

# Unsolved Problems in Visibility Graphs of Points, Segments and Polygons<sup>1</sup>

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## Abstract

In this survey paper, we present open problems and conjectures on visibility graphs of points, segments and polygons along with necessary backgrounds for understanding them.

## 1 Introduction

The visibility graph is a fundamental structure studied in the field of computational geometry and geometric graph theory, and pose some special challenges [23, 55]. Apart from theoretical interests, visibility graphs have important applications also. Some of the early applications include computing Euclidean shortest paths in the presence of obstacles [75] and decomposing two-dimensional shapes into clusters [101]. For more on the uses of this class of graphs, see [84, 104].

Let  $P$  be a set of  $n$  points in the plane (see Figure 1(a)). We say two points  $p_i$  and  $p_j$  of  $P$  are *mutually visible* if the line segment  $p_i p_j$  does not contain or pass through any other point of  $P$ . In other words,  $p_i$  and  $p_j$  are visible if  $P \cap \overline{p_i p_j} = \{p_i, p_j\}$ . If a point  $p_k \in P$  lies on the segment  $p_i p_j$  connecting two points  $p_i$  and  $p_j$  in  $P$ , we say that  $p_k$  blocks the visibility between  $p_i$  and  $p_j$ , and  $p_k$  is called a *blocker* in  $P$ . For example in Figure 1(a),  $p_4$  blocks the visibility between  $p_2$  and  $p_6$  as  $p_4$  lies on the segment  $p_2 p_6$ .

The *visibility graph* (also called the *point visibility graph*)  $G$  of  $P$  is defined by associating a vertex  $v_i$  with each point  $p_i$  of  $P$  such that  $(v_i, v_j)$  is an undirected edge of  $G$  if  $p_i$  and  $p_j$  are mutually visible (see Figure 1(b)). It can be seen that if no three points of  $P$  are

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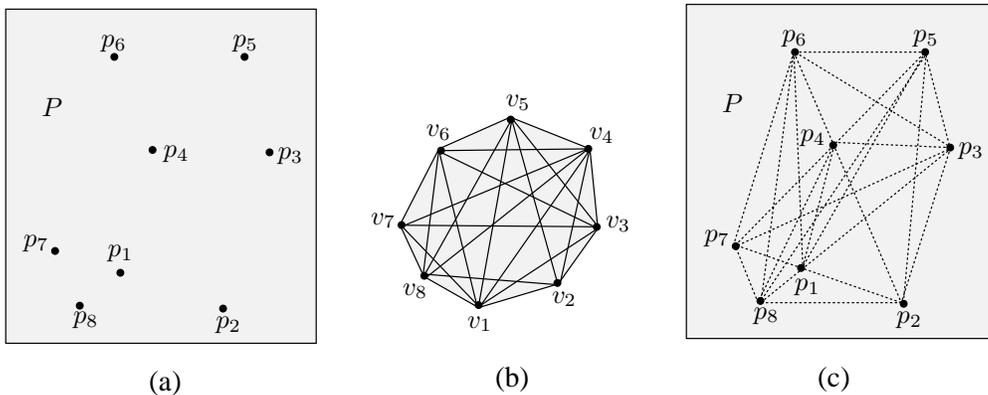


Figure 1: (a) A given set of points. (b) The visibility graph of the point set. (c) The visibility graph drawn on the point set.

collinear, i.e., there is no blocker in  $P$ , then  $G$  is a complete graph as each pair of points in  $P$  is visible. Sometimes the visibility graph is drawn directly on the point set, as shown in Figure 1(c).

Consider the problem of computing the visibility graph  $G$  of a point set  $P$ . For each point  $p_i$  of  $P$ , sort the points of  $P$  in angular order around  $p_i$ . If two points  $p_j$  and  $p_k$  are adjacent in the sorted order, check whether  $p_i$ ,  $p_j$  and  $p_k$  are collinear points. By traversing the sorted order, all points of  $P$ , that are not visible from  $p_i$ , can be located in  $O(n \log n)$  time. Hence,  $G$  can be computed from  $P$  in  $O(n^2 \log n)$  time. Using the result of Chazelle et al. [17] or Edelsbrunner et al. [34], the running time of the algorithm can be improved to  $O(n^2)$  by computing sorted angular orders for all points together in  $O(n^2)$  time.

Let  $S$  be a set of  $n$  disjoint line segments (see Figure 2(a)). The endpoints of segments  $s_1, s_2, \dots, s_n$  in  $S$  are marked as  $p_1, p_2, \dots, p_{2n}$ , where  $p_{2i-1}$  and  $p_{2i}$  are endpoints of  $s_i$ . Let  $P$  be the set of these endpoints  $p_1, p_2, \dots, p_{2n}$ . We say two points  $p_i$  and  $p_j$  of  $P$  are *mutually visible* if the line segment  $p_i p_j$  does not intersect any segment  $s_i$  in  $S$ . This definition does not allow the segment  $p_i p_j$  to pass through another endpoint  $p_k$  or graze along a segment in  $S$ . The *visibility graph* (also called the *segment visibility graph* or *segment endpoint visibility graph*)  $G$  of  $S$  is defined by associating a vertex  $v_i$  with each point  $p_i$  of  $P$  such that  $(v_i, v_j)$  is an undirected edge of  $G$  if  $p_i$  and  $p_j$  are mutually visible (see Figure 1(b)). In addition, the corresponding vertices of two endpoints of every segment in  $S$  is also connected by an edge in  $G$ . Sometimes the visibility graph is drawn directly on the segments, as shown in Figure 2(c).

Let  $P$  be a simple polygon with or without holes in the plane (see Figure 3(a)). We say

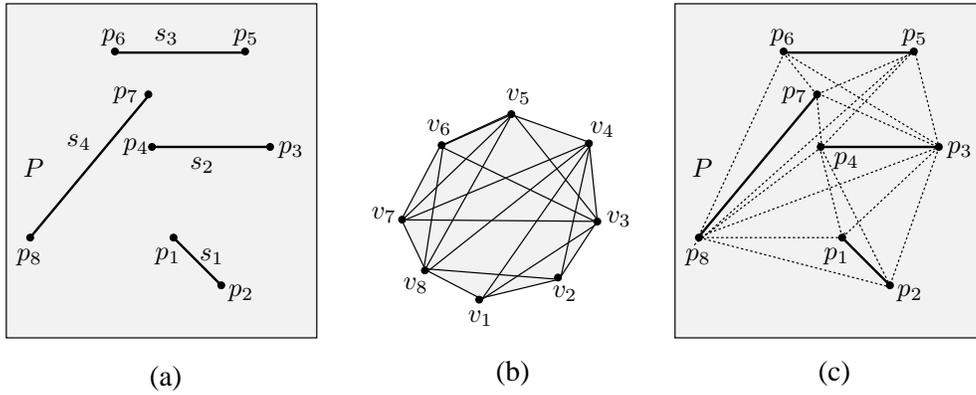


Figure 2: (a) A given set of segments. (b) The visibility graph of the set of segments. (c) The visibility graph drawn on the set of segments.

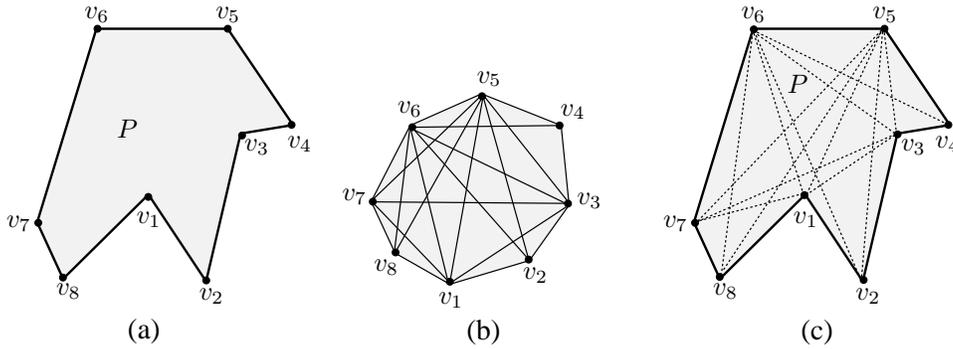


Figure 3: (a) A polygon. (b) The visibility graph of the polygon. (c) The visibility graph drawn on the polygon.

two points  $a$  and  $b$  in  $P$  are *mutually visible* if the line segment  $ab$  lies entirely within  $P$ . This definition allows the segment  $ab$  to pass through a reflex vertex or graze along a polygonal edge. The *visibility graph* (also called the *vertex visibility graph*)  $G$  of  $P$  is defined by associating a node with each vertex of  $P$  such that  $(v_i, v_j)$  is an undirected edge of  $G$  if polygonal vertices  $v_i$  and  $v_j$  are mutually visible. Figure 3(b) shows the visibility graph of the polygon in Figure 3(a). Sometimes the visibility graph is drawn directly on the polygon, as shown in Figure 3(c). It can be seen that every triangulation of  $P$  corresponds to a subgraph of the visibility graph of  $P$ .

The problem of computing the visibility graph of a polygon  $P$  (with or without holes) or a set of disjoint segments  $S$  is well studied in computational geometry [55, 71, 75, 102].

Asano et al. [10] and Welzl [117] proposed  $O(n^2)$  time algorithms for this problem. Since, at its largest, a visibility graph can be of size  $O(n^2)$ , algorithms of Asano et al. and Welzl are worst-case optimal. The visibility graph may be much smaller than its worst-case size of  $O(n^2)$  (in particular, it can have  $O(n)$  edges) and therefore, it is not necessary to spend  $O(n^2)$  time to compute it. Hershberger [63] developed an  $O(E)$  time output sensitive algorithm for computing the visibility graph of a simple polygon. Ghosh and Mount [58] presented  $O(n \log n + E)$  time,  $O(E + n)$  space algorithm for computing visibility graph of polygon with holes. Keeping the same time complexity, Pocchiola and Vegter [93] improved the space complexity to  $O(n)$ .

## 2 Visibility graph theory: Points

### 2.1 Visibility Graphs: Recognition, Characterization, and Reconstruction

We have stated earlier how to compute the visibility graph  $G$  from a given set of points  $P$ . Consider the opposite problem of determining if there is a set of points  $P$  whose visibility graph is the given graph  $G$ . This problem is called the visibility graph *recognition* problem. Identifying the set of properties satisfied by all visibility graphs is called the visibility graph *characterization* problem. The problem of actually drawing one such set of points  $P$  whose visibility graph is the given graph  $G$ , is called the visibility graph *reconstruction* problem.

**Open Problem 1** *Given a graph  $G$  in adjacency matrix form, determine whether  $G$  is the visibility graph of a set of points  $P$  in the plane.*

**Open Problem 2** *Characterize visibility graphs of point sets.*

**Open Problem 3** *Given the visibility graph  $G$  of a set of points, draw the points in the plane whose visibility graph is  $G$ .*

### 2.2 Colouring Visibility Graphs

Consider the problem of colouring the visibility graph  $G = (V, E)$  of a point set  $P$ . A  $k$ -colouring of  $G$  is a function  $f : V \rightarrow C$  for some set  $C$  of  $k$  colours such that  $f(v) \neq f(w)$  for every edge  $(v_i, v_j) \in E$ . If  $G$  can be coloured by  $k$  colours,  $G$  is called  $k$ -colourable.

