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Effect of a static phase transition on searching dynamics

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We consider a one dimensional Euclidean network which is grown using a preferential attachment. Here the jth incoming node gets attached to the ith existing node with the probability $\Pi_i \propto k_i l_{ij}^{\alpha}$, where l_{ij} is the Euclidean distance between them and k_i the degree of the ith node. This network is known to have a static phase transition point at $\alpha_c \simeq 0.5$. On this network, we employ three different searching strategies based on degrees or distances or both, where the possibility of termination of search chains is allowed. A detailed analysis shows that these strategies are significantly affected by the presence of the static critical point. The distributions of the search path lengths and the success rates are also estimated and compared for the different strategies. These distributions appear to be marginally affected by the static phase transition.

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I. INTRODUCTION

Static critical points are known to affect dynamical phemomena usually giving rise to critical slowing down, e.g., as in the relaxation of the order parameter in magnetic systems. Dynamical critical phenomena is a well studied and important topic, exploring the dynamical behaviour of systems, especially at the thermal critical point [1]. In many systems, phase transitions driven by factors other than temperature can occur, as for example the geometrical phase transition occurring in percolation. Even in magnetic systems, e.g., in the axial next nearest neighbour Ising (ANNNI) model, a phase transition occurs at zero termperature when the competing second neighbour interaction takes up a certain value. However, here a zero temperature quenching dynamics fails to carry any signature of the phase transition [2].

Recently, with the discovery of the small world effect in real networks, many theoretical models have been set up to mimic small worlds. In some of these models, interesting phase transitions have been noted, e.g., in the Watts-Strogatz model [3, 4], where one starts with nearest neighbour links only and then rewires links to long range neighbours with probability p, the small world effect is observed even as $p \to 0$. Phase transitions in models in which the linking probability is dependent on spatial and/or temporal factors have also been observed [5, 6]. There is no temperature associated with these networks. Of course, if one considers spin systems, e.g., the Ising model, on such networks, it is possible to obtain thermal phase transitions as well. Dynamical studies of such systems at both zero and non-zero temperature have shown unexpected phenomena, as for example freezing in case of the quenching dynamics of the Ising model on small world networks [7].

The idea of a small world first emerged from a real dynamical experiment made on the US population by Milgram [8] which showed that on an average there are six steps required to reach another individual. Later, a mathematical definition of small world property was proposed; by the small world property it is meant that if any two nodes in the network is separated by an average number of s steps, then $s \propto \ln(N)$, where N is the total number of nodes in the network. In some networks, even slower variation (i.e., sub-logarithmic) scaling has been observed [9].

Following Milgram's original experiment, several new experiments have been done to study the searching dynamics in real social networks [10, 11]. A considerable number of theoretical studies on searching phenomena has also been made recently [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

While the small world property is a "static" property, i.e., calculated on the basis of global knowledge and without any dynamics involved, it must be noted that it is not necessary that a navigation or search on a small world network would show the small world property, i.e., the dynamic paths s_d may not scale as $\ln(N)$. This is because searching is done using local information only. This was explicitly shown by Kleinberg [15] in a theoretical study where nodes where placed on a two dimensional Euclidean space. Each node here has connections to its nearest neighbours as well as to neighbours at a distance l with probability

$$P(l) \propto l^{-\alpha}. \quad (1)$$

Although the network is globally a small world for a range of values of α, navigation on such networks using greedy algorithm showed a small world behaviour with $s_d \sim (log(N))^2$ only at $\alpha = 2$. In general the path length s_d showed a sublinear power law increase with N.

Although searching dynamics is not comparable to dynamics of relaxation, quenching etc., still we have some indication that it bears the signature of a static phase transition from some earlier studies [15, 18]. In one dimension, the Euclidean network in which connections are made using (1) has been shown to have three phases corresponding to the scaling behaviour of the link lengths. It is a small world for for $\alpha < 1$ where the link lengths scale as N; finite dimensional network for $1 < \alpha < 2$ where the link lengths scale sublinearly with N and a regular network for $\alpha > 2$ where the link lengths are $O(1)$ [25]. Here the existence of the two phase transitions are quite confirmed and the greedy searching strategy has been shown to bear the signatures of both [18]. Correspondingly, it is expected that in two dimensions, there is a transition to the small world behaviour at $\alpha_c = 2$ and the result that the search path lengths scale uniquely at this value of α shows that the searching dynamics is indeed sensitive to it. However, this has been observed for a particular algorithm and may not be true always. In this paper, we have considered a network where different kind of algorithms can be used to check the sensitivity to a static phase transition.

