Commun. Korean Math. Soc. **27** (2012), No. 2, pp. 327–339
http://dx.doi.org/10.4134/CKMS.2012.27.2.327

A CLASSIFICATION OF (k, μ)-CONTACT METRIC MANIFOLDS

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Abstract. In this paper we study h-projectively semisymmetric, ϕ-projectively semisymmetric, h-Weyl semisymmetric and ϕ-Weyl semisymmetric non-Sasakian (k, μ)-contact metric manifolds. In all the cases the manifold becomes an η-Einstein manifold. As a consequence of these results we obtain that if a 3-dimensional non-Sasakian (k, μ)-contact metric manifold satisfies such curvature conditions, then the manifold reduces to an $N(k)$-contact metric manifold.

1. Introduction

As is well-known, the local geodesic symmetries on a locally Riemannain symmetric space are isometries and hence they are volume-preserving local diffeomorphisms. However, there are many Riemannian manifolds all of whose geodesic symmetries are volume-preserving but which are not locally symmetric. To our knowledge it is not even known if such spaces are locally homogeneous.

The notion of local symmetry of a Riemannian manifold has been weakend by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [8] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry, De, Shaikh and Biswas [4] introduced the notion of ϕ-recurrent Sasakian manifold. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by Boeckx, Buecken and Vanhecke [3] with several examples.

A $(0, p)$-tensor field T on (M, g) is called parallel when it is invariant under parallel translation, i.e., when

$$\nabla T = 0,$$

in particular, if the $(0, 4)$-Riemann-Christoffel curvature tensor R is parallel, i.e.,

$$\nabla R = 0,$$

Received December 16, 2010.
2010 *Mathematics Subject Classification.* 53C25, 53C35, 53D10.
Key words and phrases. semisymmetric spaces, (k, μ)-contact metric manifolds, non-Sasakian manifolds, η-Einstein manifolds, h-projectively semisymmetric, ϕ-projectively semisymmetric, h-Weyl semisymmetric, ϕ-Weyl semisymmetric.

then M is said to be locally symmetric.

This property justifies the name given to such manifolds [10] locally they are symmetric with respect to each of their points. If each geodesic symmetry s_p, $p \in M$, is a global isometry of M, then M is called a symmetric space. Thus $\nabla R = 0$ for every symmetric space and conversely, every complete and simply connected locally symmetric space is symmetric.

A Riemannian manifold (M^{2n+1}, g) is said to be *semi-symmetric* if its curvature tensor R satisfies $R(X,Y) \cdot R = 0$, $X, Y \in \chi(M)$, where $R(X,Y)$ acts on R as a derivation ([5], [7]).

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. Here P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.1}$$

where S is the Ricci tensor of M.

In an $(2n+1)$-dimensional Riemannian manifold, the conformal curvature tensor C is given by [11]

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z \\ &- \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &+ \frac{\tau}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{1.2}$$

where τ is a scalar curvature and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

In the present paper after introduction, in Section 2 we give some preliminary results of (k,μ)-contact metric manifolds. In Section 3, we study η-Einstein (k,μ)-contact metric manifolds. Section 4 deals with h-projectively semisymmetric non-Sasakian (k,μ)-contact metric manifolds. Section 5 is devoted to study ϕ-projectively semisymmetric non-Sasakian (k,μ)-contact metric manifolds. In Section 6, we study h-Weyl semisymmetric non-Sasakian (k,μ)-contact metric manifolds. The last section contains ϕ-Weyl semisymmetric non-Sasakian (k,μ)-contact metric manifolds. In all the cases the manifold becomes an η-Einstein manifold. As a consequence of these results we obtain that if a 3-dimensional non-Sasakian (k,μ)-contact metric manifold satisfies such curvature conditions, then the manifold reduces to an $N(k)$-contact metric manifold.

