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A class of solutions of the generalized Lund–Regge model

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A class of solutions are obtained for the generalized Lund–Regge model of Corones [J. Math. Phys. **19**, 2431 (1978)] and its Euclidean counterpart. As a consequence, a new solution is noted for the original Lund–Regge model.

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I. INTRODUCTION

In a recent paper, Corones[1] has studied the following form of coupled equations for a generalized Lund–Regge model:

$$\Theta_{11}-\Theta_{22}-4g(\Theta)+h(\Theta)(\lambda_1^2-\lambda_2^2)=0,$$
$$\lambda_{11}-\lambda_{22}=2p(\Theta)(\lambda_1\Theta_1-\lambda_2\Theta_2), \tag{1.1}$$

where $\Theta_1=\partial\Theta/\partial x'$ and so on.

Θ and λ are unspecified field variables. $g(\Theta)$, $h(\Theta)$, and $p(\Theta)$ are unspecified functions of Θ. A transformation $x^2\rightarrow ix^1$ transforms (1.1) to

$$\Theta_{11}+\Theta_{22}-4g(\Theta)+h(\Theta)(\lambda_1^2+\lambda_2^2)=0,$$
$$\lambda_{11}+\lambda_{22}=p(\Theta)(\lambda_1\Theta_1+\lambda_2\Theta_2). \tag{1.2}$$

Thus if (1.1) is a set of coupled equations in the two-dimensional space–time continuum, where x^1 is timelike and x^2 is spacelike, then (1.2) is its Euclidean counterpart, where both x^1 and x^2 are spacelike.

Equations of the form (1.1) and (1.2) occur in a number of physical problems, although the definition of the field variables Θ and λ as well as the functions $g(\Theta)$, $h(\Theta)$, and $p(\Theta)$ vary from one problem to another. Particular examples of (1.1) occur in the study of relativistic strings in a uniform external field,[2] and the nonlinear σ model in $1+1$ dimensions. Particular examples of (1.2) occur in the study of two-dimensional Heisenberg ferromagnets[3] and in the Ginzburg–Pitaevski approach to superfluids.[4] Also Corones[1] has shown that an infinite number of equations of the form (1.1), including the two above, admit soliton solutions.

The present note provides a unified approach towards obtaining some particular solutions of (1.1) and (1.2) for unspecified $g(\Theta)$, $h(\Theta)$, and $p(\Theta)$ subject to the restriction that

$$g(\Theta)\not\equiv0,\quad h(\Theta)\not\equiv0,\quad p(\Theta)\not\equiv0. \tag{1.3}$$

To obtain these solutions, we assume that

$$\Theta=\Theta(z),$$

where $z=k(x^{1^2}+\epsilon x^{2^2})+lx^1+mx^2$,

k, l, and m are constants,

and

$$\epsilon=-1 \quad \text{for Eqs. (1.1)}$$
$$=+1 \quad \text{for Eqs. (1.2)}. \tag{1.4}$$

It is obvious that Eqs. (1.1) are Lorentz-invariant and Eqs. (1.2) are invariant under arbitrary translations and rotation in two-dimensional Euclidean space.

Therefore, for Eqs. (1.1), without loss of generality, one can express z in one of the following forms.

(i) $z=x^1$ with $\Theta_z\neq0$,
(ii) $z=x^2$ with $\Theta_z\neq0$,
(iii) $z=x^1+x^2$ with $\Theta_z\neq0$,
(iv) $z=x^{1^2}-x^{2^2}$ with $\Theta_z\neq0$,
(v) $\Theta=$ const, where the form of z is immaterial.

In a similar manner for (1.2), z can be expressed in one of the following ways:

(i) $z=x^1$, $\Theta_z\neq0$,
(ii) $z=x^{1^2}+x^{2^2}$,
(iii) $\Theta=$ const, where z is immaterial.

Before we study these cases one by one we have to establish two lemmas.