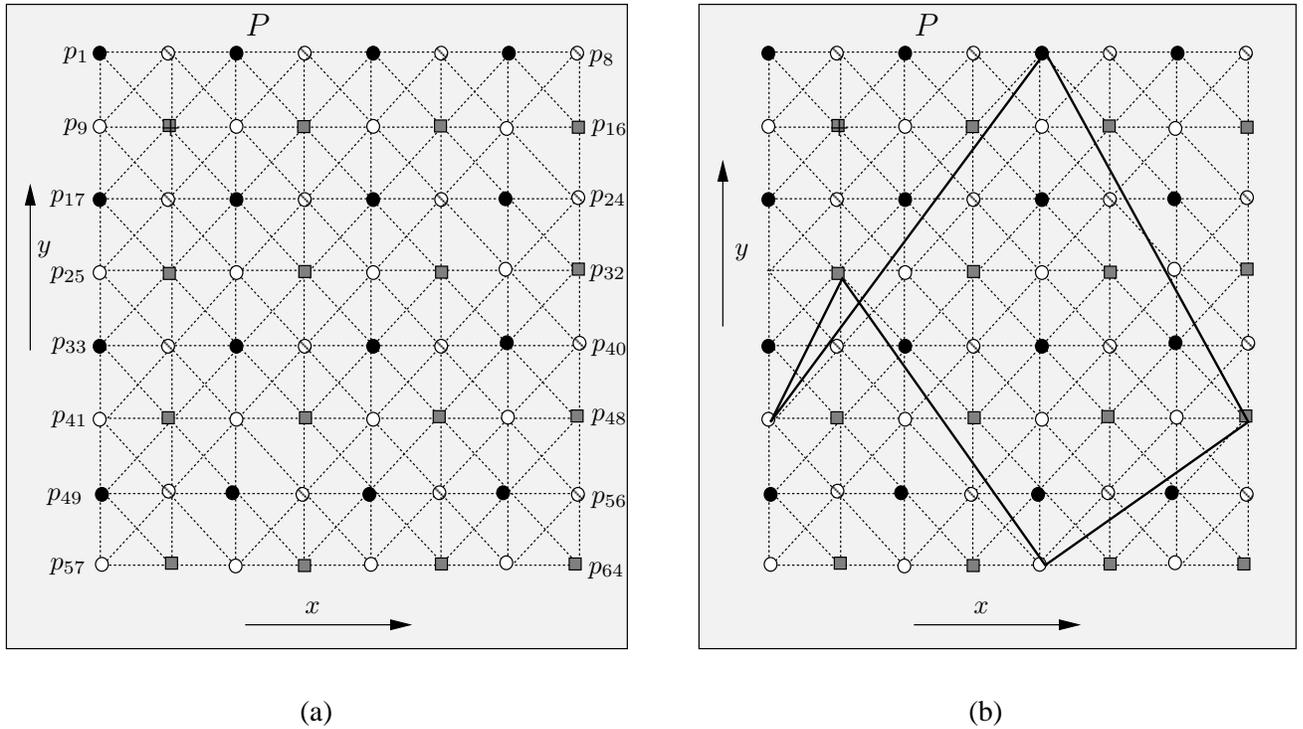


Figure 4: (a) The visibility graph of the integer lattice are coloured with four colours. Only collinear edges of the visibility graph are drawn in the figure. (b) Five lattice points form a chordless cycle.

The *chromatic number*  $\chi(G)$  is the minimum  $k$  such that  $G$  is  $k$ -colourable. The *clique number*  $\omega(G)$  is the maximum  $m$  such  $G$  contains a complete graph of  $m$  vertices as a subgraph. We start with the following lemma of Kára et al. [70].

**Lemma 1** *Let  $P = \{(x, y) : x, y \in \mathbb{Z}\}$  be the integer lattice. Then  $\chi(G) = \omega(G) = 4$ .*

It can be seen that all collinear lattice points on a line in Figure 4(a) can be coloured by two colours alternatively. Using this observation, the above lemma proves that the graph can be coloured by four colours. They also made the following observation.

**Lemma 2** *If a point set  $P \subseteq \mathbb{R}^2$  can be covered by  $m$  lines, then  $\chi(G) \leq 2m$ .*

Figure 4(a) demonstrates that the visibility graph of an integer lattice has a small chromatic number (i.e., 4) though the graph contains quadratic number of edges. Observe that the chromatic number is same as the clique number in this graph but the graph is

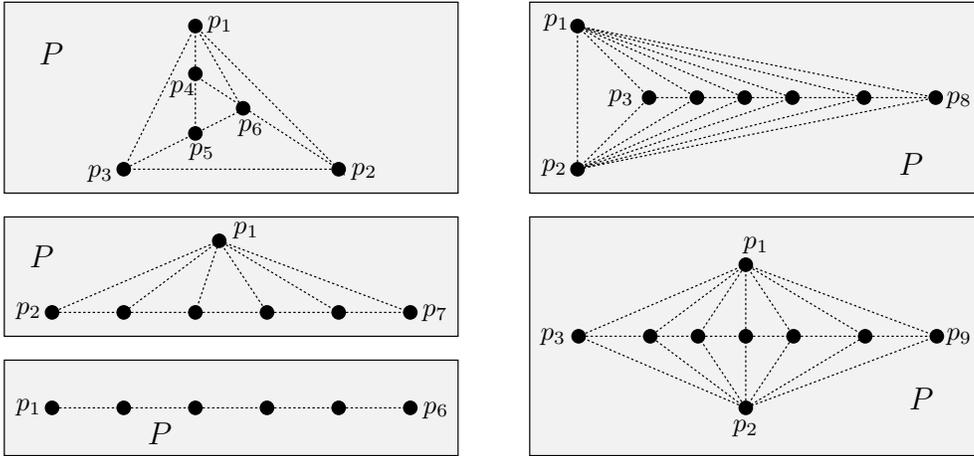


Figure 5: Visibility graphs of points with  $\omega(G) \leq 3$  are planar.

not a perfect graph as it contains a cycle of five vertices without chord. For example, the five lattice points in  $8 \times 8$  lattice with co-ordinates  $(2, 5), (1, 3), (5, 8), (8, 3), (5, 1)$  form a chordless cycle (see Figure 4(b)). However, Kára et al. felt that there is a relationship between the clique number and chromatic number in visibility graphs of points, and made the following conjecture.

**Conjecture 1** *There exists a function  $f$  such that  $\chi(G) \leq f(\omega(G))$ .*

In support of the conjecture, they proved that visibility graphs having  $\omega(G) \leq 3$  are planar [32] (see Figure 5) and they require at most 3 colours. For  $\omega(G) = 4$ , they showed an example (see Figure 6 (a)) that visibility graphs with  $\omega(G) = 4$  require 5 colours.

**Open Problem 4** *Prove that every visibility graph with  $\omega(G) \leq 4$  has  $\chi(G) \leq 5$ .*

**Open Problem 5** *Prove Conjecture 1 for visibility graphs with  $\omega(G) = 5$ .*

For visibility graphs with  $\omega(G) \geq 6$ , Pfender [91] proved that the conjecture does not hold as shown in the following lemma.

**Lemma 3** *For every  $k$ , there is a finite point set  $y \subset \mathbb{R}^2$  such that  $\chi(G) \geq k$  and  $\omega(G) = 6$ .*

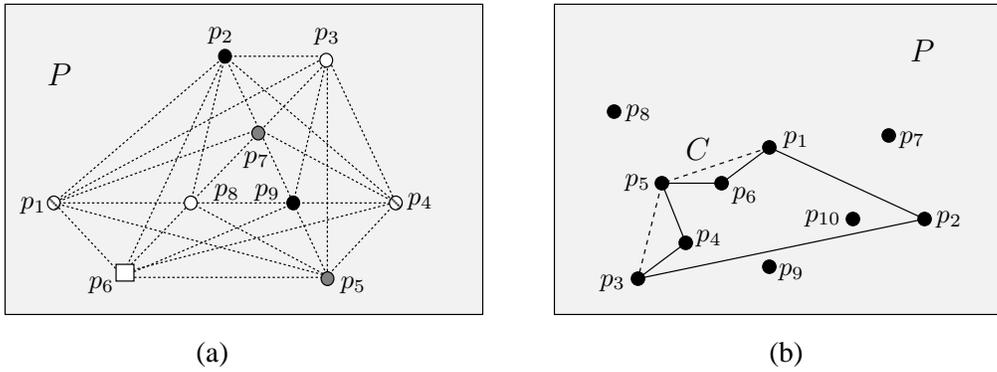


Figure 6: (a) The clique number of the visibility graph is 4 but the graph requires 5 colours. (b) Any subset of points may not form a convex polygon.

### 2.3 Big Clique in Visibility Graphs

As stated earlier, if all points of  $P$  are in general position, i.e., no three points of  $P$  are collinear, the visibility graph  $G$  of  $P$  is a complete graph, and therefore,  $\omega(G)$  is the size of  $P$ . Consider a convex polygon formed by points of  $P$ . A polygon  $C$  is said to be *convex* if the line segment joining any two points in  $C$  lie inside  $C$ . Though any subset  $X$  of  $P$  forms a complete graph in  $G$ , these points may not always form a convex polygon in  $P$  (see Figure 6(b)). Even if all points of  $X$  are in convex position forming a convex polygon  $C$ , some points of  $P$  may lie inside  $C$ . Several papers studied these problems for point sets with or without collinear points. We start with the famous result of Erdős and Szekeres [41] for points of  $P$  in general position.

**Theorem 1** *For every positive integer  $k$ , there exists a smallest integer  $g(k)$  such that any point set  $P$  of at least  $g(k)$  points in general position has a subset  $X$  of  $k$  points that are the vertices of a convex polygon  $C$ .*

It may be noted that the existence of the value  $g(k)$  runs immediately from the famous Ramsey theorem [95]. The best known bounds for  $g(k)$  are  $2^{k-2} + 1 \leq g(k) \leq \binom{2k-5}{k-2} + 2$ . The lower and upper bounds are given by Erdős and Szekeres [42], and Tóth and Valtr [113] respectively. For survey on this problem and many variants, see [13, 16, 80, 113].

Observe that some points of  $P - X$  may lie inside  $C$ , i.e.,  $C$  may not be empty (see Figure 6(a)). In this context, Erdős [39, 40] posed a problem of determining the smallest positive integer  $h(k)$  (if it exists) such that any point set  $P$  of at least  $h(k)$  points in general position in the plane has  $k$  points that are vertices of an empty convex polygon  $C$ . For

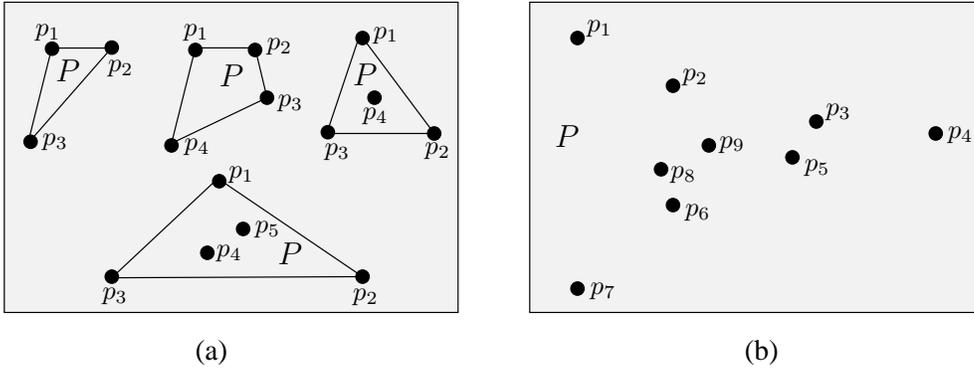


Figure 7: (a) Three points are enough for an empty triangle but four points may not always give empty quadrilateral. (b) In this set of 9 points, no subset of 5 points forms an empty convex polygon.

an empty triangle,  $h(3) = 3$  and for an empty quadrilateral, it can be seen that  $h(4) = 5$  (Figure 7(a)). For an empty pentagon, Figure 7(b) demonstrates  $h(5) \geq 10$ . In fact, Harborth [60] proved that  $h(5) = 10$ . For an empty hexagon, Gerken [51] and Nicolás [83] showed independently that  $h(6) \leq g(9) \leq 1717$  and  $h(6) \leq g(25)$  respectively. On the other hand, Overmars [88] established a lower bound based on computer experiment that  $h(6) \geq 30$ . The gap between the bounds has been reduced by Koshelev [69] by showing that  $h(6) \leq \max\{g(8), 400\} \leq 463$ . For  $k \geq 7$ ,  $h(k)$  is not bounded as shown by Horton [66].

Let us consider the other situation where point sets  $P$  contain collinear points. So, the boundary of a convex polygon  $C$  formed by a subset of points  $X$  of  $P$  may contain collinear points. Let  $Y$  be the points of  $X$  that are on corners of  $C$ . The points of  $Y$  are called *points in strictly convex position* as deletion of any point of  $Y$  reduces the area of  $C$ . The Erdős-Szekeres theorem mentioned above generalises to the following theorem [1, 77].

**Theorem 2** *For every integers  $\ell \geq 2$  and  $k \geq 3$ , there exists a smallest integer  $g(k, \ell)$  such that any point set  $P$  of at least  $g(k, \ell)$  points in the plane contains*

- (i)  $\ell$  collinear points, or
- (ii)  $k$  points in strictly convex position.

A straightforward upper bound on  $g(k, \ell)$  can be derived as given in [1]. Assume that  $P$  has  $\ell - 1$  collinear points and at most  $k - 1$  points in strictly convex position. Let  $X \subseteq P$  be any maximal set of points in strictly convex position. So, every point of  $P - X$  is collinear with two points in  $X$ . So,  $\binom{|X|}{2}$  lines cover all points of  $P$  and each line can have

at most  $\ell - 3$  points of  $P - X$ . Therefore,  $|P| \leq \binom{|X|}{2}(\ell - 3) + |X| \leq \binom{k}{2}(\ell - 3) + k - 1$ . If one more point is added to  $P$  (i.e.,  $|P| \leq \binom{k}{2}(\ell - 3) + k$ ), then  $P$  must contain  $\ell$  collinear points or  $k$  points in strictly convex position. A tighter upper bound on  $g(k, \ell)$  has been derived by Abel et al. [1].

