Topology and its Applications 160 (2013) 1603–1605

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A correspondence between ideals and z-filters for certain rings of continuous functions – some remarks

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ARTICLE INFO

Article history:
Received 26 September 2012
Received in revised form 8 June 2013
Accepted 12 June 2013

MSC:
primary 54C40
secondary 45C30

Keywords:
Rings of continuous functions
Prime ideals
Maximal ideals
z-filters
z-ultrafilters

ABSTRACT

Let X be a completely regular Hausdorff topological space and $A(X)$ a ring lying between $C^*(X)$ and $C(X)$. A correspondence $\mathcal{Z}_A$ between ideals of $A(X)$ and the z-filters on X was initiated by Redlin and Watson in 1987 and was further investigated by Byun and Watson in a paper published in Topology and its Applications in 1991. In the last mentioned paper, the authors have established a lemma which reads that for any two rings $A(X)$ and $B(X)$ lying between $C^*(X)$ and $C(X)$ with $B(X) \subseteq A(X)$ and for any ideal I of $A(X)$, $\mathcal{Z}_A[I] = \mathcal{Z}_B[I \cap B(X)]$. We point out an error in the proof of this lemma. The authors have used this lemma to prove a theorem, which says that (a) if M is a maximal ideal of $A(X)$ then $\mathcal{Z}_A[M]$ is contained in a unique z-ultrafilter on X and (b) if $\mathfrak{F}$ is a z-ultrafilter on X, then $\mathcal{Z}_A^{-1}[\mathfrak{F}]$ is a maximal ideal of $A(X)$. The authors have given a correct proof of part (b) of this result, in a more general context, in a later article [Redlin and Watson, 1997]. We give a correct proof of the above lemma and generalize part (a) of the above theorem to prime ideals. Lastly we show that if $A(X) \neq C(X)$, then there exists a non-maximal prime ideal in $A(X)$.

1. Introduction

Let X be a completely regular Hausdorff topological space and $\Sigma(X)$ be the totality of all the rings that lie between the ring $C^*(X)$ of bounded real-valued continuous functions on X and the ring $C(X)$ of all real-valued continuous functions on X. A correspondence between proper ideals of any such intermediate ring $A(X) \in \Sigma(X)$ and the z-filters on X was initiated by Redlin and Watson [7] in 1987 and was further investigated by Byun and Watson [3] in 1991. With each $A(X) \in \Sigma(X)$ and $f \in A(X)$, they have associated a collection $\mathcal{Z}_A(f)$ of subsets of X, given by $\mathcal{Z}_A(f) = \{E \in Z[X]\colon$ there exist a $g \in A(X)$ such that $fg|_{(X-E)} = 1\}$, here $Z(X)$ as usual denotes the family of all zero sets in X. For any proper ideal I of $A(X)$, they have set $\mathcal{Z}_A[I] = \bigcup_{f \in I} \mathcal{Z}_A(f)$ and for a z-filter $\mathfrak{F}$ on X,

$$\mathcal{Z}_A^{-1}[\mathfrak{F}] = \{f \in A(X)\colon\ \mathcal{Z}_A(f) \subset \mathfrak{F}\}.$$

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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.06.011

In the present article our focus is on Lemma 3.1 and Theorems 3.2 and 4.5 of [3].

We show, by way of a counterexample, that there is an error in the proof of Lemma 3.1 of [3]. The statement of Lemma 3.1 of [3] is correct, and we give a proof. We feel it is important to supply a correct proof of Lemma 3.1 of [3] because the lemma is used in the proof of Theorem 3.2 of [3] as well as elsewhere in [3], and the lemma is cited in [1, Lemma 1.6]. We mention that a correct proof of Theorem 3.2(b), in a more general context, appears in [8]. We show here that part (a) of Theorem 3.2 of [3] is also correct. Indeed, we generalize this result to prime ideals. Finally, we show that if $A(X)$ is different from $C(X)$ then there exists a prime ideal of $A(X)$ which is not maximal.

2. A proof of Lemma 3.1 of [3]

In this section we give a correct proof of Lemma 3.1 of [3].

Lemma 3.1. *Let $A(X)$ and $B(X)$ be subrings of $C(X)$ such that $B(X) \subset A(X)$. Then for any ideal I of $A(X)$, $\mathcal{Z}_A[I] = \mathcal{Z}_B[I \cap B(X)]$.*

In the proof in [3] it is claimed that if $f \in A(X)$ and $u \in C^*(X)$ and $fu \in I \cap C^*(X)$, then $\mathcal{Z}_A(f) = \mathcal{Z}_{C^*}(fu)$. If this is the case then by [3, Lemma 1.2] we have $\mathcal{Z}_C(f) = \mathcal{Z}_{C^*}(fu) \subseteq \mathcal{Z}_{C^*}(f)$, and hence $\mathcal{Z}_C(f) = \mathcal{Z}_{C^*}(f)$. But this is not true as the following counterexample shows. Let $A(X) = C(\mathbb{R})$ and let $f \in A(X)$ be defined by

$$f(x) = \frac{x}{1+x^2}, \quad x \in \mathbb{R}.$$

Clearly the zero set $E = [-1, 1]$ belongs to $\mathcal{Z}_C(f)$ but does not belong to $\mathcal{Z}_{C^*}(f)$. We also note that the set $E = [-1, 1]$ does not contain any set of the form $E_\varepsilon = \{x \in \mathbb{R}\colon |f(x)| \leqslant \varepsilon\}$, for $\varepsilon > 0$. This shows that the statement in [3, p. 48] that each zero set in $\mathcal{Z}_A(f)$ contains such a set is false.