2. (k,μ)-contact metric manifolds

A $(2n+1)$-dimensional differentiable manifold M^{2n+1} is called a contact manifold if it carries a global differentiable 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. This 1-form η is called the contact form of M^{2n+1}. A Riemannian metric g is said to be associated with a contact manifold if there exists a $(1,1)$ tensor field ϕ and a contravariant global vector field ξ, called the characteristic vector field of the manifold such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(Y, \phi X), \quad g(X,\xi) = \eta(X), \tag{2.3}$$

$$g(X, \phi Y) = d\eta(X,Y), \tag{2.4}$$

for all vector fields X, Y on M. In a contact metric manifold we define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $\mathrm{Tr}h = \mathrm{Tr}\phi h = 0$ and $h\xi = 0$. Also,

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.5}$$

holds in a contact metric manifold. A contact metric manifold is said to be η-Einstein if

$$Q = aId + b\eta \otimes \xi, \tag{2.6}$$

where a, b are smooth functions on M^{2n+1}.

D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [1] considered the (k,μ)-nullity condition on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ ([1], [6]) of a contact metric manifold M is defined by

$$N(k,\mu) : p \to N_p(k,\mu) = \{W \in T_pM \mid R(X,Y)W = (kI + \mu h)(g(Y,W)X - g(X,W)Y)\},$$

for all $X, Y \in TM$, where $(k,\mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold, we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.7}$$

Also, in a (k,μ)-contact metric manifold, the following relations hold ([1], [2]):

$$h^2 = (k-1)\phi^2, \ k \leq 1, \tag{2.8}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.9}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.10}$$

$$\begin{aligned} S(X,Y) = {} & [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ & + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \ n \geq 1, \end{aligned} \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \tag{2.12}$$

where S is the Ricci tensor of type $(0, 2)$ of the manifold.

If $\mu = 0$, the (k, μ)-nullity distribution $N(k, \mu)$ is reduced to the *k-nullity distribution* [9], where k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by

$$N(k) : p \longrightarrow N_p(k) = \{W \in T_pM \mid R(X, Y)W = k(g(Y, W)X - g(X, W)Y)\}.$$

If $\xi \in N(k)$, then we call a contact metric manifold M an $N(k)$-*contact metric manifold.*

The class of (k, μ)-contact metric manifolds contains both the class of Sasakian ($k = 1$ and $h = 0$) and non-Sasakian ($k \neq 1$ and $h \neq 0$) manifolds. For example, the unit tangent sphere bundle of a flat Riemannian manifold with the usual contact metric structure is a non-Sasakian (k, μ)-contact metric manifold. Throught the present paper we study of $(2n + 1)$-dimensional non-Sasakian (k, μ)-contact metric manifolds.

3. η-Einstein (k, μ)-contact metric manifolds

It is well known that in a Sasakian manifold the Ricci operator Q commutes with ϕ. But in a (k, μ)-contact metric manifold, Q does not commute with ϕ. In general, in a (k, μ)-contact metric manifold D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [1] proved the following:

Proposition 1. *Let M^{2n+1} be a (k, μ)-contact metric manifold. Then the relation*

$$Q\phi - \phi Q = 2[2(n - 1) + \mu]h\phi,$$

holds.

From the definition of η-Einstein manifold it follows easily that $Q\phi = \phi Q$. Hence from Proposition 1 we obtain either $\mu = -2(n - 1)$, or the manifold is Sasakian. Using $\mu = -2(n-1)$ from (2.11) we get the manifold is an η-Einstein manifold. Therefore we obtain the following:

Proposition 2. *In a non-Sasakian (k, μ)-contact metric manifold the following conditions are equivalent*: (i) *η-Einstein manifold,* (ii) $Q\phi = \phi Q$.

For $n = 1$, from Proposition 1 and Proposition 2 we obtain the following:

Corollary 1. *A 3-dimensional non-Sasakian (k, μ)-contact η-Einstein manifold is an $N(k)$-contact metric manifold.*

4. h-projectively semisymmetric non-Sasakian (k, μ)-contact metric manifolds

Definition 1. A Riemannian manifold (M^{2n+1}, g), $n > 1$, is said to be h-projectively semisymmetric if

$$P(X, Y) \cdot h = 0$$

holds on M.

Before we state our first result we need the following lemma which was proved in [1].