Observe that  $P$  with  $g(k, \ell)$  points may have  $k$  points in strictly convex position but the convex polygon  $C$  formed by these  $k$  points may not be empty. So, the visibility graph  $G$  of  $P$  having  $g(k, \ell)$  points may not have a clique of size  $k$  as some points of  $P$  lying inside  $C$  may block the visibility between vertices of  $C$ . In the following, we state the *Big-Line-Big-Clique Conjecture* of Kára et al. [70].

**Conjecture 2** *For all integers  $k \geq 2$  and  $\ell \geq 2$ , there is an integer  $h(k, \ell)$  such that any point set  $P$  of at least  $h(k, \ell)$  points in the plane contains  $\ell$  collinear points, or  $k$  mutually visible points.*

It has been shown that a natural approach to settle this conjecture using extremal graph theory fails [94]. On the other hand, the conjecture is trivially true for  $\ell \leq 3$  and for all  $k$  on any point set  $P$  having  $k$  points. Based on planar graphs shown in Figure 5, Kára et al. [70] showed that every point set  $P$  of at least  $\max\{7, \ell + 2\}$  points contains  $\ell$  collinear points or 4 mutually visible points. Using Theorem 2, Abel et al. [1] proved that the conjecture is true for  $k = 5$  and for all  $\ell$ . For weaker versions of Conjecture 2 relating chromatic number with clique size, see Pór and Wood [94].

**Open Problem 6** *Prove Conjecture 2 for  $k = 6$  or  $\ell = 4$ .*

## 2.4 Blockers of Visibility Graphs

Let  $P$  be a set of  $n$  points in the plane. Let  $Q = (q_1, q_2, \dots, q_j)$  be another set of points (or blockers) in the plane such that (i)  $P \cap Q = \emptyset$  and (ii) every segment with both endpoints in  $P$  contains at least one point of  $Q$  (see Figure 8(a)). In other words, there is no edge in the visibility graph of  $P \cup Q$  that connects two points of  $P$ . Any such set  $Q$  is called a *blocking set* for  $P$ . If all points of  $P$  are collinear (see Figure 8(b)), then one blocker  $q_i$  is placed on the midpoint of each visible pairs in  $P$ , and therefore,  $|P| - 1$  blockers are necessary and sufficient. On the other hand, what is the minimum size of blocking set  $Q$  for  $P$  having no three points collinear (see Figure 8(a))? Note that a blocker may block several pairs of visible points if it is placed on intersection points of segments connecting points of  $P$ .

Let  $b(P)$  denote the smallest size blocking set  $Q$  for  $P$ . Let  $b(n)$  denote the minimum of  $b(P)$  for all  $P$  having  $n$  points with no three points being collinear. It is obvious that

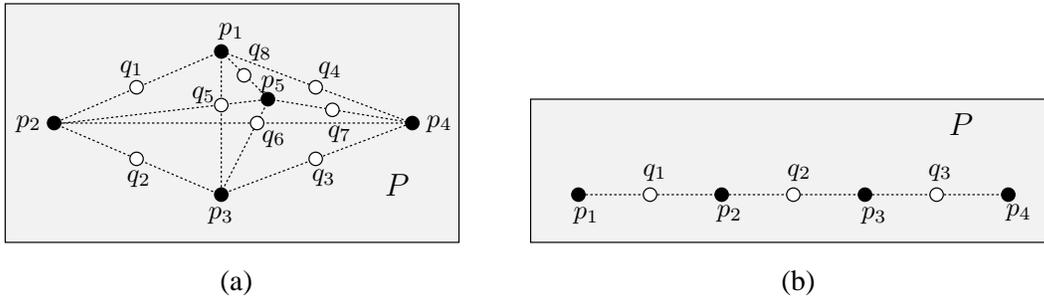


Figure 8: (a)  $Q = (q_1, q_2, \dots, q_7)$  is a blocking set for  $P = (p_1, p_2, \dots, p_5)$ . (b) Four collinear points need three blockers.

$b(n) \geq n - 1$ . A better lower bound can be derived using a triangulation of  $P$  [78]. Observe that every edge of a triangulation must contain one blocker. Since every triangulation has at least  $2n - 3$  edges, it follows that  $b(n) \geq 2n - 3$ . The lower bound is improved to  $b(n) \geq (\frac{25}{8} - o(1))n$  by Dumitrescu et al. [33].

Let us discuss upper bounds on  $b(n)$ . It is obvious that  $b(n) \leq \binom{n}{2}$ . Let  $\mu(P)$  denote the size of the set of midpoints of all  $\binom{n}{2}$  segments between points of  $P$ . Let  $\mu(n)$  denote the minimum of  $\mu(P)$  for all  $P$  having  $n$  points with no three points being collinear. Using Freiman's theorem on set addition [50], Stanchescu [108] and Pach [89] have independently shown that  $b(n) \leq \mu(n) \leq n2^{c\sqrt{\log n}}$ , where  $c$  is an absolute constant. This shows that if  $\mu(n)$  is not  $O(n)$ , it can only be slightly super-linear. Many authors [33, 62, 78, 94, 108] have stated or conjectured that every set of points  $P$  in general position requires a super-linear number of blockers as given below.

**Open Problem 7** *Prove that as  $n \rightarrow \infty$ ,  $\frac{b(n)}{n} \rightarrow \infty$ .*

Let us consider another problem of blockers in a visibility graph introduced by Aloupis et al. [7]. Let  $P$  be a set of points in the plane with some collinear points (see Figure 9). The problem is to assign  $k \geq 2$  colours to points of  $P$  such that (i) if two points are mutually visible in  $P$ , assign different colours to them, and (ii) if two points are not visible due to some collinear points in  $P$ , assign the same colour to both of them. Note that this method of colouring is different from the standard method of colouring of a graph due to the additional condition (ii). Any set of points that admits such a colouring with  $k$  colours (for a fixed  $k$ ) is called a  $k$ -blocked point set. Aloupis et al. [7] have made the following conjecture.

**Conjecture 3** *For each integer  $k$ , there is an integer  $n$  such that every  $k$ -blocked point set has at most  $n$  points.*

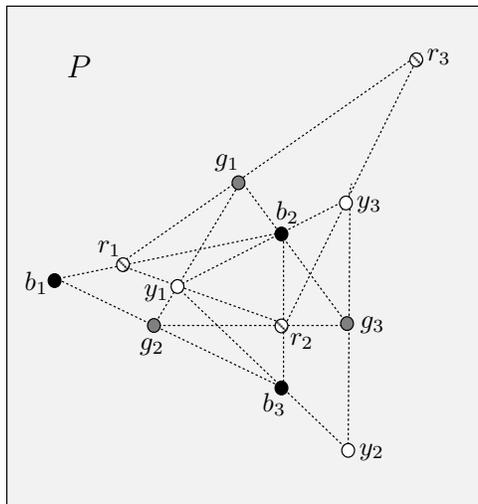


Figure 9: Since four colours are required for colouring points in  $P$ , it is a 4-blocked point set.

Let  $p_i$ ,  $p_j$  and  $p_l$  be three points in  $P$  such that  $p_i$  is not visible from both  $p_j$  and  $p_l$ . By condition (ii), all three points should be assigned the same colour. If  $p_j$  and  $p_l$  are also not mutually visible due to a collinear point in  $P$ , then the same colour can certainly be assigned to all three of them. However, if  $p_j$  and  $p_l$  are mutually visible, both conditions (i) and (ii) cannot be satisfied, and therefore, such a colouring is not possible which means that  $P$  is not a  $k$ -blocked point set. This implies that points of  $P$  that have received the same colour must be an independent set in the visibility graph of  $P$ . In other words, colour classes correspond to partition of points of  $P$  into independent sets. Following lemmas of Aloupis et al. [7] follow from the above discussion.

**Lemma 4** *At most three points are collinear in every  $k$ -blocked point set.*

**Lemma 5** *Each colour class in a  $k$ -blocked point set is in a general position.*

Let us discuss Conjecture 3 for different values of  $k$ . It can be seen from Figure 10 that every 2-blocked point set has at most 3 points [7]. It follows from the characterization of 3-colourable visibility graphs by Kára et al. [70] that every 3-blocked point set has at most 6 points (see Figure 10). Aloupis et al. [7] have proved that every 4-blocked point set has at most 12 points (see Figure 9). They also made the following conjectures on the size of blocked point sets.

**Conjecture 4** *Every  $k$ -blocked point set has  $O(k^2)$  points.*

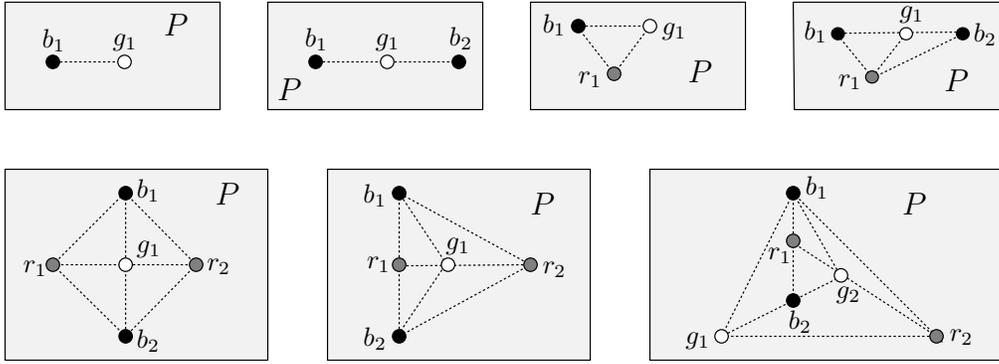


Figure 10: Every 2-blocked point set has at most 3 points, and every 3-blocked point set has at most 6 points

**Conjecture 5** *In every  $k$ -blocked point set, there are at most  $k$  points in each colour class.*

## 2.5 Obstacle Representations of Visibility Graphs

Let  $P = (p_1, p_2, \dots, p_n)$  be a set of points in the plane. Let  $Q = (Q_1, Q_2, \dots, Q_h)$  be a set of simple polygons in the plane called *obstacles*. Construct the visibility graph  $G$  such that every point  $p_i$  of  $P$  is represented as a vertex  $v_i$  of  $G$ , and two vertices  $v_i$  and  $v_j$  of  $G$  are connected by an edge in  $G$  if and only if the line segment  $p_i p_j$  does not intersect any obstacle  $Q_j$  for all  $j$ . We assume that the line segment joining any two points of  $P$  does not pass through a point of  $P$  or any vertex of an obstacle, i.e., all points of  $P$  and vertices of all obstacles are in general position. We call the pair  $(P, Q)$  as obstacle representation of  $G$ . Polygonal obstacles can be viewed as a generalization of blockers of visibility graphs discussed earlier.

Consider the problem of obstacle representation of a given graph  $G$  of  $n$  vertices, which was introduced by Alpet et al. [8]. Draw every vertex  $v_i$  of  $G$  as a point  $p_i$  in the plane and draw obstacles in such a way that every segment  $p_i p_j$  intersects an obstacle if and only if  $(v_i, v_j)$  is not an edge in  $G$ . The *obstacle number* of  $G$  is the minimum number of obstacles required in any obstacle representation of  $G$ . Since an obstacle can be placed to block the visibility between each pair of points,  $\binom{n}{2}$  is an upper bound on the obstacle number of  $G$ . We have the following question from Alpet et al. [8].

**Open Problem 8** *Is the obstacle number of a graph with  $n$  vertices bounded above by a linear function of  $n$ ?*

Regarding the lower bound on obstacle numbers, Alpet et al. [8] have showed that there exists a graph of  $n$  vertices with obstacle number  $O(\sqrt{\log n})$ , which has been improved to  $O(n/\log^2 n)$  by Mukkamala et al. [81]. The bound becomes  $O(n/\log n)$  if the obstacles are restricted to convex polygons.

**Open Problem 9** *Improve the present lower bound  $O(n/\log^2 n)$  of the obstacle number of a graph with  $n$  vertices.*

Regarding the graphs with low obstacle numbers, Alpet et al. [8] and Mukkamala et al. [81] have studied graphs with obstacle numbers 1 and 2. Mukkamala et al. [81] showed that for any positive integer  $h \geq 3$ , there is a graph with obstacle number exactly  $h$ . The following questions of Alpet et al. [8] are still open.

