Scale-free network on a vertical plane

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A scale-free network is grown in the Euclidean space with a global directional bias. On a vertical plane, nodes are introduced at unit rate at randomly selected points and a node is allowed to be connected only to the subset of nodes which are below it using the attachment probability: $\pi_i(t) \sim k_i(t)\ell^{\alpha}$. Our numerical results indicate that the directed scale-free network for $\alpha = 0$ belongs to a different universality class compared to the isotropic scale-free network. For $\alpha < \alpha_c$ the degree distribution is stretched exponential in general which takes a pure exponential form in the limit of $\alpha \to -\infty$. The link length distribution is calculated analytically for all values of α.

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It has been seen in many branches of Statistical Physics that a global directional bias in space has strong effect on the critical behaviour of simple models. Introduction of a preferred direction in the system reduces the degrees of freedom of the constituting elements of the system, which shrinks the configuration space available to the system compared to the undirected system. As a result a directed system is simpler and quite often tractable analytically. Examples include Directed percolation [1], Directed Sandpile Model [2], Directed River networks [3] and Directed Self-avoiding walks [4] etc.

Over the last few years it is becoming increasingly evident that highly complex structures of many social [5], biological [6, 7] or electronic communication [8, 9] networks etc. cannot be modeled by simple random graphs. For example in the well known random graphs by Erdös and Rényi, the degree distribution $P(k)$ is Poissonian (degree k of a vertex is the number of edges attached to it) [10]. In contrast, it has been observed recently that the nodal degree distributions of many networks, e.g., World-wide web [8] and the Internet [9] have power law tails: $P(k) \sim k^{-\gamma}$. Due to the absence of a characteristic value for the degrees these networks are called 'scale-free networks' (SFN) [11, 12, 13, 14]. Barabási and Albert (BA) generated scale-free graphs where a fixed number of vertices are added at each time and are linked with a linear attachment probability [11]. On the other hand some of these networks are directed networks whose links are meaningful only when there is a connection from one end to the other but not the opposite, e.g., the World-wide web [8], the phone-call graph [15] and the citation graph [16].

However, there are networks in which the nodes are geographically located in different positions on a two-dimensional Euclidean space e.g., electrical networks, Internet or even in postal and transport networks etc. The edges of the graphs representing these networks carry non-uniform weights which in most cases are either equal or proportional to the Euclidean lengths of the links. In these networks a relevant question is how to optimize the total cost of the connections e.g., electrical wires, Ethernet cables or say travel distances of postal carriers [17]. On the other hand a detailed knowledge of link length distribution is also important in the study of Internet's topological structure for designing efficient routing protocols and modeling Internet traffic. For example, Waxman model describes the Internet with exponentially decaying link length distribution [18]. Yook et. al. observed that nodes of the router level network maps of North America are distributed on a fractal set and the link length distribution is inversely proportional to the link lengths [19]. Other models of networks on Euclidean space are also studied in the literature [20, 21, 22].

In this paper we studied the effect of a global directional preference on the statistics of scale-free networks embedded in the Euclidean space. A typical link in this model must have a positive component along some preferred direction. Similar to the directed versions of well known models of Statistical Physics [1, 2, 3, 4] our spatially directed networks have different universal critical behaviour compared to their undirected counterparts.

A two-dimensional network is grown whose nodes are the points at randomly selected positions within an unit square on the vertical $x-y$ plane. To construct a network of $N+1$ nodes, let $(x_0, x_1, x_2, ..., x_N)$ and $(y_1, y_2, ..., y_N)$ be the $2N+1$ independent random variables identically and uniformly distributed within the interval $\{0, 1\}$. Let a specific set of values of the random variables $\{(x_0, 0), (x_1, y_1), (x_2, y_2), ..., (x_N, y_N)\}$ represent the coordinates of the of $N+1$ randomly distributed points. The growth of the network starts with only one node $(x_0, 0)$ on the bottom side of the unit square and then the other nodes are added one by one at unit rate according to their serial numbers $i = 1$ to N.