We have considered realistic searches on this network. As in [24], here also the search paths may terminate. Thus there is a success rate also involved in the study. To study the effect of the parameter α which governs the phase transition in the model (details given in the next section), we have computed the path length, the success rate and the ratio of the two as well. The last quantity shows a power law variation with the network size, $N^{-\delta}$, and can be taken as a reliable measure to compare different search strategies [24]. We find out that the variation of δ with α indeed shows the signature of the phase transition to a considerable extent for the different algorithms.

We describe the model and the algorithms in section II and the results in section III. We have computed the distribution of the path lengths and success rates for all the algorithms also which are presented in sec IV. Summary and concluding remarks are made in sec V.

II. THE NETWORK AND THE STRATEGIES

We have considered a growing Euclidean network in which the nodes are added one by one using a preferential attachment such that the probability that an incoming node j gets attached to an existing node i with k_i neighbours at that time is [5]

$$\Pi_i \propto k l_{ij}^{-\alpha}, \qquad (2)$$

where l_{ij} is the distance between the ith and the jth nodes. The nodes are generated randomly at sites x_i $(0 \leq x_i \leq 1)$ on a continuous one dimensional line and each new node gets attached to m pre-existing nodes.

For $m = 1$, this network was shown to have a phase transition at α_c : below α_c it has a stretched exponential degree distrtribution while above it the degree distribution has a power law tail with the exponent γ equal to 3.0. The value of α_c was found to be close to 0.5 [5].

It is to be noted that this model, in the limit $\alpha = 0$ is nothing but the scale-free Barabasi Albert model [26]. On scale-free networks, a degree based search is natural to adopt while the greedy algorithm appears to be the most popular one from the findings of the original Milgram experiment as well as that of [10, 11] on a Euclidean network. We have therefore employed three different search algorithms here, based on the degrees of the nodes, or distances, or both.

All the search strategies follow the basic rules :
After a source node and a target node are selected randomly, the source node will send the signal to one of its neighbouring nodes provided that node has not already taken part in the search. This is repeated till the message reaches a node which is connected to the target node and this scheme is in tune with Milgram-like experiments. In course of this search, it may happen that a node cannot pass the signal to any of its neighbour as they have already taken part in the search. In that case, the search terminates at that node. Such searches are termed *unsuccessful*. The fraction of successful searches is called the success rate ρ. The average number of steps taken to reach the target in a successful search is the average dynamic path length s_d. We also calculate the quantity μ, defined as $\mu = \frac{\rho}{s_d}$.

The choice of the neighbour to whom the signal is being passed depends on the strategy. The three search strategies considered in the present work are as follows:

(1) Highest Degree Search (HDS): Here after a source and a target pair are chosen randomly the source scans its nearest neighbours and chooses the one with the highest degree to pass on the signal.

(2) Nearest neighbour search (NNS): In this strategy, after the source-target pair is chosen randomly, the source chooses from among its nearest neighbours, the one whose Euclidean distance (l) from the target is the least. It may be noted that in conventional greedy algorithms, the strategy is to pass the message to a neighbour which is *nearer* to the target than itself. In the present case, this condition has not been imposed and therefore in an intermediate stage, the message may "proceed backward". This is analogous to allowing a system to go a higher energy configuration in simulated annealing applied to the dynamics of frustrated systems like spin glasses and to combinatorial optimisation problems.

(3) Optimised Search (OS): In this strategy, we follow an algorithm where the degree (k) of a node as well as its Euclidean distance (l) from the target are taken into account. Here, after a source-target pair is picked up at random, the ratio $\xi = k/l$ is calculated for all the nearest neighbours of the source and the one with the highest value of ξ is chosen to pass on the signal.

III. RESULTS FOR ρ, s_d AND δ

We have simulated the networks with a maximum of $N = 5000$ nodes using upto 1000 configurations. For each configuration, the searching is repeated $N/2$ times with randomly chosen source-target pairs. We have considered two cases, $m = 1$ and $m = 2$.