**Open Problem 10** *For  $h > 1$ , what is the smallest number of vertices of a graph with obstacle number  $h$ ?*

**Open Problem 11** *Does every planar graph have obstacle number 1?*

## 2.6 Connectivity of Visibility Graphs

Let us consider the problems of vertex-connectivity and edge-connectivity of visibility graphs. A graph  $H$  is called  *$k$ -vertex-connected* (or,  *$k$ -edge-connected*) if  $H$  remains connected even after deleting  $k - 1$  vertices (respectively, edges) from  $H$ . For such a graph  $H$ , there exists  $k$  number of vertex (or, edge) disjoint paths between every pair of vertices of  $H$  by Menger's theorem [28]. Let  $\kappa(H)$  and  $\lambda(H)$  denote the vertex and edge connectivity of  $H$  respectively. Let  $\delta(H)$  denote the minimum degree of  $H$ . We know that  $\kappa(H) \leq \lambda(H) \leq \delta(H)$ . Like any graph, all these properties naturally hold for a visibility graph  $G$ . Here, we state additional properties on connectivity of visibility graphs given by Payne et al. [90].

We know that the distance between two vertices in a graph is the number of edges in the shortest path between them, and the diameter of the graph is the longest path among the shortest paths between every pair of vertices in the graph. If all points of  $P$  are not collinear, then every pair points in  $P$  must be visible from some point of  $P$ . So, the shortest path between any two non-adjacent vertices in a visibility graph  $G$  passes through exactly two edges of  $G$ . Therefore, the diameter of  $G$  is of length 2 [70]. On the other hand, Plesník [92] has proved that the edge-connectivity of a graph with the diameter of length at most 2 equals its minimum degree. So, we have  $\kappa(G) \leq \lambda(G) = \delta(G)$ . This

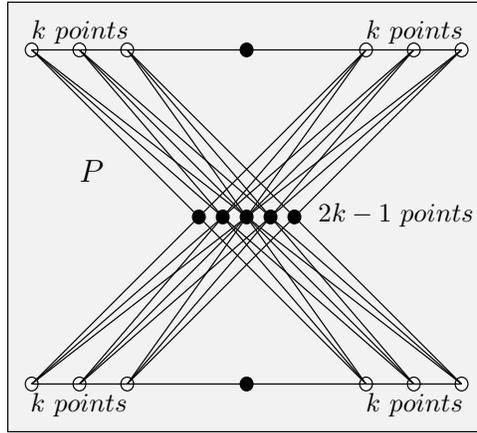


Figure 11: The black vertices are cut set for a visibility graph with vertex-connectivity  $\frac{2\delta(G)+1}{3}$ . The minimum degree  $\delta(G) = 3k + 1$  is achieved for extreme corner points. Not all edges are drawn in the figure.

result has been strengthened by Payne et al. [90] for visibility graphs as stated in the following theorems.

**Theorem 3** *Let  $v_i$  and  $v_j$  be two vertices of a visibility graph  $G$ . If  $d$  is the minimum of the degree of  $v_i$  and the degree of  $v_j$ , then there exists  $d$  number of edge disjoint paths of length at most 4 between  $v_i$  and  $v_j$  in  $G$ .*

**Theorem 4** *Let  $S$  be a set of minimum number of edges in a visibility graph  $G$  whose removal disconnects  $G$ . Then, all edges of  $S$  are incident to the same vertex of  $G$ .*

**Theorem 5** *Every visibility graph  $G$  with minimum degree  $\delta(G)$  has vertex-connectivity at least  $\frac{\delta(G)}{2} + 1$ .*

**Theorem 6** *Every visibility graph  $G$  with minimum degree  $\delta(G)$  has vertex-connectivity at least  $\frac{2\delta(G)+1}{3}$  if the number of collinear points of  $P$  on a line is restricted to 4.*

**Conjecture 6** *Every visibility graph  $G$  with minimum degree  $\delta(G)$  has vertex-connectivity at least  $\frac{2\delta(G)+1}{3}$  (see Figure 11).*

### 3 Visibility graph theory: Segments

#### 3.1 Visibility Graphs: Recognition, Characterization, and Reconstruction

In Section 1, we have defined the segment visibility graph  $G$  for a given set of disjoint line segments  $S$  (see Figure 2(b)). We have also stated how to compute  $G$  from  $S$  efficiently. Consider the opposite problem of determining if there is a set of disjoint segment  $S$  whose visibility graph is the given graph  $G$ . This problem is called the segment visibility graph *recognition* problem. Identifying the set of properties satisfied by all segment visibility graphs is called the segment visibility graph *characterization* problem. The problem of actually drawing one such set of segment  $S$  whose visibility graph is the given graph  $G$ , is called the segment visibility graph *reconstruction* problem.

All three above problems are open for segment visibility graphs. Only characterization known is for a sub-class given by Everette et al. [46]. They have characterized those segment visibility graphs that do not have  $K_5$  (a complete graph of five vertices) as a minor. A graph  $M$  is called a *minor* of a graph  $G$  if  $M$  can be obtained from  $G$  by a sequence of vertex deletions, edge deletions and edge contractions. Their characterization gives a straightforward polynomial time algorithm for recognizing this class of graphs.

**Open Problem 12** *Given a graph  $G$  in adjacency matrix form, determine whether  $G$  is the segment visibility graph of a set of disjoint segments  $S$  in the plane.*

**Open Problem 13** *Characterize the segment visibility graphs.*

**Open Problem 14** *Given a segment visibility graph  $G$ , draw the segments  $S$  in the plane whose visibility graph is  $G$ .*

#### 3.2 Hamiltonian Cycles in Visibility Graphs

Let  $G$  be the segment visibility graph of a given set  $S$  of disjoint line segments. Consider the problem of identifying a Hamiltonian cycle  $C$  in  $G$  (see Figure 12). A cycle in  $G$  is called a *Hamiltonian cycle* if the cycle passes through all vertices of  $G$  exactly once. There can be two types of Hamiltonian cycles  $C$  in  $G$ . Assume that  $G$  is drawn directly on the segments of  $S$  and call this embedded segment visibility graph as  $G'$ . If no two segments in  $G'$  corresponding to edges of a cycle  $C$  intersect, then  $C$  forms the boundary of a simple polygon in  $G'$ . Such cycles are called *Hamiltonian circuits* [96] (see Figure

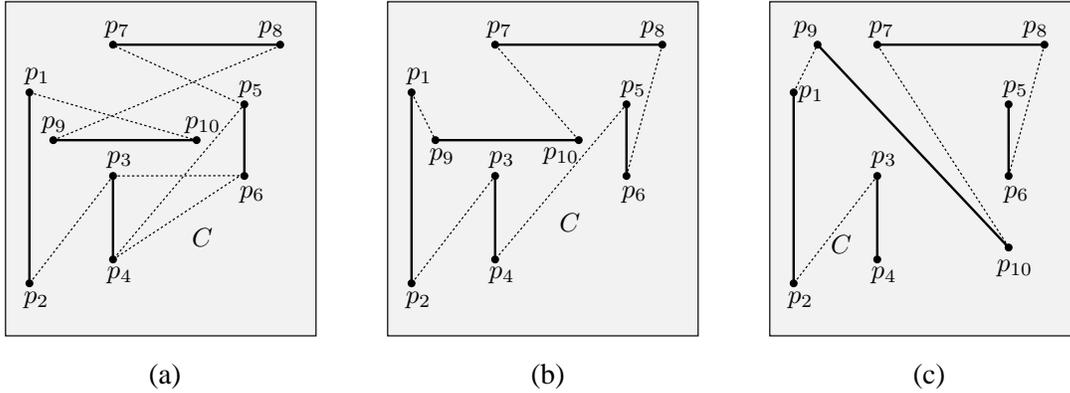


Figure 12: (a) The cycle  $C = (p_1, p_{10}, p_9, p_8, p_7, p_5, p_4, p_6, p_3, p_2, p_1)$  is Hamiltonian but is self-intersecting. (b) The cycle  $C = (p_1, p_9, p_{10}, p_7, p_8, p_6, p_5, p_4, p_3, p_2, p_1)$  is a Hamiltonian circuit as it forms a boundary of a simple polygon. (c) There is no alternating cycle as both endpoints of the segment  $p_9p_{10}$  belong to the convex hull of the segments [97].

12(b)), and the corresponding polygons are called *spanning polygons* [79] or *Hamiltonian polygons* [65]. Otherwise,  $C$  corresponds to the boundary of a self-crossing polygon in  $G'$  (see Figure 12(a)). A Hamiltonian polygon  $Q$  is called a *circumscribing polygon* if it has an additional property that no segment of  $S$  lies to the exterior of  $Q$ , i.e., each segment of  $S$  is either an edge on the boundary of  $Q$  or an internal chord of  $Q$  [79] (see Figure 12(b)). Mirzaian [79] made the following conjectures.

**Conjecture 7** *Every segment visibility graph  $G$  contains a Hamiltonian cycle  $C$ .*

**Conjecture 8** *Every segment visibility graph  $G$  contains a Hamiltonian cycle  $C$  that corresponds to a Hamiltonian circuit in the embedded segment visibility graph  $G'$ .*

**Conjecture 9** *Every segment visibility graph  $G$  contains a Hamiltonian cycle  $C$  that corresponds to the boundary of a circumscribing polygon in the embedded segment visibility graph  $G'$ .*

Observe that if Conjecture 9 is true, then both Conjectures 8 and 7 are also true. However, Urbe and Watanabe [114] gave a counter-example to Conjecture 9. For special classes of segments in  $S$ , Conjectures 8 was proved by Mirzaian [79] and O'Rourke and J. Rippel [86]. Later, Conjectures 8 was proved for all classes of segments in  $S$  by Hoffmann and Tóth [65] using the result of Bose et al. [15], and they presented an  $O(n \log n)$  time algorithm

for locating a Hamiltonian circuit in  $G'$ . Since Conjectures 8 is true, Conjectures 7 is also true.

Observe that the algorithm of Hoffmann and Tóth [65] takes  $S$  as an input and then locates a Hamiltonian circuit in the embedded segment visibility graph  $G'$ . Suppose  $S$  is not given as an input but only  $G$  is given, then it is not clear how to identify a Hamiltonian cycle in  $G$  in polynomial time as there is no known algorithm for segment visibility graph reconstruction problem. So, we have the following problems.

**Open Problem 15** *Given a segment visibility graph  $G$  in adjacency matrix form, identify a Hamiltonian cycle in  $G$  in polynomial time.*

**Open Problem 16** *Given a segment visibility graph  $G$  in adjacency matrix form, identify the edges of  $G$  that correspond to segments of  $S$ .*

Consider the problem of identifying a special type of Hamiltonian circuit  $C$  in  $G'$  where every alternate edge of  $C$  is a segment of  $S$  (see Figure 12(b)). It has been shown by Rappaport et al. [97] that such an *alternating cycle* may not always exist in  $G'$  (see Figure 12(c)). On the other hand, they showed that an alternating cycle always exists if one endpoint of every segment in  $S$  belongs to the convex hull of  $S$ . For this special class of segments, they gave an  $O(n \log n)$  time algorithm for constructing an alternating cycle.

If  $G'$  does not contain an alternating cycle, it is natural to ask for a longest alternating path that is present in  $G'$  (see Figure 12(c)). Urratia [115] made the following conjecture which was proved by Hoffmann and Tóth [64].

**Conjecture 10** *In the embedded segment visibility graph  $G'$  of a set  $S$  of  $n$  disjoint segments, there exists an alternating path containing at least  $O(\log n)$  segments of  $S$ .*

### 3.3 Bar Visibility Graphs

The idea of representing a graph using a visibility relation was introduced in the 1980s as a model tool for VLSI layout problems [31, 100]. A graph  $G$  is called a *bar visibility graph* if its vertices  $v_1, v_2, \dots, v_n$  can be associated with a set  $S$  of disjoint line segments (or, horizontal bars)  $s_1, s_2, \dots, s_n$  in the plane such that  $v_i$  and  $v_j$  are joined by an edge in  $G$  if and only if there exists an unobstructed vertical line of sight between  $s_i$  with  $s_j$  [112]. The set  $S$  is called a *bar visibility representation* of  $G$  (see Figure 13). If each line of sight is required to be a rectangle of positive width, then  $S$  is an  $\epsilon$ -*visibility representation* of  $G$  (see Figure 13(b)). If each line of sight is a segment (i.e. width is 0), then  $S$  is a *strong*

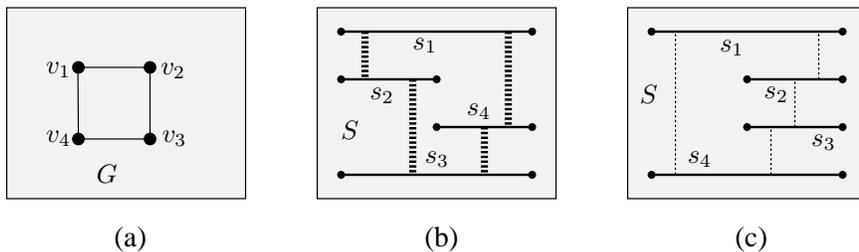


Figure 13: (a) A given graph  $G$ . (b) An  $\epsilon$ -visibility representation of  $G$ . (c) A strong visibility representation of  $G$ .

