omega_0, q^2)}{2(2i\omega_0 - \beta P^* + \mu)} \\
 B_2 &= -\frac{\theta_0 \bar{H}(i\omega_0, q^2)}{2(2i\omega_0 - \beta P^* + \mu + 4q^2)}
 \end{aligned} \tag{62}$$

Let us now study the third perturbation equation. The third perturbation equation for the considered system will be

$$M \begin{pmatrix} P_3 \\ Z_3 \end{pmatrix} = \begin{pmatrix} -\omega_1 P_{2\tau} - \omega_2 P_{1\tau} - \frac{2\gamma}{k} P_1 P_2 - \alpha(P_1 Z_2 + P_2 Z_1) \\ \Theta \end{pmatrix} \tag{63}$$

where

$$\begin{aligned}
 \Theta &= -\omega_1 Z_{2\tau} - \omega_2 Z_{1\tau} + \beta(P_1 Z_2 + P_2 Z_1) - \theta_1 Z_1(F ** P_1)^0 - \theta_0 Z_1(F ** P_1)_\varepsilon^0 \\
 &\quad - \theta_1 Z^*(F ** P_1)_\varepsilon^0 - \theta_0 Z^*(F ** P_1)_{\varepsilon\varepsilon}^0 - (\theta_1 Z^* + \theta_0 Z_1)(F ** P_2)^0 \\
 &\quad - \theta_0 Z^*(F ** P_2)_\varepsilon^0 - \theta_2 Z^*(F ** P_1)^0 - Z_2 \theta_0 (F ** P_1)^0
 \end{aligned}$$

But from the previous analysis we have $\omega_1 = \theta_1 = 0$ and consequently $(F ** P_1)_\varepsilon^0 = 0$. With these conditions we have

$$M \begin{pmatrix} P_3 \\ Z_3 \end{pmatrix} = \begin{pmatrix} -\omega_2 P_{1\tau} - \frac{2\gamma}{k} P_1 P_2 - \alpha(P_1 Z_2 + P_2 Z_1) \\ \hat{\Theta} \end{pmatrix} \tag{64}$$

where

$$\begin{aligned}
 \hat{\Theta} &= -\omega_2 Z_{1\tau} + \beta(P_1 Z_2 + P_2 Z_1) - \theta_0 Z^*(F ** P_1)_{\varepsilon\varepsilon}^0 \\
 &\quad - \theta_0 Z_1(F ** P_2)^0 - \theta_0 Z_2(F ** P_1)^0 - \theta_2 Z^*(F ** P_2)^0
 \end{aligned}$$

The solvability condition for (64) gives

$$\begin{aligned}
 &[i\omega_2(a_1 \bar{d}_1 + 1) + 2i\omega_2 a_1 \theta_0 Z^* \bar{H}_s(i\omega_0, q^2)] \int_0^\pi \int_0^{2\pi} \cos^2 \xi d\xi d\tau \\
 &+ \left[\frac{2a_1 \bar{d}_1 \gamma}{k} + A_2 \alpha a_1 \bar{d}_1 + \alpha A_1 \bar{d}_1 - \beta a_1 A_2 - \beta A_1 \right]
 \end{aligned}$$

$$\begin{aligned}
& + \theta_0 \bar{H}(i\omega_0, q^2)(A_1 + \alpha_1 A_2) \int_0^\pi \int_0^{2\pi} \cos^2 \xi e^{2i\tau} d\xi d\tau \\
& + \left[\frac{2\gamma a_1}{k} B_1 + \alpha \bar{d}_1 (a_1 B_2 + B_1) - \beta a_1 B_2 - \beta B_1 + \theta_0 \bar{H}(i\omega_0, q^2)(B_1 + a_1 B_2) \right] \\
& \times \int_0^\pi \int_0^{2\pi} \cos^2 \xi \cos 2\xi e^{2i\tau} d\xi d\tau \\
& + \theta_2 Z^* \bar{H}(i\omega_0, q^2) \int_0^\pi \int_0^{2\pi} (A_1 + B_1 \cos 2\xi) \cos \xi e^{i\tau} d\xi d\tau = 0 \quad (65)
\end{aligned}$$

which may be written as

$$\Psi(q^2, \xi, \tau, a_1, \bar{d}_1, \omega_0, \omega_2) + \theta_2 = 0$$

where $\Psi(q^2, \xi, \tau, a_1, \bar{d}_1, \omega_0, \omega_2) \neq 0$. From (65) we may write

$$\theta_2 = \text{Re}\{-\Psi(q^2, \xi, \tau, a_1, \bar{d}_1, \omega_0, \omega_2)\} \quad (66)$$

Now, if $\text{Re } \Psi < 0$, the bifurcation is supercritical, that is a stable bifurcation of the system takes place. If $\text{Re } \Psi > 0$ then the bifurcation will be subcritical and hence in this case, the bifurcating periodic solutions are not stable.

6. Discussion

The phenomenon of planktonic blooms has been receiving much attention among experimental ecologists as well as mathematical ecologists. Though the process by which these population outbursts occur is not clearly understood, researchers are attempting to explain bloom phenomenon by different approaches. Change in temperature due to seasonal succession has been suggested as a cause. Some others are trying to establish that viral infection on plankton plays an important role for regulation of blooms [30]. But the identification of viruses is a difficult task and also its control mechanism is still a matter of debate. Some others [24] are considering the effect of toxic substances released by some planktons as a controlling factor of blooms.

In the present analysis, we have considered a phytoplankton-zooplankton system together with the toxication effect of phytoplankton. To model the spatial non-homogeneity, which is an inherent characteristic of the aquatic environment, we have introduced diffusion of different planktons into our model equations. To our knowledge, the existing models describing bloom phenomenon do not take into account this spatial effect. The time lag in the toxin liberation process, as is observed experimentally [24], is incorporated into our model as a distributed time delay.

From the stability analysis in Section 3, we have obtained a critical value θ_0 of the toxication parameter θ , below which the system remains asymptotically stable and above which it becomes unstable. This emphasizes the role, toxication plays, in shaping the dynamics of the system. Our study in Section 4 revealed that the system will undergo a Hopf-bifurcation for $\theta > \theta_0$ when the parameter restrictions given in (33) are satisfied. Hence our model is able to exhibit the periodic nature of blooms. This is one of the most important findings of our analysis. In their analysis, Chattopadhyay *et al.* [24] were unable to exhibit the periodic nature of blooms by considering the toxication delay to be of distributed nature, though in the literature of biological delay system, distributed delay is considered to be more general in nature. In Section 5 of the paper, we have applied the perturbation technique of Poincaré-Lindstedt to construct the bifurcating solutions. The solution up to the second order is not significantly different from the first order solution. The overall shape of the spatial wave solution is determined by the wave number q . The nature of the solution is periodic (Hopf-type) whose stability condition is determined by (66).

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