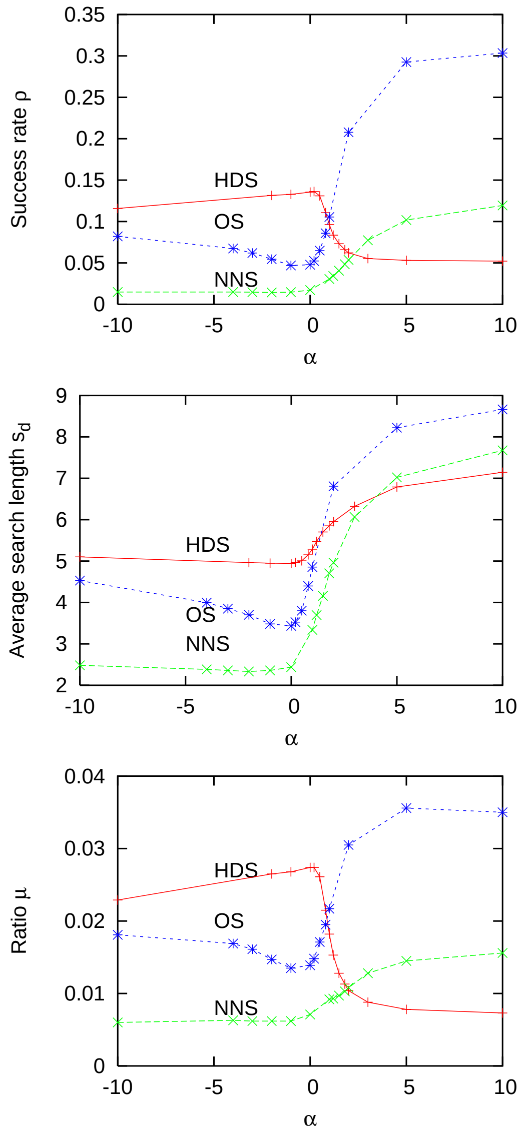

FIG. 1: Variation of ρ, s_d and μ with α for the three search strategies for $N = 1000$. The parameter α is varied from -10.0 to $+10.0$. A transitional behaviour is observed for all the strategies around the static critical point of the system, i.e., near $\alpha = \alpha_c \simeq 0.5$

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A. Case I, m = 1

Here the tunable parameter α has been varied from -10.0 to $+10.0$. Once the network is generated following eq. (2), and the navigation has been simulated following one of the three strategies described in the last section, the success rate ρ and the average search length s_d are evaluated and their variation with α and N is noted.

First we have made a comparison of the three strategies by analysing the variation of ρ, s_d and μ with α for a fixed system size N. We show in Fig. 1 these variations for $N = 1000$.

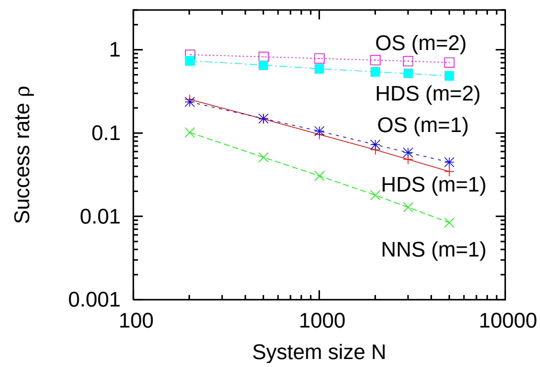

FIG. 2: Variations of ρ with system size N are shown for $\alpha = 1.0$ for the three search strategies for $m = 1$ and $m = 2$. The success rate decreases with increasing system size.

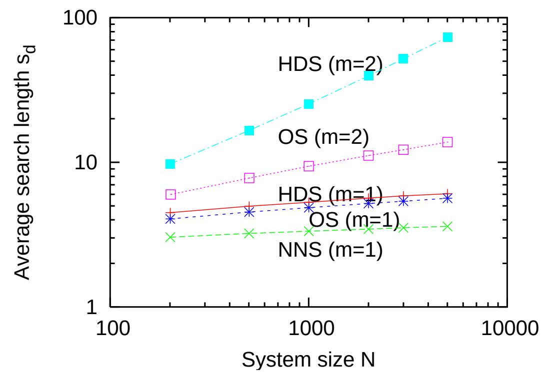

FIG. 3: Variations of the average path length s_d with system size N for the three different strategies are shown for $\alpha = 1.0$ for $m = 1$ and $m = 2$. It is observed that the average search length increases with increasing N.