*visibility representation* of  $G$  (see Figure 13(c)). Note that if a vertical segment between two horizontal segments (say,  $s_1$  and  $s_3$  in Figure 13(c)) passes through an endpoint of another horizontal segment (i.e.,  $s_2$ ), it is considered that the line of sight is obstructed by the middle horizontal segment. We have the following theorems on the characterizations and representations of bar visibility graphs [9, 68, 76, 99, 112, 118].

**Theorem 7** *A graph  $G$  admits an  $\epsilon$ -visibility representation if and only if there is a planar embedding of  $G$  such that all cutpoints of  $G$  appear on the boundary of the external face in the embedding.*

**Theorem 8** *An  $\epsilon$ -visibility representation of a 2-connected planar graph  $G$  of  $n$  vertices can be done in  $O(n)$  time.*

**Theorem 9** *Let  $G$  be a 2-connected planar graph. If  $G$  admits a strong visibility representation, then there is no pair of non-adjacent vertices  $v_i$  and  $v_j$  of  $G$  such that the removal of  $v_i$  and  $v_j$  separates  $G$  into four or more components.*

*Proof:* Let  $c_1, c_2, \dots, c_k$  for  $k \geq 4$  be the connected component of  $G$  after removing  $v_i$  and  $v_j$  (see Figure 14(a)). Assume on the contrary that  $G$  admits a strong visibility representation (say,  $R$ ). Let  $s_m$  denote the horizontal segment in  $R$  corresponding to  $v_m$  of  $G$ .

Consider the situation where there exists a vertical line  $L$  between  $s_i$  and  $s_j$  of  $R$  that does not intersect either of them (see Figure 14(b)). Since every component  $c_l$  connects  $v_i$  to  $v_j$  in  $G$ , there exists a vertex  $v_m$  of  $c_l$  for all  $l$  such that  $L$  intersects the horizontal segment  $s_m$  of  $R$ . This intersection of  $L$  with  $s_m$  for every component implies that vertices of different components are connected by edges in  $G$ , which is a contradiction.

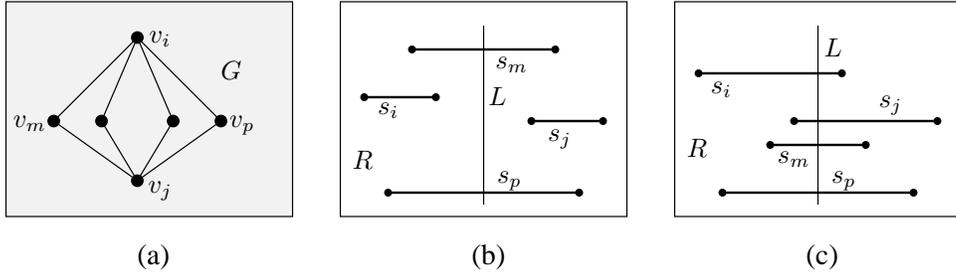


Figure 14: (a) A given graph  $G$ . (b)  $L$  does not intersect  $s_i$  and  $s_j$ . (c)  $L$  intersects both  $s_i$  and  $s_j$ .

Consider the other situation where there exists a vertical line  $L$  intersecting both  $s_i$  and  $s_j$  of  $R$  (see Figure 14(c)). So, there are three parts of  $L$  due to these intersections. If any of these parts of  $L$  intersects two horizontal segments  $s_m$  and  $s_p$  in  $R$  where  $v_m$  and  $v_p$  belong to different component in  $G$ , then there must be an edge between  $v_m$  and  $v_p$  in  $G$ , which is a contradiction. So, each part of  $L$  can intersect horizontal segments of  $R$  coming only from the same component. Therefore,  $L$  cannot intersect horizontal segments of  $R$  coming from more than three different components. Hence,  $G$  does not admit a strong visibility representation, a contradiction.  $\square$

**Theorem 10** *There exists a 3-connected planar graph  $G$  that does not admit a strong visibility representation.*

**Theorem 11** *Every 4-connected planar graph  $G$  admits a strong visibility representation.*

**Theorem 12** *A strong visibility representation of a 4-connected planar graph  $G$  of  $n$  vertices can be done in  $O(n^3)$  time.*

Let us consider variations of bar visibility graphs. While representing vertices of  $G$  as bars, there is no restriction on the length of horizontal bars. Suppose, a restriction is imposed that all bars in a visibility representation must have the same length. In that case, we get another type of visibility graphs which are known as *unit bar visibility graphs*. Though there are characterizations for special classes of unit bar visibility graphs [18, 25, 27], no characterization is known for general graphs. We have the following problems.

**Open Problem 17** *Characterize unit bar visibility graphs.*

**Open Problem 18** *Is recognition of unit bar visibility graphs NP-complete?*

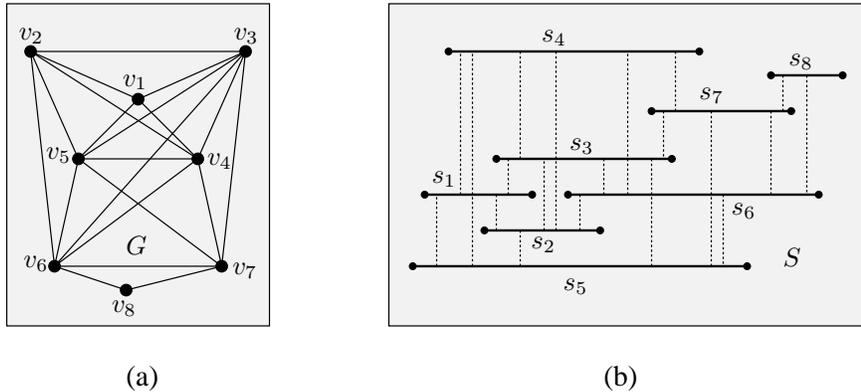


Figure 15: (a) A given graph  $G$ . (b) Representation shows that  $G$  is a bar 1-visibility graph.

Dean et al. [24] introduced another variation of bar visibility graphs called *bar  $k$ -visibility graphs*, where bars are allowed to see vertically through at most  $k$  bars under strong visibility (see Figure 15). It means that standard bar visibility graphs become bar 0-visibility graphs. If bars are allowed to see vertically through all other bars (i.e. bar  $\infty$ -visibility graphs), then the bar representation gives an interval graph representation. Formally, a graph  $G$  is called a *bar  $k$ -visibility graph* if its vertices  $v_1, v_2, \dots, v_n$  can be associated with a set  $S$  of disjoint line segments (or, horizontal bars)  $s_1, s_2, \dots, s_n$  in the plane such that  $v_i$  and  $v_j$  are joined by an edge in  $G$  if and only if a vertical segment between  $s_i$  and  $s_j$  intersects at most  $k$  segments of  $S$  [24, 49, 61].

Unlike bar visibility graphs, bar  $k$ -visibility graphs have not been completely characterized. For understanding this class of graphs, Dean et al. [24] considered the problem of bounding the number of edges of a bar  $k$ -visibility graph  $G$ . They proved an upper bounds of  $(k + 1)(3n - 7/2k - 5) - 1$  edges for  $G$  and conjectured an improve edge bound of  $(k + 1)(3n - 4k - 6)$  which was proved later by Hartke et al. [61]. They also studied  $d$ -regular  $k$ -visibility graphs. Dean et al. [24] also proved that  $6k + 6$  is an upper bound on the chromatic number of  $G$ . We have the following problems [24, 61].

**Open Problem 19** *Characterize bar  $k$ -visibility graphs.*

**Open Problem 20** *Triangle-free non-planar graphs are forbidden subgraphs of bar  $k$ -visibility graphs. Is there any other class of forbidden subgraphs of bar  $k$ -visibility graphs?*

**Open Problem 21** *Are there  $(2k + 2)$ -regular bar  $k$ -visibility graphs for  $k \geq 5$ ?*

**Open Problem 22** *Are there  $d$ -regular bar  $k$ -visibility graphs with  $d \geq 2k + 3$ ?*

**Open Problem 23** *Improve the current upper bound of  $6k + 6$  on the chromatic number of a bar  $k$ -visibility graph.*

Dean et al. [24] also studied the thickness of bar  $k$ -visibility graphs. The *thickness* of a graph  $G$  is defined as the minimum number of planar graphs whose union is  $G$ . Exact values of thickness is known for very few classes of graphs [82]. Dean et al. proved an upper bound of 4 for the thickness of bar 1-visibility graphs, and conjectured that bar 1-visibility graphs actually have thickness of at most 2. The conjecture was disproved by Felsner and Massow [49] by constructing a bar 1-visibility graph having thickness 3. For a special class of bar 1-visibility graphs, Felsner and Massow [49] presented an algorithm for partitioning the edges into two planar graphs showing that the thickness of this special class of graphs is 2. We have the following problems [24].

**Open Problem 24** *It has been shown that the thickness of a bar  $k$ -visibility graph is bounded by  $2k(9k - 1)$ . Can this upper bound be improved?*

**Open Problem 25** *The crossing number of a graph is the minimum possible number of crossings with which the graph can be drawn in the plane. What is the largest crossing number of a bar  $k$ -visibility graph?*

**Open Problem 26** *The genus of a graph is the minimal integer  $g$  such that the graph can be embedded on a surface of genus  $g$ . What is the largest genus of a bar  $k$ -visibility graph?*

Bar visibility graphs have been generalized to rectangle visibility graphs by considering both vertical and horizontal visibility among bars having non-zero thickness [14, 26, 85, 98, 111]. A graph  $G$  is called a *rectangle visibility graph* if it can be realized by closed isothetic rectangles in the plane, with pairwise disjoint interiors, with vertices representing rectangles in such a way that two vertices  $v_i$  and  $v_j$  of  $G$  are connected by an edge if and only if their corresponding rectangles are vertically or horizontally visible from each other by a beam of unobstructed visibility of finite width. Unlike bar visibility graphs, no characterization of rectangle visibility graphs are known, and moreover, Shermer [105] has shown that the problem of recognizing them is NP-complete. We have the following problem.

**Open Problem 27** *Characterize rectangle visibility graphs.*

## 4 Visibility graph theory: Polygons

### 4.1 Visibility Graph Recognition

In Section 1, we have defined the vertex visibility graph  $G$  for a given simple polygon  $P$  (see Figure 3(b)). We have also stated how to compute  $G$  from  $P$  efficiently. Consider the opposite problem of determining if there is a simple polygon  $P$  whose visibility graph is the given graph  $G$ . This problem is called the visibility graph *recognition* problem for polygons. The general problem of recognizing a given graph  $G$  as the visibility graph of a simple polygon  $P$  is yet to be solved. However, this problem has been solved for visibility graphs of *spiral* polygons [43, 44] and *tower* polygons [19].

**Open Problem 28** *Given a graph  $G$  in adjacency matrix form, determine whether  $G$  is the visibility graph of a simple polygon  $P$ .*

**Open Problem 29** *Is the problem of recognizing visibility graphs in NP?*

Ghosh [53, 55] presented three necessary conditions for recognizing visibility graphs  $G$  of a simple polygon  $P$  under the assumption that a Hamiltonian cycle  $C$  of  $G$ , which corresponds to the boundary of  $P$ , is given as input along with  $G$ . It can be seen that this problem is easier than the actual recognition problem as the edges of  $G$  corresponding to boundary edges of  $P$  have already been identified. Assume that the vertices of  $G$  are labeled with  $v_1, v_2, \dots, v_n$  and  $C = (v_1, v_2, \dots, v_n)$  is in counterclockwise order. An edge in  $G$  connecting two non-adjacent vertices of a cycle is called a *chord* of the cycle. A cycle  $w_1, w_2, \dots, w_k$  in  $G$  is called *ordered* if the vertices  $w_1, w_2, \dots, w_k$  follow the order in  $C$ . The Hamiltonian cycle  $C$  is an ordered cycle of all  $n$  vertices in  $G$ .