Lemma 1 ([1]). *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the (k, μ)-nullity distribution. Then for any vector fields X, Y, Z*

$$\begin{aligned} &R(X,Y)hZ - hR(X,Y)Z \\ &= \{k[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)] \\ &\quad + \mu(k-1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\}\xi \\ &\quad + k\{g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY + g(Z,\phi hY)\phi X - g(Z,\phi hX)\phi Y \\ &\qquad + \eta(Z)[\eta(X)hY - \eta(Y)hX]\} \\ &\quad - \mu\{\eta(Y)[(1-k)\eta(Z)X + \mu\eta(X)hZ] \\ &\qquad - \eta(X)[(1-k)\eta(Z)Y + \mu\eta(Y)hZ] + 2g(X,\phi Y)\phi hZ\}. \end{aligned} \tag{4.1}$$

Theorem 1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a non-Sasakian (k, μ)-contact metric manifold. If M is h-projectively semisymmetric, then M is an η-Einstein manifold.*

Proof. Let M be a $(2n+1)$-dimensional h-projectively semisymmetric non-Sasakian (k, μ)-contact metric manifold. The condition $P(X,Y) \cdot h = 0$ turns into

$$(P(X,Y)\cdot h)Z = P(X,Y)hZ - hP(X,Y)Z = 0, \tag{4.2}$$

for any vector fields X, Y, Z. Using (1.1) and (4.1) in (4.2), we have

$$\begin{aligned} &\{k[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)] \\ &+ \mu(k-1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\}\xi \\ &+ k\{g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY + g(Z,\phi hY)\phi X - g(Z,\phi hX)\phi Y \\ &\quad + \eta(Z)[\eta(X)hY - \eta(Y)hX]\} - \mu\{\eta(Y)[(1-k)\eta(Z)X + \mu\eta(X)hZ] \\ &\quad - \eta(X)[(1-k)\eta(Z)Y + \mu\eta(Y)hZ] + 2g(X,\phi Y)\phi hZ\} \\ &+ \frac{1}{2n}[S(Y,Z)hX - S(X,Z)hY + S(X,hZ)Y - S(Y,hZ)X] = 0. \end{aligned} \tag{4.3}$$

Replacing X by hX and using symmetry property of h, we obtain from (4.3)

$$\begin{aligned} &- kg(hX,hZ)\eta(Y)\xi + \mu(k-1)g(hX,Z)\eta(Y)\xi \\ &+ k\{g(Y,\phi Z)\phi h^2X - g(hX,\phi Z)\phi hY + g(Z,\phi hY)\phi hX \\ &\quad - g(Z,\phi h^2X)\phi Y - \eta(Z)\eta(Y)h^2X + \mu(k-1)\eta(Y)\eta(Z)hX \\ &\quad + 2g(hX,\phi Y)\phi hZ\} \\ &+ \frac{1}{2n}[S(Y,Z)h^2X - S(hX,Z)hY + S(hX,hZ)Y - S(Y,hZ)hX] = 0. \end{aligned} \tag{4.4}$$

Now using (2.8) and (2.11) in (4.4), we get

$$
\begin{aligned}
& k\{g(Y,hZ)hX + (k-1)g(X,Z)Y - (k-1)\eta(X)\eta(Z)Y \\
& \quad + (k-1)g(Y,Z)X - (k-1)g(Y,Z)\eta(X)\xi + g(hX,Z)hY\} \\
& + \mu(k-1)\{g(Y,hZ)\eta(X)\xi - g(Y,hZ)X + g(X,Z)hY \\
& - \eta(X)\eta(Z)hY + g(Y,hZ)hX + (k-1)g(X,Z)Y \\
& - (k-1)g(X,Z)\eta(Y)\xi \\
& - (k-1)\eta(X)\eta(Z)Y + (k-1)\eta(X)\eta(Y)\eta(Z)\xi\} \\
& + \frac{1}{2n}\{(k-1)S(Y,Z)\eta(X)\xi - (k-1)S(Y,Z)X \\
& \qquad - S(hX,Z)hY - S(Y,hZ)hX + S(hX,hZ)Y\} = 0.
\end{aligned} \tag{4.5}
$$