II. TWO LEMMAS

Lemmas: If χ is a function of any two independent variables ξ and η satisfying

$$\chi_{\xi\xi}\mp\chi_{\eta\eta}=U(\xi)\chi_\xi, \tag{2.1a}$$
$$\chi_\xi^2\mp\chi_\eta^2=V(\xi), \tag{2.1b}$$
$$U(\xi)\not\equiv0,\quad V(\xi)\not\equiv0,$$

then

$$\chi_\eta=\text{const}. \tag{2.2}$$

We shall prove this for the negative sign in (2.1). The other is similar. From (2.1b),

$$\chi_\xi\chi_{\xi\eta}-\chi_\eta\chi_{\eta\eta}=0. \tag{2.3}$$

From (2.1a) and (2.3)

$$\chi_\eta\chi_{\xi\xi}-\chi_\xi\chi_{\xi\eta}=U(\xi)\chi_\xi\chi_\eta. \tag{2.4}$$

If $\chi_\eta\equiv0$, the (2.2) is obviously satisfied. On the other hand, if $\chi_\eta\not\equiv0$, one can divide (2.4) by χ_η^2 and integrate to obtain

$$\chi_\xi=\chi_\eta\exp\left[\int U(\xi)\,d\xi\right]W(\eta),$$

where $W(\eta)$ is some function of η.

From this equation and (2.1b) one can solve χ_ξ and χ_η in terms of $U(\xi)$, $V(\xi)$, and $W(\eta)$. The condition that

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$\partial\chi_\xi/\partial\eta=\partial\chi_\eta/\partial\xi$ then gives (2.2). Therefore, (2.2) holds in either case.

III. SOLUTIONS OF (1.1)

Case (i) $z=x^1$, $\Theta_z\neq 0$

Here

$$\Theta=\Theta(x^1). \tag{3.1}$$

Equation (1.1) is now of the form (2.1) with the negative sign, where λ is taken as χ, x^1 as ξ, and χ^2 as a. Therefore, using (3.1) and the lemma, we obtain from (1.1b)

$$\lambda=K^{\prime}x^2+A^{\prime}\int\exp\left[2\int p(\Theta)\,d\Theta\right]dx^1.$$

Formula (1.1a) then gives

$$\int\frac{d\Theta}{(B^{\prime}+8\int g(\Theta)\,d\Theta-2\int h(\Theta)\{A^{\prime 2}\exp[4\int p(\Theta)\,d\Theta]-K^{\prime 2}\}\,d\Theta)^{1/2}}=x^1, \tag{3.2}$$

where $A^{\prime}$, $B^{\prime}$, and $K^{\prime}$ are constants of integration.

Case (ii) $z=x^2$ with $\Theta_z\neq 0$

Here

$$\Theta=\Theta(x^2). \tag{3.3}$$

Similarly, here

$$\lambda=K^{\prime\prime}x^1+A^{\prime\prime}\int\exp\left[2\int p(\Theta)\,d\Theta\right]dx^2,$$

$$\int\frac{d\Theta}{(B^{\prime\prime}-8\int g(\Theta)\,d\Theta-2\int h(\Theta)\{A^{\prime\prime 2}\exp[4\int p(\Theta)\,d\Theta]-K^{\prime\prime 2}\}\,d\Theta)^{1/2}}=x^2. \tag{3.4}$$

Case (iii) $z=x^1+x^2$, $\Theta\neq$ const

Define

$$\alpha=x^1+x^2,\qquad \beta=x^1-x^2. \tag{3.5}$$

Therefore,

$$\Theta=\Theta(\alpha), \tag{3.6}$$

Formula (1.1) reduces to

$$-g(\Theta)+h(\Theta)\lambda_\alpha\lambda_\beta=0, \tag{3.7a}$$

$$\lambda_{\alpha\beta}=p(\Theta)\lambda_\beta\Theta_\alpha. \tag{3.7b}$$

Equation (3.7b) can be readily integrated to give

$$\lambda=\psi+\phi\exp\left[\int p(\Theta)\,d\Theta\right],\qquad \text{where}\quad \psi=\psi(\alpha),\quad \phi=\phi(\beta). \tag{3.8}$$

Formula (3.7a) then gives

$$\phi_\beta\exp\left[\int p(\Theta)\,d\Theta\right]\left\{\psi_\alpha+\phi\exp\left[\int p(\Theta)\,d\Theta\right]p(\Theta)\Theta_\alpha\right\}=g(\Theta)/h(\Theta). \tag{3.9}$$