We assume that the global directional bias is the gravity and acts along the $-y$ direction which restricts the choice of the link: a new node can only be connected to a node positioned below this node. In practice when

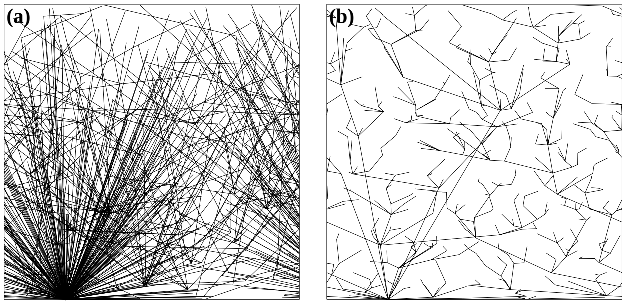

FIG. 1: Pictures of the networks generated from the same distribution of 513 points within the unit box. A large degree node is visible for a DSFN in (a) and long length early links are observed for a DMGN in (b).

the t-th node is introduced, we consider the subset S_t of the nodes situated below the t-th node. The t-th node is then connected to any node of this subset using some specific attachment rule. In addition, we assume that from each node only one link comes out but any number of links can terminate on this node. This condition ensures that the network is a singly connected tree graph. Initially the 0-th point is assigned the degree $k_0(0) = 1$. Link lengths are measured using the periodic boundary condition imposed only along the x direction because of the anisotropy. Depending on how a node from the subset S_t is selected for connection we consider the following two models:

(a) *Directed scale-free Network (DSFN):* The t-th node is randomly connected to a node i of the subset S_t using an attachment probability which is linearly proportional to the degree $k_i(t)$ of the node i at time t as: $\pi_i(t) \sim k_i(t)$.

(b) *Directed minimal growing network (DMGN):* The t-th node is connected with probability one to the nearest node in the subset S_t. Pictures of typical network configurations are shown in Fig. 1.

A continuous tuning between these two different models is possible by the choice of a suitable tunable parameter α. This is achieved by modulating the attachment probability in DSFN by a link length ℓ dependent factor like:

$$\pi_i(t) \sim k_i(t)\ell^{\alpha}. \tag{1}$$

This introduces a competition between the roles played by the degree as well as the link length on the attachment probability. The limiting extreme cases are the above two models. In the case with α=0 the link lengths do not play any role and therefore the model corresponds to $DSFN$. On the other hand when $\alpha = -\infty$ only the shortest link is selected with probability one irrespective of the degree of the node and therefore the model corresponds to $DMGN$. First we study these two limiting cases.

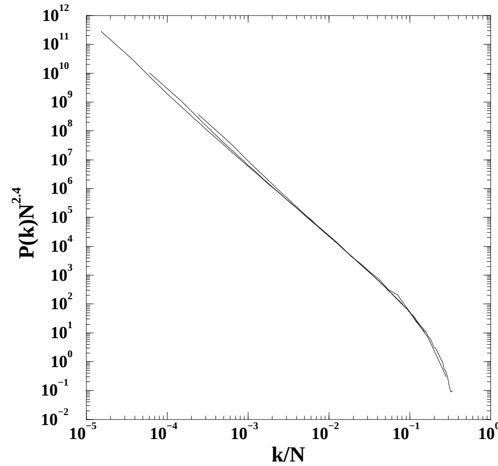

FIG. 2: The scaled degree distribution for DSFN for network sizes $N = 2^{12}, 2^{14}$ and 2^{16}. The collapse of the data at large k values imply that the degree distribution exponent $\gamma \approx 2.4$.

For a scale-free network the nodal degree distribution has a power law tail: $P(k) \sim k^{-\gamma}$ and it obeys a finite size scaling form:

$$P(k,N) \sim N^{-\eta}\mathcal{F}(k/N^{\zeta}). \tag{2}$$

We numerically find that the degree distribution of DSFN (excluding the node on the bottom line) indeed follows such a scaling form with $\eta \approx 2.4$ and $\zeta \approx 1$ (Fig. 2). This gives $\gamma_{DFSN} = \eta/\zeta \approx 2.4$. This value of γ_{DSFN} is compared with $\gamma = 3$ for the BA model of SFN [11] and therefore it seems that DSFN belongs to a new universality class different from BA SFN. On the other hand the degree distribution for the DMGN is found to decay exponentially as: $P(k) \sim exp(-\kappa k)$ with $\kappa \approx 0.74$.