Taking the inner product with W in (4.5) and then using symmetry property of h, we get

$$
\begin{aligned}
& k\{g(Y,hZ)g(hX,W) + (k-1)g(X,Z)g(Y,W) \\
& - (k-1)g(Y,W)\eta(X)\eta(Z) + (k-1)g(Y,Z)g(X,W) \\
& - (k-1)g(Y,Z)\eta(X)\eta(W) + g(X,hZ)g(hY,W)\} \\
& + \mu(k-1)\{g(Y,hZ)\eta(X)\eta(W) - g(Y,hZ)g(X,W) \\
& + g(X,Z)g(hY,W) - g(hY,W)\eta(X)\eta(Z) + g(hY,Z)g(hX,W) \\
& + (k-1)g(X,Z)g(Y,W) - (k-1)g(X,Z)\eta(Y)\eta(W) \\
& - (k-1)g(Y,W)\eta(X)\eta(Z) + (k-1)\eta(X)\eta(Y)\eta(Z)\eta(W)\} \\
& + \frac{1}{2n}\{(k-1)S(Y,Z)\eta(X)\eta(W) - (k-1)S(Y,Z)g(X,W) \\
& - S(X,hZ)g(hY,W) - S(Y,hZ)g(hX,W) + S(hX,hZ)g(Y,W)\} = 0.
\end{aligned} \tag{4.6}
$$

Let $\tilde{e}_i$, $i = 1, \dots, 2n+1$, be an orthonormal ϕ-basis of vector fields in M^{2n+1}. If we put $X = W = \tilde{e}_i$ in (4.6) and sum up with respect to i, then using (2.11), we obtain

$$
\begin{aligned}
& \left[\frac{(2n-1)}{2n}(k-1)\right] S(Y,Z) \\
& = [(2n-1)(k-1)k + \mu(k-1)^2]g(Y,Z) \\
& \quad + \mu[2n(1-k)g(Y,hZ) - (k-1)^2\eta(Y)\eta(Z)].
\end{aligned} \tag{4.7}
$$

Again using (2.11) in (4.7), we obtain

$$
S(Y,Z) = A_1 g(Y,Z) + B_1 \eta(Y)\eta(Z), \tag{4.8}
$$

where

$$
A_1 = \frac{[2(n-1)+\mu][2nk(2n-1) + 2n(k-1)\mu] + 4n^2\mu[2(n-1) - n\mu]}{4n^2\mu + (2n-1)[2(n-1)+\mu]},
$$

and

$$B_1 = \frac{4n^2\mu[2(1-n)+n(2k+\mu)] - 2n\mu(k-1)[2(n-1)+\mu]}{4n^2\mu + (2n-1)[2(n-1)+\mu]}.$$

Thus M is an η-Einstein manifold. □

Now from Corollary 1 we can state the following:

Corollary 2. *If a 3-dimensional non-Sasakian (k,μ)-contact metric manifold is h-projectively semisymmetric, then the manifold is an $N(k)$-contact metric manifold.*

5. ϕ-projectively semisymmetric non-Sasakian (k,μ)-contact metric manifolds

Definition 2. A Riemannian manifold (M^{2n+1}, g), $n > 1$, is said to be ϕ-projectively semisymmetric if

$$P(X,Y)\cdot\phi = 0$$

holds on M.

Now we need the following:

Lemma 2 ([1])**.** *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then for any vector fields X, Y, Z*

$$\begin{aligned}
&R(X,Y)\phi Z - \phi R(X,Y)Z \\
&= \{(1-k)[g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y)] \\
&\quad + (1-\mu)[g(\phi hY,Z)\eta(X) - g(\phi hX,Z)\eta(Y)]\}\xi \\
&\quad - g(Y+hY,Z)(\phi X+\phi hX) + g(X+hX,Z)(\phi Y+\phi hY) \\
&\quad - g(\phi Y+\phi hY,Z)(X+hX) + g(\phi X+\phi hX,Z)(Y+hY) \\
&\quad - \eta(Z)\{(1-k)[\eta(X)\phi Y - \eta(Y)\phi X] \\
&\quad + (1-\mu)[\eta(X)\phi hY - \eta(Y)\phi hX]\}.
\end{aligned} \tag{5.1}$$

Theorem 2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a non-Sasakian (k,μ)-contact metric manifold. If M is ϕ-projectively semisymmetric, then M is an η-Einstein manifold.*