For the HDS strategy, it is observed that ρ shows a slow increase as α increases from -10.0 upto 0.0 after which it falls sharply till $\alpha = 2.0$ and finally tends to saturate beyond $\alpha = 5.0$. The value of s_d for this strategy however remains constant from $\alpha = -10.0$ to $\alpha = 0.0$ and increases slowly from this value also showing a tendency to saturate at large values of α.

For the NNS strategy, ρ and s_d remain very small for $\alpha < 0$; ρ shows a gradual increase between $\alpha = 0.0$ and $\alpha = 2.0$. The values of s_d however increase quite rapidly between $\alpha = 0.0$ and $\alpha = 2.0$.

For the OS strategy, ρ has a slow decrease from $\alpha = -10.0$ upto $\alpha = 0.0$ and then it increases quite sharply between $\alpha = 0.0$ and $\alpha = 5.0$ beyond which it saturates. Similarly s_d decreases slowly between $\alpha = -10.0$ and $\alpha = 0.0$, then increases very sharply upto $\alpha = 2.0$.

A saturation of both ρ and s_d is expected for all the algorithms as the network approaches the behaviour of a growing network in which links are made to the

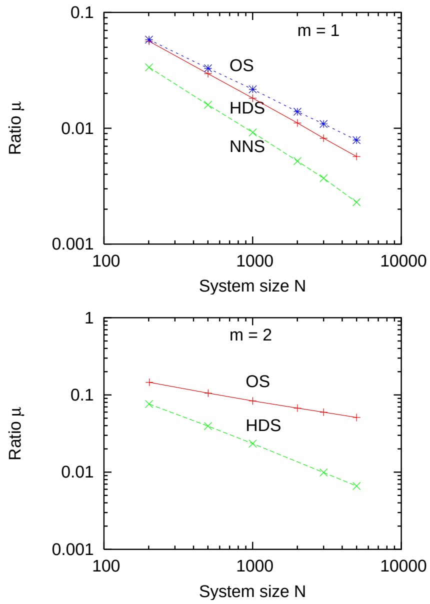

FIG. 4: The ratio $\mu = \rho/s_d$ vs N plots at $\alpha = 1.0$ for $m = 1$ and $m = 2$.

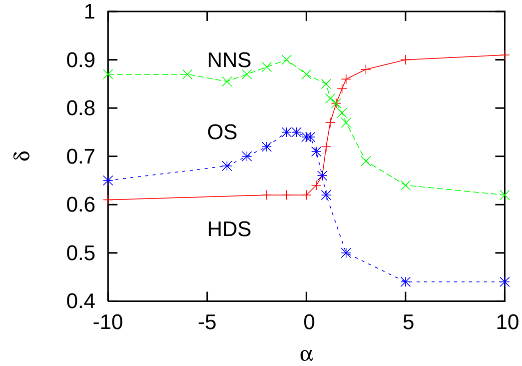

FIG. 5:

Comparison of the different search strategies showing the variation of the exponent δ with α. All the three search strategies show transitional behaviour close to the static phase transition point $\alpha \simeq 0.5$. $m = 1$ here.

nearest neighbours (i.e., the $\alpha \to \infty$ limit) for large α. Similar saturation behaviour for the static properties was observed in [5]

Looking at s_d alone, it would seem that the HDS is still the best strategy even at $\alpha >> 1$, when the network is not scale-free. However, ρ for HDS becomes very low here indicating that very few chains are completed, in which case chains tend to be 'short'. This explains the above observation for s_d. Similarly, for $\alpha < 0$, NNS would seem to be the best from the values of s_d. On the other hand, from the ρ plots, OS seems best for $\alpha >> 1$, while HDS seems best for $\alpha < 0$. As in [24], here also we compute μ, which incorporates both ρ and s_d, to comment on the relative capabilities of the three strategies. From the μ vs α plots above it is apparent that HDS works best upto $\alpha \sim 0.5$ while OS is best for $\alpha > 0.5$ for this particular value of N. NNS works rather poorly for $\alpha < 0$ and performs relatively better for $\alpha > 0$.