From (1.3) and (3.9),

$$\phi_\beta\neq 0. \tag{3.10}$$

Therefore, (3.9) can be rewritten as

$$\frac{\psi_\alpha\exp[-\int p(\Theta)q\,d\Theta]}{p(\Theta)\Theta_\alpha}+\phi=\frac{g(\Theta)\exp[-2\int p(\Theta)\,d\Theta]}{h(\Theta)p(\Theta)\Theta_\alpha}\cdot\frac{1}{\phi_\beta}. \tag{3.11}$$

Since ψ, $g(\Theta)$, $p(\Theta)$, and $h(\Theta)$ are functions of α and ϕ is a function of β, differentiating (3.11) with respect to α and β, we have

$$\left[\frac{g(\Theta)\exp[-2\int p(\Theta)\,d\Theta]}{h(\Theta)p(\Theta)\Theta_\alpha}\right]_\alpha\left(\frac{1}{\phi_\beta}\right)_\beta=0.$$

However, if $(1/\phi_\beta)_\beta=0$, i.e., $1/\phi_\beta=$ const, then the right-hand side of (3.11) is a function of α only and hence so is the left-hand side which is possible only if ϕ is a constant which violates (3.10). Therefore,

$$\frac{g(\Theta)\exp[-2\int p(\Theta)\,d\Theta]}{h(\Theta)p(\Theta)\Theta_\alpha}=\text{const}. \tag{3.12}$$

Then the right-hand side of (3.11) is a function of β and so must be the left-hand side; therefore,

$$\frac{\psi_\alpha\exp[-\int p(\Theta)\,d\Theta]}{p(\Theta)\Theta_\alpha}=\text{const}. \tag{3.13}$$

Using (3.8), (3.11), (3.12), and (3.13), the complete solutions of (3.7), which are the complete solutions of (1.1) for Θ as a function of x^1+x^2, this can be written as

$$\begin{aligned}
&B\int\frac{h(\Theta)p(\Theta)\exp[2\int p(\Theta)\,d\Theta]}{g(\Theta)}\,d\Theta=x^1+x^2,\\
&\psi=A\int\exp\left[\int p(\Theta)\,d\Theta\right]p(\Theta)\,d\Theta,\\
&\phi=-A\pm\overline{2B(x^1-x^2)+C},\\
&\lambda=\psi(x^1+x^2)+\phi(x^1-x^2)\exp\left[\int p(\Theta)\,d\Theta\right],
\end{aligned} \tag{3.14}$$

where A, B, and C are constants of integration.

Case (iv) $z = x'^2 - x^{2'}$, $\Theta \neq$ **const**

Define

$$y = \ln(x^{1^2} - x^{2^2}),$$
$$s = \ln \frac{x^1 + x^2}{x^1 - x^2}. \tag{3.15}$$

Therefore,

$$\Theta = \Theta(y). \tag{3.16}$$

Formula (1.1) can then be rewritten as

$$\lambda_{yy} - \lambda_{ss} = 2p(\Theta)\Theta_y \lambda_y \;, \tag{3.17a}$$

$$\lambda_y^2 - \lambda_s^2 = \frac{e^y g(\Theta) - \Theta_{yy}}{h(\Theta)} \;. \tag{3.17b}$$

Therefore, using (3.16) and Sec. II,

$$\lambda_s = \text{const}. \tag{3.18}$$

Using (3.18), one can integrate (3.17a) to give

$$\lambda = M\exp\left(2\int p \, d\Theta\right) + N \ln \frac{x^1 + x^2}{x^1 - x^2}. \tag{3.19a}$$

(3.17b) then becomes

$$\left[M^2 \exp\left(4\int p \, d\Theta\right) - N^2\right] h(\Theta) = e^y g(\Theta) - \Theta_{yy}, \tag{3.19b}$$

where

$$\Theta = \Theta(y) \tag{3.19c}$$

and

$$y = \ln(x^{1^2} - x^{2^2}). \tag{3.19d}$$

If p, h, and g are known functions of Θ, then (3.19b) is an ordinary second order differential equation for Θ as a function of y which can be solved numerically by computer. λ and y are then given respectively by (3.19a) and (3.19d).

Case (v) $\Theta =$ **const**

Here (1.1) gives

$$\lambda_1^2 - \lambda_2^2 = \text{const},$$
$$\lambda_{11} - \lambda_{22} = 0.$$

The solution can be readily seen as

$$\lambda = A'''x^1 + B'''x^2, \tag{3.20}$$

where A''' and B''' are constants satisfying

$$A'''^2 - B'''^2 = 4g(\Theta)/h(\Theta)$$

and Θ is a constant.