We now give a proof of this lemma.

Proof of Lemma 3.1. The hypothesis is that I is an ideal of a ring $A(X)$ lying between $C^*(X)$ and $C(X)$; it is enough to check that $\mathcal{Z}_A[I] \subseteq \mathcal{Z}_{C^*}[I \cap C^*(X)]$. If $E \in \mathcal{Z}_A[I]$, then there exists an $f \in I$ and a $g \in A(X)$ such that $f.g|_{(X-E)} = 1$. Therefore the function $h = \frac{2f.g}{1+|f.g|}$ belongs to $C^*(X) \cap I$ and $h|_{(X-E)} = 1$. This shows that $E \in \mathcal{Z}_{C^*}(h)$ and hence $E \in \mathcal{Z}_{C^*}[I \cap C^*(X)]$. □

3. Prime ideals in $A(X)$

In this section we prove two results on prime ideals in $A(X)$.

It is shown in [3] that if M is a maximal ideal of $A(X)$ then $\mathcal{Z}_A[M]$ is contained in a unique z-ultrafilter on X. We generalize this result to prime ideals.

Theorem 3.1. *If $A(X)$ is any ring between $C^*(X)$ and $C(X)$, then for any prime ideal P of $A(X)$, $\mathcal{Z}_A[P]$ is contained in a unique z-ultrafilter on X.*

Proof. Suppose $\mathcal{Z}_A[P]$ is contained in distinct z-ultrafilters $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Since the map $\mathcal{Z}_A^{-1}$ is a bijection [2, Theorem 3.3(a)], it follows that $\mathcal{Z}_A^{-1}[\mathfrak{F}_1] = M_1$ and $\mathcal{Z}_A^{-1}[\mathfrak{F}_2] = M_2$ are distinct maximal ideals in $A(X)$, and of course $P \subset M_1 \cap M_2$. But this is impossible since a prime ideal in $A(X)$ is contained in a unique maximal ideal [3, Corollary 1.9 *ff*]. □

Rings in which every prime ideal is contained in a unique maximal ideal are called Gelfand rings [6].

It is shown in [3, Theorem 4.5] that if every prime ideal in $A(X)$ is maximal then X is a P-space. In the next theorem we show that if $A(X)$ is properly contained in $C(X)$ then $A(X)$ must contain a prime ideal which is not maximal.

Theorem 3.2. *If $A(X)$ is a ring containing $C^*(X)$ and properly contained in $C(X)$, then there exists at least one prime ideal in $A(X)$ which is not maximal* [*see also* [4]].

To prove this theorem we need to recall the following fact from Willard [9, 44C].

A subalgebra $\mathcal{A}$ of $C(X)$ is called a star subalgebra if it is closed in the uniform topology, contains the constant functions and is inverse closed in the sense that if $f \in \mathcal{A}$ and $Z(f) = \phi$, then f is invertible in $\mathcal{A}$. A star subalgebra $\mathcal{A}$ of $C(X)$ is all of $C(X)$ if and only if $C^*(X) \subseteq \mathcal{A}$.

Proof of Theorem 3.2. The hypothesis implies in view of the above fact that $A(X)$ is not a star subalgebra of $C(X)$. This means that there exists an $f \in A(X)$ for which $Z(f) = \phi$ but f is not invertible in $A(X)$. Therefore by Theorem 1 of [7], $\mathcal{Z}_A(f)$ is a z-filter on X. Furthermore $\bigcap \mathcal{Z}_A(f) = Z(f)$ [see [3], the remarks immediately following Lemma 1.5]. Thus $\mathcal{Z}_A(f)$ turns out to be a free z-filter on X. Consequently the set $S[\mathcal{Z}_A(f)]$ of cluster points of $\mathcal{Z}_A(f)$ in βX is a non-empty subset of $\beta X - X$ [see [5], Chapter 6]. If $p \in S[\mathcal{Z}_A(f)]$, then it follows from Theorem 3.3 of [3] that $f \in M_A^p$, the maximal ideal in $A(X)$ corresponding to the point p. On the other hand the simple fact that X is dense in βX clearly implies that no neighbourhood of p in βX is contained in $S[\mathcal{Z}_A(f)]$, this means that $f \notin O_A^p = \{g \in A(X)\colon\ p \in int\, S[\mathcal{Z}_A(g)]\}$ [see Section 4, [3]]. Thus $M_A^p \neq O_A^p$. Hence from Theorem 4.2 of [3], it follows that there exists at least one non-maximal prime ideal in $A(X)$. □

Acknowledgements

The authors thankfully acknowledge several suggestions of the learned referees, which have helped us a lot towards improvement of this paper. The authors also acknowledge Dr. Dibyendu De, an Assistant Professor, Department of Mathematics, Kalyani University, West Bengal, India and a former Ph.D. student of the first author, for kindly suggesting us to take up the problem in the last Section 3 of this present article. Finally the second author is thankful to NBHM Ph.D. Research Grant (sanction No. 39/7/2010/R&D-II/6959), Anushakti Bhavan, CSM Marg, Mumbai-400001, India for financial assistance.

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