We find that the behaviour of μ in general closely follows that of the success rates ρ. This may indicate that rather than the path lengths, which are "small" in all cases, the success rate decides the quality of the search strategy here.

Next we discuss the behaviour of the above quantities with N.

We show typical plots of ρ and s_d against N (Figs 2,3) for a fixed value of $\alpha = 1.0$. ρ clearly shows a power law decay with N. s_d apparently has a power law increase, with a very small exponent (~ 0.01). However here one expects $s_d \sim \ln(N)$ as the network has a tree structure for $m = 1$. Indeed, we find that the exponent tends to decrease at larger N, consistent with this.

From Fig. 4, we find that μ shows a power law variation with the system size N,

$$\mu \sim N^{-\delta}, \tag{3}$$

where the value of δ varies with α.

We have computed δ for different values of α (from $\alpha = -10.0$ to $\alpha = +10.0$) and shown its variation against α for all the three strategies in Fig. 5. A smaller value of δ indicates a more successful strategy. For values of $\alpha \leq 0.5$, i.e., when the network is still scale free, the strategy purely dependent on degree works better compared to those dependent on distance. For $\alpha >> 0$ on the other hand, the distance dependent searches perform better than the purely degree dependent search. In fact, the OS appears to be the best strategy immediately beyond the static critical point while the NNS works better than the HDS only when $\alpha > 1.5$. For $\alpha < \alpha_c$ the network is scale free and there are several high degree nodes present so that the HDS strategy wins over the other two. However this strategy becomes inefficient beyond $\alpha > \alpha_c$, when the system is no longer scale free and high degree nodes are no longer available. On the other hand, in this region, distance-based search strategies work more efficiently as the network has nodes linked to closer neighbours and both the algorithms, NNS and OS are greedy algorithms as far as distances are concerned. Although for $\alpha > \alpha_c$, the OS strategy works best, the exponent δ is never very close to zero, which means that the dynamic small world effect [24] is absent here. For all the strategies however, $0 < \delta < 1$, consistent with the boundary values obtained in [24]. It is observed that for a narrow region of values of $\alpha > \alpha_c$, HDS is still better than NNS which indicates that the relevance of the degree of a node reduces gradually, once the network becomes non-scale free. The fact that the OS performs best even for very large values of α also suggests that the degree is never totally irrelevant.

For both HDS and OS, δ shows a drastic increase/decrease, indicating a sharp transition at $\alpha \simeq 0.5$, which is the static phase transition point. Unlike HDS and OS, there is no sharp change in behaviour in δ for NNS and it is affected by the static phase transition point to a lesser extent compared to the other two strategies.

NNS at $\alpha = 0$ is nothing but a random search. δ for NNS remains almost a constant for $-\infty < \alpha < 0$, showing that it is never better than a random search strategy for this region. For negative values of α, nodes at large distance are linked up, but it does not help a greedy algorithm. Incorporating k in the algorithm surely helps as OS is better than NNS here. In fact δ shows a variation with α for $\alpha < 0$ only for the OS strategy. For HDS, the plot of δ versus α is close to a perfect sigmoid, showing accountable variation only around $\alpha = \alpha_c$.

B. Case II, m = 2

As long as $m = 1$, the network cannot have any loop and the path from one node to another is unique. To introduce loops to the lowest order, we have next considered searches on networks generated using (2) once again where each incoming node can get two links ($m = 2$). It is expected that the static phase transition point remains same for $m = 2$.

With loops, the success rate should be higher but the search lengths may increase several times. In the last subsection, we found that the HDS and OS are the more effective strategies and we have used only these two in the present study. The variation of ρ, s_d and μ with N for $m = 2$ have been plotted in figures 2, 3 and 4 along with the $m = 1$ plots. All these show power law variations with N. As expected, we find a slower decay of ρ compared to that for $m = 1$ while s_d increases clearly with a power law compared to the logarithmic increase obtained for $m = 1$. The results show that δ for $\alpha = 0.0$ and $\alpha = -1$ are very close for OS and HDS; in fact for both these values of α, $\delta \simeq 0.7$ for the two strategies. This is greater than that of the $m = 1$ case. A higher value of δ indicates a deterioration in performance, the rapid increase in the path lengths being the reason behind this deterioration.