**Necessary condition 1.** *Every ordered cycle of  $k \geq 4$  vertices in a visibility graph  $G$  of a simple polygon  $P$  has at least  $k - 3$  chords.*

*Proof:* Since an ordered cycle of  $k$  vertices in  $G$  corresponds to a sub-polygon  $P'$  of  $k$  vertices in  $P$ , the ordered cycle must have at least  $k - 3$  chords in  $G$  as  $P'$  needs  $k - 3$  diagonals for triangulation of  $P'$ .  $\square$

A pair of vertices  $(v_i, v_j)$  in  $G$  is a *visible pair* (or *invisible pair*) if  $v_i$  and  $v_j$  are adjacent (respectively, not adjacent) in  $G$ . The vertices from  $v_i$  to  $v_j$  on  $C$  in counterclockwise order are denoted as  $chain(v_i, v_j)$ . Let  $v_a$  be a vertex of  $chain(v_i, v_j)$  for  $i < j$  such that no two vertices  $v_k \in chain(v_i, v_{a-1})$  and  $v_m \in chain(v_{a+1}, v_j)$  are connected by an edge in  $G$ . Then  $v_a$  is called a *blocking vertex* for the invisible pair  $(v_i, v_j)$  (see Figure 16(a) and

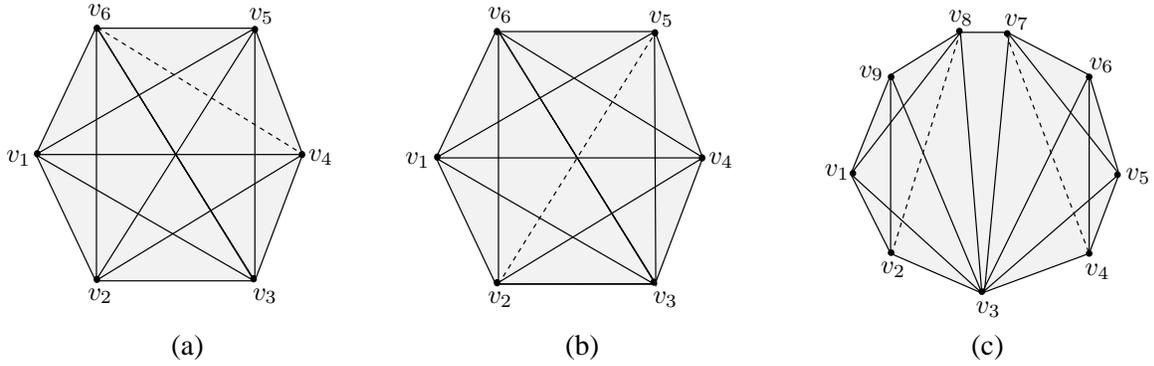


Figure 16: (a) For the invisible pair  $(v_4, v_6)$ ,  $v_5$  is the blocking vertex. (b) The invisible pair  $(v_2, v_5)$  does not have any blocking vertex. (c) The blocking vertex  $v_3$  cannot simultaneously block the visibility of invisible pairs  $(v_2, v_8)$  and  $(v_4, v_7)$ .

Figure 16(b)). Intuitively, blocking vertices correspond to reflex vertices of the polygon though all blocking vertices in  $G$  may not be reflex vertices in  $P$ .

**Necessary condition 2.** *Every invisible pair  $(v_i, v_j)$  in the visibility graph  $G$  of a simple polygon  $P$  has at least one blocking vertex.*

*Proof:* Since  $(v_i, v_j)$  is an invisible pair in  $G$ , the *Euclidean shortest path* in  $P$  between  $v_i$  and  $v_j$  makes turns at reflex vertices of  $P$ , and therefore, each of these reflex vertices is a blocking vertex for  $(v_i, v_j)$  in  $G$ .  $\square$

Let  $v_a$  be a blocking vertex in  $G$  for two invisible pairs  $(v_i, v_j)$  and  $(v_k, v_l)$ . Traverse the Hamiltonian cycle  $C$  from  $v_a$  in counterclockwise order. If both  $v_k$  and  $v_l$  are encountered before  $v_i$  and  $v_j$  during the traversal, then  $(v_i, v_j)$  and  $(v_k, v_l)$  are referred as *separable* with respect to  $v_a$ . In Figure 16(c), invisible pairs  $(v_2, v_8)$  and  $(v_4, v_7)$  are separable with respect to the blocking vertex  $v_3$ .

**Necessary condition 3.** *Two separable invisible pairs  $(v_i, v_j)$  and  $(v_k, v_l)$  in the visibility graph  $G$  of a simple polygon  $P$  must have distinct blocking vertices.*

*Proof:* Let  $(v_i, v_j)$  and  $(v_k, v_l)$  be two separable invisible pairs and the vertex  $v_a$  is their sole blocking vertex. So,  $v_a$  must be a reflex vertex in  $P$ . Since the visibility in  $P$  between  $v_i$  and  $v_j$  as well as between  $v_k$  and  $v_l$  can only be blocked by  $v_a$  and the sub-polygons of  $P$  corresponding to ordered cycles  $v_i, v_a, v_j, \dots, v_i$  and  $v_a, v_k, \dots, v_l, v_a$  are disjoint,  $v_a$  cannot simultaneously block the visibility between  $v_i$  and  $v_j$  and between  $v_k$  and  $v_l$  in  $P$ .  $\square$

It has been pointed out by Everett and Corneil [43, 45] that these three conditions are not sufficient as there are graphs that satisfy the three necessary conditions but are not visibility graphs of any simple polygon. These counterexamples can be eliminated once the third necessary condition is strengthened. It has been shown by Srinivasraghavan and Mukhopadhyay [107] that the stronger version of the third necessary condition proposed by Everett [43] is in fact necessary.

**Necessary condition 3'.** *There is an assignment in a visibility graph such that no blocking vertex  $v_a$  is assigned to two or more minimal invisible pairs that are separable with respect to  $v_a$ .*

On the other hand, the counterexample given by Abello, Lin and Pisupati [5] shows that the three necessary conditions of Ghosh [53] are not sufficient even with the stronger version of the third necessary condition. In a later paper by Ghosh [54], another necessary condition is identified which circumvents the counterexample of Abello, Lin and Pisupati [5].

**Necessary condition 4.** *Let  $D$  be any ordered cycle of the visibility graph  $G$  of a simple polygon  $P$ . For any assignment of blocking vertices to all minimal invisible pairs in  $G$ , the total number of vertices of  $D$  assigned to the minimal invisible pairs between the vertices of  $D$  is at most  $|D| - 3$ .*

*Proof:* Let  $P'$  be the subpolygon of  $P$  such that the boundary of  $P'$  corresponds to  $D$ . If every blocking vertex  $v_a \in D$  is assigned to some minimal invisible pair between vertices of  $D$ ,  $v_a$  becomes a reflex vertex in  $P'$ . So, the sum of internal angles of  $P'$  is more than  $(|D| - 2)180^\circ$  contradicting the fact that the sum of internal angles of any simple polygon of  $|D|$  vertices is  $(|D| - 2)180^\circ$ .  $\square$

It has been shown later by Streinu [109, 110] that these four necessary conditions are also not sufficient. It is not clear whether another necessary condition is required to circumvent the counter example. For more details on the recognition of visibility graphs, see Ghosh [55].

**Open Problem 30** *Given a graph  $G$  in adjacency matrix form along with a Hamiltonian cycle  $C$  of  $G$ , determine whether  $G$  is the visibility graph of a simple polygon  $P$  whose boundary corresponds to the given Hamiltonian cycle  $C$ .*

Everett [43] presented an  $O(n^3)$  time algorithm for testing Necessary Condition 1 which was later improved by Ghosh [54, 55] to  $O(n^2)$  time. Ghosh also gave an  $O(n^2)$  time algorithm for testing Necessary Condition 2. Das, Goswami and Nandy [22] showed that Necessary Condition 3' can be tested in  $O(n^4)$  time. We have the following theorem.

**Theorem 13** *Given a graph  $G$  of  $n$  vertices and a Hamiltonian cycle  $C$  in  $G$ , Necessary Conditions 1, 2 and 3' can be tested in  $O(n^4)$  time.*

**Open Problem 31** *Design an algorithm for testing Necessary Condition 4 in polynomial time.*

## 4.2 Visibility Graph Characterization

The problem of identifying the set of properties satisfied by all visibility graphs of simple polygons is called the visibility graph *characterization* problem for polygons. Let us state some results on the problems of characterizing visibility graphs for special classes of simple polygons. The earliest result is from ElGindy [38] who showed that every *maximal outerplanar graph* is a visibility graph of a simple polygon, and he suggested an  $O(n \log n)$  algorithm for reconstruction. If all reflex vertices of a simple polygon occur consecutively along its boundary, the polygon is called a *spiral polygon*. Everett and Corneil [43, 44] characterized visibility graphs of spiral polygons by showing that these graphs are a subset of *interval graphs* which lead to an  $O(n)$  time algorithm. Choi, Shin and Chwa [19] characterized funnel-shaped polygons, also called *towers*, and gave an  $O(n)$  time recognition algorithm. Visibility graphs of towers are also characterized by Colley, Lubiw and Spinrad [20] and they have shown that visibility graphs of towers are bipartite permutation graphs with an added Hamiltonian cycle. If the internal angle at each vertex of a simple polygon is either 90 or 270 degrees, then the polygon is called a *rectilinear polygon*. If the boundary of a rectilinear polygon is formed by a staircase path with two other edges, the polygon is called a *staircase polygon*. Visibility graphs of staircase polygons have been characterized by Abello, Egecioglu, and Kumar [2]. Lin and Chen [73] have studied visibility graphs that are *planar*.

For the characterization of visibility graphs of arbitrary simple polygons, Ghosh has shown that visibility graphs do not possess the characteristics of *perfect graphs*, *circle graphs* or *chordal graphs*. On the other hand, Coullard and Lubiw [21] have proved that every triconnected component of a visibility graph satisfies *3-clique ordering*. This property suggests that structural properties of visibility graphs may be related to well-studied graph classes, such as *3-trees* and *3-connected graphs*. Everett and Corneil [43, 45] have shown that there is no finite set of forbidden induced subgraphs that characterize visibility graphs. Abello and Kumar [3, 4] have suggested a set of necessary conditions for recognizing visibility graphs. However, it has been shown in [54] that this set of conditions follow from the last two necessary conditions of Ghosh. For more details on the characterization of visibility graphs, see Ghosh [55].

**Open Problem 32** *Characterize visibility graphs of simple polygons.*

### 4.3 Visibility Graph Reconstruction

The problem of actually drawing a simple polygon  $P$  whose visibility graph is the given graph  $G$ , is called the visibility graph *reconstruction* problem for polygons. Let us mention some of the approaches on the visibility graph reconstruction problem. It has been shown by Everett [43] that visibility graph reconstruction is in  $PSPACE$ . This is the only upper bound known on the complexity of the problem. Abello and Kumar [4] studied the relationship between visibility graphs and oriented matroids, Lin and Skiena [74] studied the equivalent order types, and Streinu [109, 110] and O'Rourke and Streinu [87] studied psuedo-line arrangements. Everett and Corneil [43, 45] have solved the reconstruction problem for the visibility graphs of *spiral* polygons and the corresponding problem for the visibility graph of *tower* polygons has been solved by Choi, Shin and Chwa [19]. Reconstruction problem with added information has been studied by Coullard and Lubiw [21], Disser, Mihalák, and Widmayer [29], Everett, Hurtado, and Noy [47], Everett, Lubiw, and O'Rourke [48], Jackson and Wismath [67].

**Open Problem 33** *Draw a simple polygon whose visibility graph is the given graph  $G$ .*

### 4.4 Hamiltonian Cycle in Visibility Graphs

A *Hamiltonian cycle* is a cycle in an undirected graph which visits each vertex exactly once and also returns to the starting vertex. The Hamiltonian cycle problem is to determine whether a Hamiltonian cycle exists in a given graph  $G$ . Observe that  $G$  may contain several Hamiltonian cycles, and  $G$  may be visibility graph for a Hamiltonian cycle and is not a valid visibility graph for another Hamiltonian cycle in  $G$ .

**Open Problem 34** *Given the visibility graph  $G$  of a simple polygon  $P$ , determine the Hamiltonian cycle in  $G$  that corresponds to the boundary of  $P$ .*

### 4.5 Minimum Dominating Set in Visibility Graphs

A *dominating set* for a graph  $G = (V, E)$  is a subset  $D$  of  $V$  such that every vertex not in  $D$  is joined to at least one member of  $D$  by some edge. The minimum dominating set problem in visibility graphs corresponds to the art gallery problem in polygons which has been shown to be NP-hard [72, 74]. Following the approximation algorithm for the

art gallery problem for polygons given by Ghosh [52, 56], a minimum dominating set of visibility graph can be computed with an approximation ratio of  $O(\log n)$ .

**Open Problem 35** *Design a constant factor approximation algorithm for computing minimum dominating set of visibility graphs.*

## 4.6 Maximum Hidden Set in Visibility Graphs

An *independent set* is a set of vertices in a graph with no two of which are adjacent. Independent sets in visibility graphs are known as *hidden vertex sets*. Shermer [103] has proved that the maximum hidden vertex set problem on visibility graphs is also NP-hard. However, the problem may not remain NP-hard if the Hamiltonian cycle corresponding to the boundary of the simple polygon is given as an input along with the visibility graph. With this additional input, Ghosh, Shermer, Bhattacharya and Goswami [59] have shown that it is possible to compute in  $O(ne)$  time the maximum hidden vertex set in the visibility graph of a very special class of simple polygons called *convex fans*, where  $n$  and  $e$  are the number of vertices and edges of the input visibility graph of the convex fan respectively. Hidden vertex sets are also studied by Eidenbenz [35, 36], Ghosh et al. [57] and Lin and Skiena [74].