Proof. Let M be a $(2n+1)$-dimensional ϕ-projectively semisymmetric non-Sasakian (k,μ)-contact metric manifold. The condition $P(X,Y)\cdot\phi = 0$ turns into

$$(P(X,Y)\cdot\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0, \tag{5.2}$$

for any vector fields X, Y, Z. Using (1.1) and (5.1) in (5.2), we have

$$\begin{aligned}
&\{(1-k)[g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y)] \\
&\ +(1-\mu)[g(\phi hY,Z)\eta(X) - g(\phi hX,Z)\eta(Y)]\}\xi \\
&-g(Y+hY,Z)(\phi X+\phi hX) + g(X+hX,Z)(\phi Y+\phi hY)
\end{aligned} \tag{5.3}$$

$$-g(\phi Y+\phi hY,Z)(X+hX)+g(\phi X+\phi hX,Z)(Y+hY)$$
$$-\eta(Z)\{(1-k)[\eta(X)\phi Y-\eta(Y)\phi X]+(1-\mu)[\eta(X)\phi hY-\eta(Y)\phi hX]\}$$
$$-\frac{1}{2n}[S(Y,\phi Z)X-S(X,\phi Z)Y+S(X,Z)\phi Y-S(Y,Z)\phi X]=0.$$

Replacing X by ϕX, we obtain from (5.3)

$$\begin{aligned}&(1-k)[g(X,Z)\eta(Y)\xi-\eta(Y)\eta(Z)X]\\&+(\mu-1)[g(hX,Z)\eta(Y)\xi-\eta(Y)\eta(Z)hX]\\&+g(Y,Z)X-g(Y,Z)\eta(X)\xi+g(hY,Z)X-g(hY,Z)\eta(X)\xi\\&-g(Y,Z)hX-g(hY,Z)hX+g(\phi X,Z)\phi Y+g(\phi X,Z)\phi hY\\&+g(h\phi X,Z)\phi Y+g(h\phi X,Z)\phi hY-g(\phi Y,Z)\phi X-g(\phi Y,Z)h\phi X\\&-g(X,Z)Y+\eta(X)\eta(Z)Y-g(X,Z)hY+\eta(X)\eta(Z)hY\\&+g(hX,Z)Y-g(hX,Z)hY-\frac{1}{2n}[S(Y,\phi Z)\phi X-S(\phi X,\phi Z)Y\\&+S(\phi X,Z)\phi Y+S(Y,Z)X-S(Y,Z)\eta(X)\xi]=0.\end{aligned}\tag{5.4}$$

Taking the inner product with W in (5.4) and then using symmetry property of h, we get

$$\begin{aligned}&(1-k)[g(X,Z)\eta(Y)\eta(W)-g(X,W)\eta(Y)\eta(Z)]\\&+(\mu-1)[g(X,hZ)\eta(Y)\eta(W)-g(hX,W)\eta(Y)\eta(Z)]\\&+g(Y,Z)g(X,W)-g(Y,Z)\eta(X)\eta(W)+g(hY,Z)g(X,W)\\&-g(hY,Z)\eta(X)\eta(W)-g(Y,Z)g(hX,W)-g(hY,Z)g(hX,W)\\&+g(\phi X,Z)g(\phi Y,W)+g(\phi X,Z)g(\phi hY,W)+g(\phi X,hZ)g(\phi Y,W)\\&+g(\phi X,hZ)g(\phi hY,W)-g(\phi Y,Z)g(\phi X,W)-g(\phi Y,Z)g(\phi X,hW)\\&-g(X,Z)g(Y,W)+g(Y,W)\eta(X)\eta(Z)-g(X,Z)g(hY,W)\\&+g(hY,W)\eta(X)\eta(Z)+g(X,hZ)g(Y,W)-g(X,hZ)g(hY,W)\\&-\frac{1}{2n}[S(Y,\phi Z)g(\phi X,W)-S(\phi X,\phi Z)g(Y,W)\\&+S(\phi X,Z)g(\phi Y,W)+S(Y,Z)g(X,W)-S(Y,Z)\eta(X)\eta(W)]=0.\end{aligned}\tag{5.5}$$

Let $\tilde{e}_i$, $i=1,\dots,2n+1$, be an orthonormal ϕ-basis of vector fields in M^{2n+1}. If we put $X=W=\tilde{e}_i$ in (5.5) and sum up with respect to i, then using (2.2), (2.8) and (2.12), we obtain

$$\begin{aligned}&\left[\frac{n-1}{n}\right]S(Y,Z)-[2(n+k)-4]g(Y,Z)\\&-\left[2n-2-\frac{2(n-1)+\mu}{n}\right]g(hY,Z)\\&-[(2n-4)(k-1)]\eta(Y)\eta(Z)=0.\end{aligned}\tag{5.6}$$