For $\alpha = 1$ on the other hand, when the network is no longer a scale-free network, δ values are drastically different for the two strategies. For OS, it is much smaller, ~ 0.3, while for HDS, it is around 0.8. The reason behind this is, with the OS strategy, the path lengths scale with a much smaller exponent in the non scale-free region (Fig. 3).

IV. DISTRIBUTIONS

In the last section, we have compared some search strategies on a growing Euclidean network by computing the quantities like success rates and search lengths, where the mean value of these quantities have been used to obtain the scaling behaviour. We have also computed

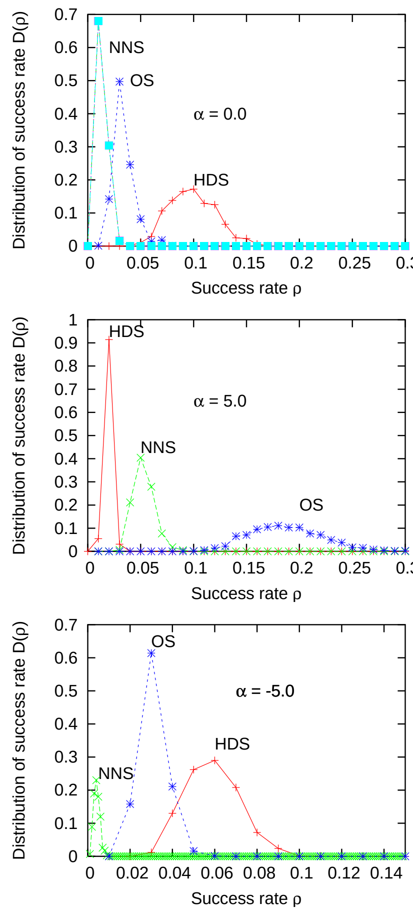

FIG. 6: Plots of the distributions of ρ for the three strategies for $\alpha = 0.0$, 5.0 and -5.0 are shown for $m = 1$. The system size used is $N = 2000$.

the distributions of these quantities to see whether the presence of the phase transition has any effect on these. The results for three values of α, $\alpha < 0$, $\alpha = 0$ and $\alpha > \alpha_c$ are reported here.

The distributions of the success rate ρ for the three strategies with $m = 1$ show the following general features (Fig. 6)

1. All of them have a well defined peak.
2. They are symmetric.
3. There is no long tail.
4. Distributions are skewed when the mean value is small, in fact very few points with non-zero value appear here. However, when the mean value is larger, there is a sufficient broadening, no matter which strategy is being used.

Overall, we do not find any indication that the static phase transition point has a significant influence on the form of the distributions. Since the data points are few, we do not attempt a fitting but in all probability these distributions are gaussian or nearly gaussian.

The distribution of the path length s_d, on the other hand, is definitely not symmetric (Fig. 7) for any of the strategies at any value of α. For $m = 1$, it has a broad peak. None of the distributions have a long tail. For

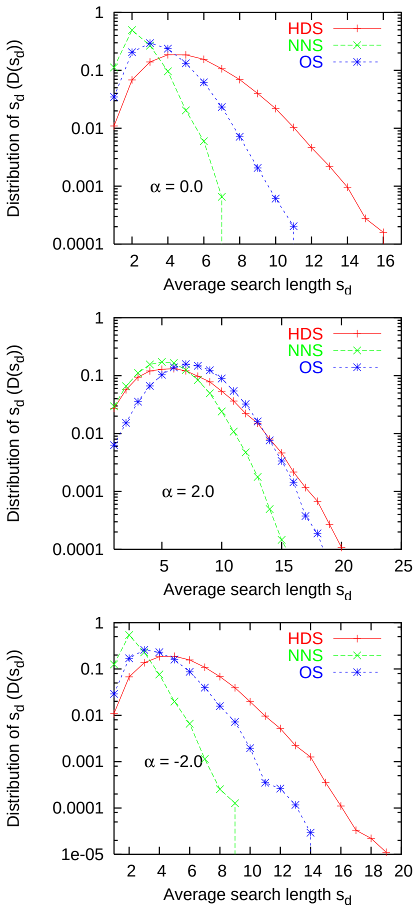

FIG. 7: The distributions of s_d for the three strategies for $\alpha = 0.0$, 2.0 and -2.0 are shown in a log-linear scale. The system size used is $N = 2000$. $m = 1$ here.