**Open Problem 36** *Given the visibility graph  $G$  of a simple polygon  $P$  along with the Hamiltonian cycle in  $G$  corresponding to the boundary of  $P$ , determine the maximum hidden set of  $G$ .*

**Open Problem 37** *Design an approximation algorithm for computing maximum hidden set of visibility graph.*

## 4.7 Maximum Clique in Visibility Graphs

A *clique* in a graph is a set of pairwise adjacent vertices. The problem of computing the maximum clique in the visibility graph is not known to be NP-hard. Observe that the maximum clique in a visibility graph corresponds to the largest empty convex polygon inside the corresponding polygon. Algorithms for computing largest empty convex polygons has been reported by several authors [12, 30, 37]. However, for each of these algorithms, the input is either a polygon [37] or a point set [12, 30]. Spinrad [106] has discussed possible approaches for computing maximum clique using the notion of *triangle-extendible ordering* which is essentially a transitive orientation of the graph.

**Open Problem 38** *Given a visibility graph  $G$  in adjacency matrix form, compute a maximum clique of  $G$ .*

**Open Problem 39** *Determine whether a set of vertices of a visibility graph has a triangle-extendible ordering in polynomial time.*

If the Hamiltonian cycle in a visibility graph corresponding to the boundary of the polygon is given along with the visibility graph as an input, Ghosh, Shermer, Bhattacharya and Goswami [59] have presented an  $O(n^2e)$  time algorithm for computing the maximum clique in the visibility graph  $G$  of a simple polygon  $P$ . Here  $n$  and  $e$  are number of vertices and edges of  $G$  respectively.

## 4.8 Counting Visibility Graphs

Two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are *isomorphic* if and only if there is a bijection  $f$  that maps vertices of  $V_1$  to the vertices of  $V_2$  such that an edge  $(v, w) \in E_1$  if and only if the edge  $(f(v), f(w)) \in E_2$ . It has been shown by Ghosh [55] that the number of *non-isomorphic* visibility graphs of simple polygons of  $n$  vertices is at least  $2^{n-4}$ . On the other hand, a straightforward application of Warren's theorem [116] shows that the number of visibility graphs is at most  $2^{O(n \log n)}$  [106].

**Open Problem 40** *Improve the lower and upper bounds on the number of non-isomorphic visibility graphs of simple polygons.*

Let  $G_1$  and  $G_2$  be the visibility graphs of simple polygons  $P_1$  and  $P_2$  respectively. Let  $C_1$  (or  $C_2$ ) denote the Hamiltonian cycles in  $G_1$  (respectively,  $G_2$ ) that corresponds to the boundary of  $P_1$  (respectively,  $P_2$ ). Polygons  $P_1$  and  $P_2$  are called *similar* if and only if there is a bijection  $f$  that maps adjacent vertices on the boundary of  $P_1$  to that of boundary of  $P_2$  such that  $f(G_1) = G_2$  [74]. It has been shown [11] that similarity of  $P_1$  and  $P_2$ , each of  $n$  vertices, can be determined in  $O(n^2)$  time. Therefore, given  $G_1$  and  $G_2$  along with  $C_1$  and  $C_2$ , the corresponding visibility graph similarity problem can also be solved in  $O(n^2)$  time. It has been shown by Lin and Skiena [74] that two simple polygons with isomorphic visibility graphs may not be similar polygons.

## 4.9 Representing Visibility Graphs

Although the most natural form of representation for visibility graphs would be to use coordinates of the points, this is not useful if we are looking for a space efficient representation. Lin and Skiena [74] have proved that visibility graphs require endpoints to have

exponential sized integers. However, it is not known whether singly exponential sized integers are sufficient. It is important because if we could guarantee that the number of bits in the integer is polynomial, then visibility graph recognition is in  $NP$  [106].

**Open Problem 41** *Can all endpoints of a visibility graph be assigned integer coordinates such that the integers use a polynomial number of bits?*

A natural form of storage is studied by Agarwal et al. [6] which uses a relatively small number of bits to store a visibility graph. However, the representation is neither space optimal, nor adjacency information can be retrieved in constant time. However, it is the most significant reduced space representation which is currently known. The authors consider the problem of representing a visibility graph as a covering set of cliques and complete bipartite graphs so that every graph in the set is a subset of  $G$ , and every edge is contained in at least one of the graphs of the covering set. Their proposed algorithm constructs a covering set which has  $O(n \log^4 n)$  bits. It can be shown that any covering set requires  $\Omega(n \log^2 n)$  bits on some visibility graphs [106]. Given this gap between upper and lower bounds on this natural form of representation, we have a number of problems.

**Open Problem 42** *Give a tight bound (with respect to order notation) on the number of bits used in an optimal covering set of a visibility graph.*

**Open Problem 43** *Find a covering set which matches the above bound in polynomial time.*

## References

- [1] Z. Abel, B. Ballinger, P. Bose, S. Collette, V. Dujmović, F. Hurtado, S. D. Kominers, S. Langerman, A. Pór, and D. R. Wood. Every large point set contains many collinear points or an empty pentagon. *Graphs and Combinatorics*, 27(1):47–60, 2011.
- [2] J. Abello, O. Egecioglu, and K. Kumar. Visibility graphs of staircase polygons and the weak Bruhat order, I: from visibility graphs to maximal chains. *Discrete & Computational Geometry*, 14:331–358, 1995.
- [3] J. Abello and K. Kumar. Visibility graphs and oriented matroids. In *Proceeding of Graph Drawing*, volume 894 of *Lecture Notes in Computer Science*, pages 147–158. Springer-Verlag, 1995.

- [4] J. Abello and K. Kumar. Visibility graphs and oriented matroids. *Discrete & Computational Geometry*, 28:449–465, 2002.
- [5] J. Abello, H. Lin, and S. Pisupati. On visibility graphs of simple polygons. *Congressus Numerantium*, 90:119–128, 1992.
- [6] P. K. Agarwal, N. Alon, B. Aronov, and S. Suri. Can visibility graphs be represented compactly? *Discrete & Computational Geometry*, 12:347–365, 1994.
- [7] G. Aloupis, B. Ballinger, S. Collette, S. Langerman, A. Pór, and D.R. Wood. Blocking coloured point sets. In *Proceedings of the 26th European Workshop Computational Geometry*, pages 29–32, 2010.
- [8] H. Alpert, C. Koch, and J. D. Laison. Obstacle numbers of graphs. *Discrete & Computational Geometry*, 44:223–244, 2010.
- [9] T. Andreae. Some results on visibility graphs. *Discrete Applied Mathematics*, 40:5–18, 1992.
- [10] T. Asano, T. Asano, L. J. Guibas, J. Hershberger, and H. Imai. Visibility of disjoint polygons. *Algorithmica*, 1:49–63, 1986.
- [11] D. Avis and H. ElGindy. A combinatorial approach to polygon similarity. *IEEE Transactions on Information Theory*, IT-2:148–150, 1983.
- [12] D. Avis and D. Rappaport. Computing the largest empty convex subset of a set of points. In *Proceedings of the 1st ACM Symposium on Computational Geometry*, pages 161–167, 1985.
- [13] I. Bárány and G. Károlyi. Problems and results around the Erdős-Szekeres convex polygon theorem. In *JCDCG '00: Revised Papers from the Japanese Conference on Discrete and Computational Geometry, Tokyo, 2000.*, pages 91–105. Springer, Berlin, 2001.
- [14] P. Bose, A. Dean, J. P. Hutchinson, and T. C. Shermer. On rectangle visibility graphs. In *Proceedings of the 4th International Conference on Graph Drawing*, volume 1190 of *Lecture Notes in Computer Science*, pages 25–44. Springer-Verlag, 1997.
- [15] P. Bose, M. E. Houle, and G. T. Toussaint. Every set of disjoint line segments admits a binary tree. *Discrete & Computational Geometry*, 26:387–410, 2001.

- [16] P. Brass, W. O. J. Moser, and J. Pach. *Research Problems in Discrete Geometry*. Springer, New York, 2005.
- [17] B. Chazelle, L. J. Guibas, and D.T. Lee. The power of geometric duality. *BIT*, 25:76–90, 1985.
- [18] G. Chen, J. Hutchinson, K. Keating, and J. Shen. Characterization of  $[1,k]$ -bar visibility trees. *The Electronic Journal of Combinatorics*, 13:1–13, 2006.
- [19] S.-H. Choi, S. Y. Shin, and K.-Y. Chwa. Characterizing and recognizing the visibility graph of a funnel-shaped polygon. *Algorithmica*, 14(1):27–51, 1995.
- [20] P. Colley, A. Lubiw, and J. Spinrad. Visibility graphs of towers. *Computational Geometry: Theory and Applications*, 7:161–172, 1997.
- [21] C. Coullard and A. Lubiw. Distance visibility graphs. *International Journal of Computational Geometry and Applications*, 2:349–362, 1992.
- [22] S. Das, P. Goswami, and S. Nandy. *Testing necessary conditions for recognizing visibility graphs of simple polygons*. Manuscript, Indian Statistical Institute, Kolkata, 2002.
- [23] M. de Berg, O. Cheong, M. Kreveld, and M. Overmars. *Computational Geometry, Algorithms and Applications*. Springer-Verlag, 3rd edition, 2008.
- [24] A. Dean, W. Evans, E. Gethner, J. D. Laison, M. A. Safari, and W. T. Trotter. Bar  $k$ -visibility graphs. *Journal of Graph Algorithms and Applications*, 11:45–59, 2007.
- [25] A. Dean, E. Gethner, and J. Hutchinson. Unit bar-visibility layouts of triangulated polygons. In *Proceedings of the 11th International Conference on Graph Drawing*, volume 3383 of *Lecture Notes in Computer Science*, pages 111–121. Springer-Verlag, 2004.
- [26] A. Dean and J. P. Hutchinson. Rectangle-visibility representations of bipartite graphs. *Discrete Applied Mathematics*, 75:9–25, 1997.
- [27] A. Dean and N. Veytsel. Unit bar-visibility graphs. *Congressus Numerantium*, 160:161–175, 2003.
- [28] R. Diestel. *Graph theory*. Springer, 2010.
- [29] Y. Disser, M. Mihalák, and P. Widmayer. A polygon is determined by its angles. *Computational Geometry: Theory and Applications*, 44:418–426, 2011.

- [30] D. P. Dobkin, H. Edelsbrunner, and M. H. Overmars. Searching for empty convex polygons. *Algorithmica*, 5:561–571, 1990.
- [31] P. Duchet, Y. Hamidoune, M. L. Vergnas, and H. Meyniel. Representing a planar graph by vertical lines joining different levels. *Discrete Mathematics*, 46:319–321, 1983.
- [32] V. Dujmović, D. Eppstein, M. Suderman, and D. R. Wood. Drawings of planar graphs with few slopes and segments. *Computational Geometry: Theory and Applications*, 38:194–212, 2007.
- [33] A. Dumitrescu, J. Pach, and G. Tóth. A note on blocking visibility between points. *Geombinatorics*, 19:67–73, 2009.
- [34] H. Edelsbrunner, J. O’Rourke, and R. Seidel. Constructing arrangements of lines and hyperplanes with applications. *SIAM Journal on Computing*, 15:341–363, 1986.
- [35] S. Eidenbenz. *In-approximability of visibility problems on polygons and terrains*. Ph. D. Thesis, Institute for Theoretical Computer Science, ETH, Zurich, 2000.
- [36] S. Eidenbenz. In-approximability of finding maximum hidden sets on polygons and terrains. *Computational Geometry: Theory and Applications*, 21:139–153, 2002.
- [37] S. Eidenbenz and C. Stamm. Maximum clique and minimum clique partition in visibility graphs. In *Proceedings of IFIP TCS, Lecture Notes in Computer Science*, volume 1872, pages 200–212. Springer-Verlag, 2000.
- [38] H. ElGindy. *Hierarchical decomposition of polygons with applications*. Ph. D. Thesis, McGill University, Montreal, 1985.
- [39] P. Erdős. Some more problems in elementary geometry. *Australian Mathematical Society Gazette*, 5:52–54, 1978.
- [40] P. Erdős. Some applications of graph theory and combinatorial methods to number theory and geometry. In *Algebraic methods in graph theory, vols. I, II, Szeged 1978. Colloquia Mathematica Societatis Janos Bolyai*, volume 25, pages 137–148. North-Holland, Amsterdam, 1981.
- [41] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.
- [42] P. Erdős and G. Szekeres. On some extremum problems in elementary geometry. *Annales Universitatis Scientiarum Budapestinensis de Rolando Etvos Nominatae Sectio Mathematica*, 3-4:53–62, 1961.