Now using (2.11) in (5.6), we get

$$S(Y,Z)=A_2g(Y,Z)+B_2\eta(Y)\eta(Z),$$

where

$$A_2 = \frac{[2n+2k-4][2(n-1)+\mu]n - [2(n-1)^2-\mu][2(n-1)-n\mu]}{n\mu},$$

and

$$B_2 = \frac{n(2n-4)(k-1)[2(n-1)+\mu] - [2(n-1)^2-\mu][2(1-n)+n(2k+\mu)]}{n\mu}.$$

Hence M is an η-Einstein manifold. □

So from Corollary 1 we can give the following:

Corollary 3. *If a 3-dimensional non-Sasakian (k,μ)-contact metric manifold is ϕ-projectively semisymmetric, then the manifold is an $N(k)$-contact metric manifold.*

6. h-Weyl semisymmetric non-Sasakian (k,μ)-contact metric manifolds

Definition 3. A Riemannian manifold (M^{2n+1}, g), $n > 1$, is said to be h-Weyl semisymmetric if

$$C(X,Y)\cdot h = 0$$

holds on M.

Theorem 3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a non-Sasakian (k,μ)-contact metric manifold. If M is h-Weyl semisymmetric, then M is an η-Einstein manifold.*

Proof. Let M be a $(2n+1)$-dimensional h-Weyl semisymmetric non-Sasakian (k,μ)-contact metric manifold. The condition $C(X,Y)\cdot h = 0$ turns into

$$(C(X,Y)\cdot h)Z = C(X,Y)hZ - hC(X,Y)Z = 0, \tag{6.1}$$

for any vector fields X, Y, Z. Using (1.2) and (4.1) in (6.1), we have

$$\begin{aligned}
&\{k[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)]\\
&+\mu(k-1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\}\xi\\
&+k\{g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY + g(Z,\phi hY)\phi X\\
&-g(Z,\phi hX)\phi Y + \eta(Z)[\eta(X)hY - \eta(Y)hX]\}\\
&-\mu\{\eta(Y)[(1-k)\eta(Z)X + \mu\eta(X)hZ]\\
&-\eta(X)[(1-k)\eta(Z)Y + \mu\eta(Y)hZ] + 2g(X,\phi Y)\phi hZ\}\\
&-\frac{1}{2n-1}\{S(Y,hZ)X - S(X,hZ)Y + g(Y,hZ)QX - g(X,hZ)QY\}\\
&+\frac{\tau}{2n(2n-1)}\{g(Y,hZ)X - g(X,hZ)Y - g(Y,Z)hX + g(X,Z)hY\} = 0.
\end{aligned} \tag{6.2}$$

Replacing X by hX, taking the inner product with W and using symmetry property of h, we obtain from (6.2)

$$\begin{aligned}
&k(k-1)\{g(X,Z)\eta(Y)\eta(W)-\eta(X)\eta(Z)\eta(Y)\eta(W)\}\\
&+\mu(k-1)g(hX,Z)\eta(Y)\eta(W)\\
&k\{g(hX,\phi Z)g(hY,\phi W)-(k-1)g(X,\phi W)g(Z,\phi Y)\\
&+g(hY,\phi Z)g(hX,\phi W)-(k-1)g(X,\phi Z)g(\phi Y,W)\\
&+(k-1)g(X,W)\eta(Y)\eta(Z)-(k-1)\eta(X)\eta(Z)\eta(Y)\eta(W)\}\\
&+\mu(k-1)g(hX,W)\eta(Y)\eta(Z)+2\mu g(hX,\phi Y)g(hZ,\phi W)\\
&-\frac{1}{2n-1}\{S(Y,hZ)g(hX,W)-S(hX,hZ)g(Y,W)+S(hX,W)g(hY,Z)\\
&+(k-1)g(X,Z)S(Y,W)-(k-1)S(Y,W)\eta(X)\eta(Z)\\
&+(k-1)S(Y,Z)g(X,W)+(k-1)S(Y,Z)\eta(X)\eta(W)\\
&+S(hX,Z)g(hY,W)-S(hX,hW)g(Y,Z)+S(Y,hW)g(X,hZ)\}\\
&+\frac{\tau}{2n(2n-1)}\{g(Y,hZ)g(hX,W)+(k-1)g(X,Z)g(Y,W)\\
&-(k-1)g(Y,W)\eta(X)\eta(Z)+(k-1)g(X,W)g(Y,Z)\\
&-(k-1)g(YZ)\eta(X)\eta(W)+g(hX,Z)g(hY,W)\}=0.
\end{aligned} \tag{6.3}$$