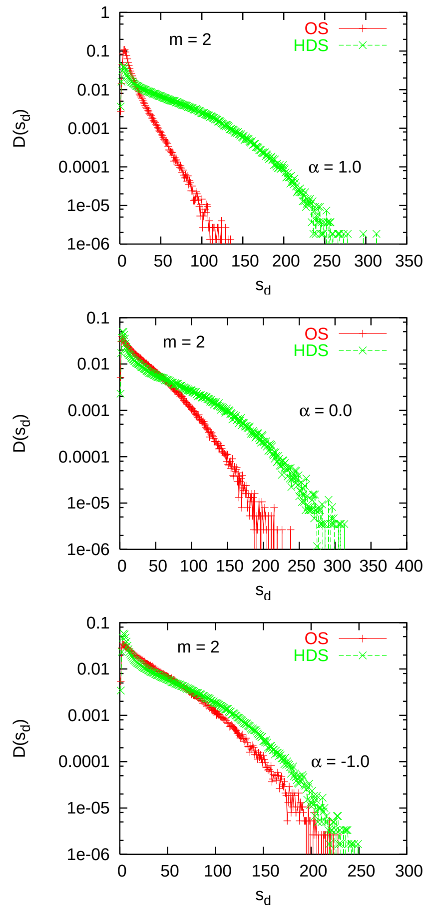

FIG. 8: The distributions of average search lengths for OS and HDS strategy at $\alpha = 1, 0, -1$ when $m = 2$ are shown. $N = 2000$ here.

$m = 2$, when the success rate becomes much higher, we find the peaks shifted towards lesser values of s_d (Fig. 8), consistent with the observation of [14]. The presence of a peak at smaller values of s_d for both $m = 1$ and $m = 2$ also show that shorter paths are more probable [11].

Again, for $m = 1$, the number of data points are few and larger fluctuations exist making it difficult to fit the data to any familiar functional form. For $m = 2$, there is a larger number of points and one can immediately see that the optimised search strategy has a clear-cut exponential decay when $\alpha > \alpha_c$ whereas for $\alpha < \alpha_c$, it has a slower than exponential decay. However, no such change in behaviour is observed for the HDS strategy, it is slower than exponential in each case.

V. SUMMARY AND CONCLUSIONS

In this work, we have applied different search strategies to a network which undergoes a phase transition from a scale-free to a non scale-free phase. One of our aims was to investigate whether such a phase transition significantly affects the search or not as in purely Euclidean networks, such an indication is there [15, 18]. We find that the search strategies indeed show a significant change in behaviour at or near the phase transition to different extents. It appears that the degree based searches are more sensitive to the phase transition.

The searching scheme used here allows termination of messages as the restriction that the message can be passed only once by any messenger has been imposed. In reality, of course, several other reasons may exist for a termination [11]. The analysis of the results therfore has been made based on an approach recently suggested by one of us [24], in which both search paths and success rates are taken into consideration.

Searching phenomena is vastly studied in social networks and the present study also uses ideas relevant to Milgram-like searches. Most social networks being non scale-free, our results for the network in its non scale-free phase is important in the context of social searches. Here the best performance is shown by the optimised search (OS) strategy, in which a node sends the signal to its neighbour having the largest value of k/l (k is its degree and l the distance from the target, see sec II). We have used here three strategies and for none of them we observe a dynamical small world effect, i.e., δ is never very close to zero. However, our list of strategies is by no means exhaustive. The OS scheme can be generalised by making the message passing rule that a node sends the signal to a neighbour with the largest value of k^a/l^b, introducing tunable parameters a and b. In this paper, we have only considered the limiting cases $a \to \infty$ (HDS), $b \to \infty$ (NNS) and $a = 1, b = 1$ (OS). It may be an interesting future study to find out whether in the $a - b$ plane, one obtains regions of dynamic small world effect.

A more detailed study for $m = 2$ (or more) can also be done for which we have presented results at some specific values of α only.

We have also estimated the distributions for the success rate and path lengths. The static phase transition seems to seriously affect only the distribution for the path lengths for the optimised strategy when $m = 2$.

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