IV. SOLUTIONS OF (1.2)

Case (i) $z = x^1$, $\Theta_z \neq 0$

This case is similar to case (i) of Sec. III. Proceeding in a similar way, we get the solution as

$$\lambda = K_0 x^2 + A_0 \int \exp\left(2\int p(\Theta)\, d\Theta\right) dx^1,$$
$$\int \frac{d\Theta}{(B_0 + 8\int g(\Theta)\, d\Theta - 2\int h(\Theta)\{A_0^2 \exp[4\int p(\Theta)\, d\Theta] + K^2\}\, d\Theta)^{1/2}} = x^1. \tag{4.1}$$

where A_0, B_0, and K_0 are constants of integration.

Case (ii) $z = x^{1^2} + x^{2^2}, \Theta_z \neq 0$

This case is similar to case (iv) of Sec. III. Proceeding in a similar way, we get

$$\lambda = M_0 \exp\left(2\int p \, d\Theta\right) + 2N_0 \tan^{-1}\frac{x^2}{x^1}, \tag{4.2a}$$

$$\left[M_0^2 \exp\left(4\int p \, d\Theta\right) + N_0^2\right] h(\Theta) = e^{y_0} g(\Theta) + \Theta_{y_0 y_0}, \tag{4.2b}$$

where

$$\Theta = \Theta(y_0), \tag{4.2c}$$

$$y_0 = \ln(x^{1^2} + x^{2^2}) \;, \tag{4.2d}$$

M_0 and N_0 are constants of integration.

As before, (4.2b) is an ordinary second order differential equation for Θ as a function of y_0 if p, h, and g are known functions of Θ. This equation can be solved numerically. λ and y_0 are given respectively by (4.2a) and (4.2d).

Case (iii) $\Theta =$ **const**

This case is similar to case (v) of Sec. III.

The solutions are

$$\lambda = A_1 x^1 + B_1 x^2, \tag{4.3a}$$

where A_1 and B_1 are constants satisfying

$$A_1^2 + B_1^2 = 4g(\Theta)/h(\Theta) \tag{4.3b}$$

and Θ is a constant.

V. SOLUTIONS FOR ORIGINAL LUND–REGGE MODEL

The original Lund–Regge model[2] is a special case of (1.1), where

$$g(\Theta) = -(c \sin\Theta\cos\Theta)/4,$$
$$h(\Theta) = \cos\Theta/\sin^3\Theta, \tag{5.1}$$
$$p(\Theta) = 1/\sin\Theta\cos\Theta.$$

In this case all the solutions for $\Theta = \Theta(x^1)$, $\Theta = \Theta(x^2)$, $\Theta = \Theta(x^1 - x^2)$, and $\Theta =$ const already appear in the literature.[5] However, in the case of $\Theta = \Theta(x^{1^2} - x^{2^2})$, i.e., in case (iv) of Sec. III, we observe some new solutions for the original Lund–Regge model.[2]

For the original Lund–Regge model the equations for case (iv) in Sec. III, i.e., (3.19), reduce to

$$\lambda = M \tan^2\Theta + N \ln \frac{x^1 + x^2}{x^1 - x^2}, \tag{5.2a}$$

$$(M^2\tan^2\Theta - N^2)\frac{\cos\Theta}{\sin^3\Theta} = e^y g(\Theta) - \Theta_{yy}\,, \tag{5.2b}$$

$$\Theta = \Theta(y), \tag{5.2c}$$

$$y = \ln(x^{1^2} - x^{2^2}) \tag{5.2d}$$

Equation (5.1b) can be solved numerically as noted before and λ and y are given by (5.1a) and (5.1d). The special case of (5.1) where $M = 0$ and the special case of (5.1) where $N = 0$ already appear in the literature.[5]

VI. CONCLUSION

In conclusion, all the solutions of (1.1) subject to (1.4) are given by (3.2), (3.4) (3.14), (3.19), and (3.20). In the particular case of the original Lund–Regge model this indicates a new solution (5.2). Solutions of (1.2) subject to (1.4) are given by (4.1) and (4.2).

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