Now taking $Y=W=\xi$ in (6.3), we get

$$\begin{aligned}
S(hX,hZ)= & \left[\frac{4n^2k-2nk(2n-1)-\tau}{2n(2n-1)}\right](k-1)g(X,Z)\\
& -(2n-1)\mu(k-1)g(hX,Z)\\
& +\left[\frac{2nk(2n-1)-4n^2k+\tau}{2n(2n-1)}\right](k-1)\eta(X)\eta(Z).
\end{aligned} \tag{6.4}$$

Again replacing X by hX and Z by hZ in (6.4) and using (2.1) and (2.8), we have

$$\begin{aligned}
S(X,Z)= & -\left[\frac{4n^2k-2nk(2n-1)-\tau}{2n(2n-1)}\right]g(X,Z)-(2n-1)\mu g(hX,Z)\\
& +\left[\frac{4n^2k-2nk(2n-1)-\tau}{2n(2n-1)}+2nk\right]\eta(X)\eta(Z).
\end{aligned} \tag{6.5}$$

Now using (2.11) in (6.5), we get

$$S(X,Z)=\tilde{A}_1 g(X,Z)+\tilde{B}_1\eta(X)\eta(Z),$$

where

$$\tilde{A}_1=\frac{[2(n-1)+\mu](\tau-2nk)+(2n-1)\mu[2(n-1)-n\mu]}{2(n-1+n\mu)},$$

and

$$\tilde{B}_1=\frac{[2(n-1)+\mu](4nk-\tau)+(2n-1)\mu[2(1-n)+n(2k+\mu)]}{2(n-1+n\mu)}.$$

So M is an η-Einstein manifold. □

Thus from Corollary 1 we have the following:

Corollary 4. *If a 3-dimensional non-Sasakian (k,μ)-contact metric manifold is h-Weyl semisymmetric, then the manifold is an $N(k)$-contact metric manifold.*

7. ϕ-Weyl semisymmetric non-Sasakian (k,μ)-contact metric manifolds

Definition 4. A Riemannian manifold (M^{2n+1},g), $n>1$, is said to be ϕ-Weyl semisymmetric if

$$C(X,Y)\cdot\phi=0$$

holds on M.

Theorem 4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a non-Sasakian (k,μ)-contact metric manifols. If M is ϕ-Weyl semisymmetric, then M is an η-Einstein manifold.*

Proof. Let M be a $(2n+1)$-dimensional ϕ-Weyl semisymmetric non-Sasakian (k,μ)-contact metric manifold. The condition $C(X,Y)\cdot\phi=0$ turns into

$$(C(X,Y)\cdot\phi)Z=C(X,Y)\phi Z-\phi C(X,Y)Z=0, \tag{7.1}$$

for any vector fields X, Y, Z. Using (1.2) and (5.1) in (7.1), we have

$$\begin{aligned}
&(1-k)[g(\phi Y,Z)\eta(X)\xi-g(\phi X,Z)\eta(Y)\xi]\\
&+(1-\mu)[g(\phi hY,Z)\eta(X)\xi-g(\phi hX,Z)\eta(Y)\xi]\\
&-g(Y+hY,Z)(\phi X+\phi hX)+g(X+hX,Z)(\phi Y+\phi hY)\\
&-g(\phi Y+\phi hY,Z)(X+hX)+g(\phi X+\phi hX,Z)(Y+hY)\\
&-\eta(Z)\{(1-k)[\eta(X)\phi Y-\eta(Y)\phi X]+(1-\mu)[\eta(X)\phi hY-\eta(Y)\phi hX]\}\\
&-\frac{1}{2n-1}[S(Y,\phi Z)X-S(X,\phi Z)Y+g(Y,\phi Z)QX-g(X,\phi Z)QX\\
&-S(Y,Z)\phi X+S(X,Z)\phi Y-g(Y,Z)\phi QX+g(X,Z)\phi QX]\\
&+\frac{\tau}{2n(2n-1)}[g(Y,\phi Z)X-g(X,\phi Z)Y-g(Y,Z)\phi X+g(X,Z)\phi X]=0.
\end{aligned} \tag{7.2}$$