- [43] H. Everett. *Visibility graph recognition*. Ph. D. Thesis, University of Toronto, Toronto, January 1990.
- [44] H. Everett and D. G. Corneil. Recognizing visibility graphs of spiral polygons. *Journal of Algorithms*, 11:1–26, 1990.
- [45] H. Everett and D. G. Corneil. Negative results on characterizing visibility graphs. *Computational Geometry: Theory and Applications*, 5:51–63, 1995.
- [46] H. Everett, C.T. Hoang, K. Kilakos, and M. Noy. Planar segment visibility graphs. *Computational Geometry: Theory and Applications*, 16:235–243, 2000.
- [47] H. Everett, F. Hurtado, and M. Noy. Stabbing information of a simple polygon. *Discrete Applied Mathematics*, 91:67–92, 1999.
- [48] H. Everett, A. Lubiw, and J. O’Rourke. Recovery of convex hulls from external visibility graphs. In *Proceedings of the 5th Canadian Conference on Computational Geometry*, pages 309–314, 1993.
- [49] S. Felsner and M. Massow. Thickness of bar 1-visibility graphs. In *Proceedings of the 14th International Conference on Graph Drawing*, volume 4372 of *Lecture notes in Computer Science*, pages 330–342. Springer-Verlag, 2007.
- [50] G. A. Freiman. *Foundations of a structural theory of set addition*. Translations of Mathematical Monographs, vol. 37, American Mathematical Society, Providence, 1973.
- [51] T. Gerken. Empty convex hexagons in planar point sets. *Discrete & Computational Geometry*, 39:239–272, 2008.
- [52] S. K. Ghosh. Approximation algorithms for art gallery problems in polygons. In *Proceedings of the Canadian Information Processing Society Congress*, pages 429–436, 1987.
- [53] S. K. Ghosh. On recognizing and characterizing visibility graphs of simple polygons. In *Report JHU/EECS-86/14, The Johns Hopkins University, 1986. Also in the proceedings of Scandinavian Workshop on Algorithm Theory, Lecture Notes in Computer Science, Springer-Verlag*, pages 96–104, 1988.
- [54] S. K. Ghosh. On recognizing and characterizing visibility graphs of simple polygons. *Discrete & Computational Geometry*, 17:143–162, 1997.
- [55] S. K. Ghosh. *Visibility Algorithms in the Plane*. Cambridge University Press, 2007.

- [56] S. K. Ghosh. Approximation algorithms for art gallery problems in polygons. *Discrete Applied Mathematics*, 158:718–722, 2010.
- [57] S. K. Ghosh, A. Maheshwari, S. P. Pal, S. Saluja, and C. E. Veni Madhavan. Characterizing and recognizing weak visibility polygons. *Computational Geometry: Theory and Applications*, 3:213–233, 1993.
- [58] S. K. Ghosh and D. M. Mount. An output-sensitive algorithm for computing visibility graphs. *SIAM Journal on Computing*, 20:888–910, 1991.
- [59] S. K. Ghosh, T. Shermer, B. K. Bhattacharya, and P. P. Goswami. Computing the maximum clique in the visibility graph of a simple polygon. *Journal of Discrete Algorithms*, 5:524–532, 2007.
- [60] H. Harborth. Konvexe fnfecke in ebenen punktmengen. *Elemente der Mathematik*, 33:116–118, 1978.
- [61] S. G. Hartke, J. Vandenbussche, and P. Wenger. Further results on bar k-visibility graphs. *SIAM Journal on Discrete Mathematics*, 21:523–531, 2007.
- [62] A. Hernandez-Barrera, F. Hurtado, J. Urrutia, and C. Zamora. On the midpoints of a plane point set. Unpublished manuscript, 2001.
- [63] J. Hershberger. Finding the visibility graph of a polygon in time proportional to its size. *Algorithmica*, 4:141–155, 1989.
- [64] M. Hoffmann and C. Tóth. Alternating paths through disjoint line segments. *Information Processing Letters*, 87:287–294, 2003.
- [65] M. Hoffmann and C. Toth. Segment endpoint visibility graphs are hamiltonian. *Computational Geometry: Theory and Applications*, 26:47–68, 2003.
- [66] J.D. Horton. Sets with no empty 7-gons. *Canadian Mathematical Bulletin*, 26:482–484, 1983.
- [67] L. Jackson and S. K. Wismath. Orthogonal polygon reconstruction from stabbing information. *Computational Geometry: Theory and Applications*, 23:69–83, 2002.
- [68] D. Kirkpatrick and S. Wismath. Determining bar-representability for ordered weighted graphs. *Computational Geometry: Theory and Applications*, 6:99–122, 1996.
- [69] V.A. Koshelev. On the Erdős - Szekeres problem in combinatorial geometry. *Electronic Notes in Discrete Mathematics*, 29:175–177, 2007.

- [70] J. Křá, A. Pór, and D. R. Wood. On the chromatic number of the visibility graph of a set of points in the plane. *Discrete & Computational Geometry*, 34(3):497–506, 2005.
- [71] D. T. Lee. Proximity and reachability in the plane. Technical Report ACT-12 and Ph. D. Thesis, Coordinated Science Laboratory, University of Illinois, Urbana-Champaign, IL, 1978.
- [72] D. T. Lee and A. K. Lin. Computational complexity of art gallery problems. *IEEE Transactions on Information Theory*, IT-32(2):276–282, 1986.
- [73] S. Y. Lin and C. Y. Chen. Planar visibility graphs. In *Proceedings of the 6th Canadian Conference on Computational Geometry*, pages 30–35, 1994.
- [74] S. Y. Lin and S. Skiena. Complexity aspects of visibility graphs. *International Journal of Computational Geometry and Applications*, 5:289–312, 1995.
- [75] T. Lozano-Perez and M. A. Wesley. An algorithm for planning collision-free paths among polyhedral obstacles. *Communications of ACM*, 22:560–570, 1979.
- [76] F. Luccio, S. Mazzone, and C. K. Wong. A note on visibility graphs. *Discrete Mathematics*, 64:209–219, 1987.
- [77] J. Matousek. *Lecture on Discrete Geometry*, volume 212 of *Graduate Text in Mathematics*. Springer, 2002.
- [78] J. Matousek. Blocking visibility for points in general position. *Discrete & Computational Geometry*, 42:219–223, 2009.
- [79] A. Mirzaian. Hamiltonian triangulations and circumscribing polygons of disjoint line segments. *Computational Geometry: Theory and Applications*, 2:15–30, 1992.
- [80] W. Morris and V. Soltan. The Erdős-Szekeres problem on points in convex position - A survey. *Bulletin of the American Mathematical Society*, 37:437–458, 2000.
- [81] P. Muckkamala, J. Pach, and D. Sariöz. Graphs with large obstacles numbers. In *Proceedings of the 36th International Workshop on Graph Theoretic Concepts in Computer Science*, number 6410 in *Lecture Notes in Computer Science*, pages 203–215. Springer-Verlag, 2010.
- [82] P. Mutzel, T. Odenthal, and M. Scharbrodt. The thickness of graphs: A survey. *Graphs and Combinatorics*, 14:59–73, 1998.

- [83] C. M. Nicolás. The empty hexagon theorem. *Discrete & Computational Geometry*, 38:389–397, 2007.
- [84] J. O’Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, New York, 1987.
- [85] J. O’Rourke. Open problems in the combinatorics of visibility and illumination. In B. Chazelle, J. E. Goodman, and R. Pollack, editors, *Advances in Discrete and Computational Geometry*, pages 237–243. (Contemporary Mathematics) American Mathematical Society, 1998.
- [86] J. O’Rourke and J. Rippel. Two segment classes with Hamiltonian visibility graphs. *Computational Geometry: Theory and Applications*, 4:209–218, 1994.
- [87] J. O’Rourke and I. Streinu. Vertex-edge pseudo-visibility graphs: characterization and recognition. In *Proceedings of the 13th Annual ACM Symposium on Computational Geometry*, pages 119–128, 1997.
- [88] M. H. Overmars. Finding sets of points without empty convex 6-gons. *Discrete & Computational Geometry*, 29:153–158, 2003.
- [89] J. Pach. Midpoints of segments induced by a point set. *Geombinatorics*, 13:98–105, 2003.
- [90] M. S. Payne, A. Pór, P. Valtr, and D. R. Wood. On the connectivity of visibility graphs. *ArXiv e-prints*, arXiv:1106.3622, June 2011.
- [91] F. Pfender. Visibility graphs of point sets in the plane. *Discrete & Computational Geometry*, 39(1):455–459, 2008.
- [92] J. Plesník. Critical graphs of given diameter. *Acta Fac. Rerum Natur. Univ. Comenian. Math.*, 30:71–93, 1975.
- [93] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. *Discrete & Computational Geometry*, 16:419–453, 1996.
- [94] A. Pór and D. R. Wood. On visibility and blockers. *Journal of Computational Geometry*, 1:29–40, 2010.
- [95] F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society, Series 2*, 30:264–286, 1930.
- [96] D. Rappaport. Computing simple circuits from a set of line segments is NP-complete. *SIAM Journal on Computing*, 18:1128–1139, 1989.

- [97] D. Rappaport, H. Imai, and G. T. Toussaint. Computing simple circuits from a set of line segments. *Discrete & Computational Geometry*, 5:289–304, 1990.
- [98] K. Romanlk. Directed rectangle-visibility graphs have unbounded dimension. *Discrete Applied Mathematics*, 73:35–39, 1997.
- [99] P. Rosenstiehl and R. E. Tarjan. Rectilinear planar layouts and bipolar orientations of planar graphs. *Discrete & Computational Geometry*, 1:343–353, 1986.
- [100] M. Schlag, F. Luccio, P. Maestrini, D.T. Lee, and C.K. Wong. A visibility problem in VLSI layout compaction. In F.P. Preparata, editor, *Advances in Computing Research*, pages 259–282. JAI Press Inc., 1985.
- [101] L.G. Shapiro and R.M. Haralick. Decomposition of two-dimensional shape by graph-theoretic clustering. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-1:10–19, 1979.
- [102] M. Sharir and A. Schorr. On shortest paths in polyhedral spaces. *SIAM Journal on Computing*, 15:193–215, 1986.
- [103] T. Shermer. Hiding people in polygons. *Computing*, 42:109–131, 1989.
- [104] T. Shermer. Recent results in art galleries. *Proceedings of the IEEE*, 80:1384–1399, 1992.
- [105] T. Shermer. On rectangle visibility graphs, III: External visibility and complexity. In *Proceedings of the 8th Canadian Conference on Computational Geometry*, pages 234–239, 1996.
- [106] J. P. Spinrad. *Efficient Graph Representations*. American Mathematical Society, 2003.
- [107] G. Srinivasaraghavan and A. Mukhopadhyay. A new necessary condition for the vertex visibility graphs of simple polygons. *Discrete & Computational Geometry*, 12:65–82, 1994.
- [108] Y. V. Stanchescu. Planar sets containing no three collinear points and non-averaging sets of integers. *Discrete Mathematics*, 256:387–395, 2002.
- [109] I. Streinu. Non-stretchable pseudo-visibility graphs. In *Proceedings of the 11th Canadian Conference on Computational Geometry*, pages 22–25, 1999.
- [110] I. Streinu. Non-stretchable pseudo-visibility graphs. *Computational Geometry: Theory and Applications*, 31:195–206, 2005.

- [111] I. Streinu and S. Whitesides. Rectangle visibility graphs: characterization, construction, and compaction. In *Proceedings of the 20th Annual Symposium on Theoretical Aspects of Computer Science*, volume 2607 of *Lecture Notes of Computer Science*, pages 26–37. Springer-Verlag, 2003.
- [112] R. Tamassia and I. Tollis. A unified approach to visibility representations of planar graphs. *Discrete & Computational Geometry*, 1:321–341, 1986.
- [113] G. Tóth and P. Valtr. The Erdős-Szekeres theorem: upper bounds and related results. In J. E. Goodman, J. Pach, and E. Welzl, editors, *Combinatorial and Computational Geometry*, volume 52, pages 557–568. Cambridge University Press, Cambridge, 2005.
- [114] M. Urabe and M. Watanabe. On a counterexample to a conjecture of Mirzaian. *Computational Geometry: Theory and Applications*, 2:51–53, 1992.
- [115] J. Urrutia. Open problems in computational geometry. In *Proceedings of the 5th Latin American Symposium on Theoretical Informatics*, volume 2286 of *Lecture Notes in Computer Science*, pages 4–11. Springer-Verlag, 2002.
- [116] H. Warren. Lower bounds for approximation by nonlinear manifolds. *Transactions of the AMS*, 133:167–178, 1968.
- [117] E. Welzl. Constructing the visibility graph for  $n$  line segments in  $O(n^2)$  time. *Information Processing Letters*, 20:167–171, 1985.
- [118] S. K. Wismath. Characterizing bar line-of-sight graphs. In *Proceedings of the 1st ACM Symposium on Computational Geometry*, pages 147–152, 1985.