Replacing X by ϕX, taking the inner product with W and using (2.1) and symmetry property of h, we obtain from (7.2)

$$\begin{aligned}
&(k-1)[-g(X,Z)\eta(Y)\eta(W)+\eta(X)\eta(Z)\eta(Y)\eta(W)]\\
&+(\mu-1)g(hX,Z)\eta(Y)\eta(W)+g(Y,Z)g(X,W)-g(Y,Z)\eta(X)\eta(W)\\
&-g(Y,Z)g(hX,W)+g(hY,Z)g(X,W)-g(hY,Z)\eta(X)\eta(W)\\
&-g(hY,Z)g(hX,W)-g(X,\phi Z)g(\phi Y,W)-g(X,\phi Z)g(\phi hY,W)\\
&+g(X,h\phi Z)g(\phi Y,W)+g(X,h\phi Z)g(\phi hY,W)-g(\phi Y,Z)g(\phi X,W)\\
&-g(\phi Y,Z)g(\phi X,hW)-g(\phi hY,Z)g(\phi X,W)+g(hY,\phi Z)g(\phi X,hW)\\
&-g(X,Z)g(Y,W)+g(Y,W)\eta(X)\eta(Z)-g(X,Z)g(hY,W)\\
&+g(hY,W)\eta(X)\eta(Z)+g(hX,Z)g(Y,W)+g(hX,Z)g(hY,W)
\end{aligned} \tag{7.3}$$

$$+(k-1)g(X,W)\eta(Y)\eta(Z)+(k-1)\eta(X)\eta(Z)\eta(Y)\eta(W)$$
$$+(\mu-1)g(hX,W)\eta(Y)\eta(Z)-\frac{1}{2n-1}[S(Y,\phi Z)g(\phi X,W)$$
$$-S(\phi X,\phi Z)g(Y,W)+S(\phi X,W)g(Y,\phi Z)-S(Y,W)g(X,Z)$$
$$+S(Y,W)\eta(X)\eta(Z)+S(Y,Z)g(X,W)-S(Y,Z)\eta(X)\eta(W)$$
$$+S(\phi X,Z)g(\phi Y,W)-S(\phi X,\phi W)g(Y,Z)+g(X,Z)S(Y,\phi W)]$$
$$+\frac{\tau}{2n(2n-1)}[g(Y,\phi Z)g(\phi X,W)-g(X,Z)g(Y,W)$$
$$+g(Y,W)\eta(X)\eta(Z)+g(X,W)g(Y,Z)-g(Y,Z)\eta(X)\eta(W)$$
$$-g(X,\phi Z)g(\phi Y,W)]=0.$$

Replacing Y and W by ξ in (7.3), we obtain

(7.4)
$$S(X,Z)=[\frac{\tau}{2n}-k]g(X,Z)+[6n-4+\mu]g(hX,Z)+[2nk-\frac{\tau}{2n}+k]\eta(X)\eta(Z).$$

Now using (2.11) in (7.4), we get

$$S(X,Z)=\tilde{A}_2 g(X,Z)+\tilde{B}_2\eta(X)\eta(Z),$$

where

$$\tilde{A}_2=\frac{[2(n-1)+\mu](\tau-2nk)-2n(6n-4+\mu)[2(n-1)-n\mu]}{4n(1-2n)},$$

and

$$\tilde{B}_2=\frac{[2(n-1)+\mu](2n(2n+1)k-\tau)-2n(6n-4+\mu)[2(1-n)+n(2k+\mu)]}{4n(1-2n)}.$$

Thus M is an η-Einstein manifold. □

Hence from Corollary 1 we get the following:

Corollary 5. *If a 3-dimensional non-Sasakian (k,μ)-contact metric manifold is ϕ-Weyl semisymmetric, then the manifold is an $N(k)$-contact metric manifold.*

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