For a tree graph, the branch size distribution is very important and the associated exponent may be used to characterize the graph. On a tree structure, each edge connects two branches of the tree. If an edge is selected randomly, the probability Prob(s) that any one of the two branches supports s nodes also decays with a power law tail, Prob(s) $\sim s^{-\tau}$ and follows a scaling form:

$$\text{Prob}(s) \sim N^{-\eta_b}\mathcal{G}(s/N^{\zeta_b}). \tag{3}$$

For DSFN, we obtain $\eta_b \approx 2.15$ and $\zeta_b \approx 1$ which implies that $\tau_{DSFN} \approx 2.15$ compared to its exact value 2 for the BA scale-free network [23]. On the other hand for DMGN we find $\eta_b \approx 2$ and $\zeta_b \approx 1$ so that $\tau_{DMGN} \approx 2$.

The probability density distribution $D(\ell)$ gives the probability $D(\ell)d\ell$ that an arbitrarily selected link has a length between ℓ and $\ell + d\ell$. For the undirected scale-free Euclidean networks we saw that $D(\ell)$ has a power law variation $D(\ell) \sim \ell^{\delta}$ [19, 22]. $D(\ell)$ can be calculated exactly for both DSFN as well as DMGN in the following way. Let us try to assign a link to the $(n+1)$-th node

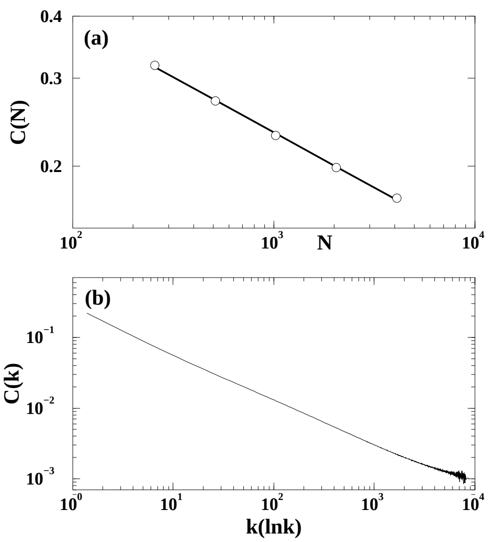

FIG. 3: Variations of the average clustering co-efficients for DSFN: (a) Over the whole network, $C(N) \sim N^{-\beta_N}$ and (b) Over the subset of nodes having degree k only $C(k) \sim [k(\ln k)]^{-\beta_k}$. Our estimates are $\beta_N \approx 0.23$ and $\beta_k \approx 0.64$.

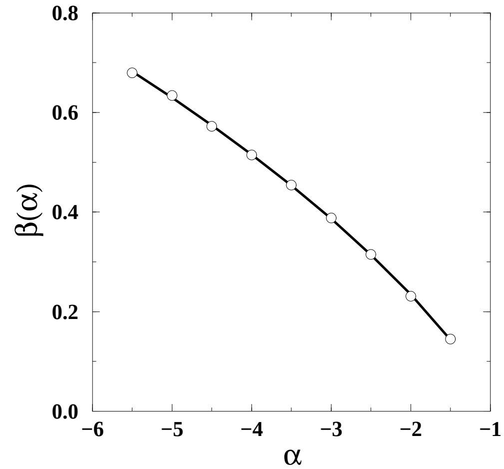

FIG. 4: Variation of the exponent $\beta(\alpha)$ characterizing the stretched exponential degree distribution. The continuous curve is a fit to the data points (circles) obtained by simulation to a form: $\beta(\alpha) = a(-\alpha^{\nu}) - b$ such that β is extrapolated to zero at $\alpha_c \approx -0.85$.

and denote $y = y_{n+1}$. Let n_1 points be positioned below the y level and $n_2 = n - n_1$ points be above this level.

We first calculate $D_{DMGN}(\ell)$. The probability that out of n_1 nodes the node which is nearest to the $(n+1)$-th node is positioned at a distance between ℓ and $\ell + d\ell$ has two distinct contributions: One from the case of all $y > \ell$ and the other for all $y < \ell$ (due to the presence of the boundary at $y = 0$). For the first case, the probability of the $(n+1)$-th point being at a particular height y is given by

$$\begin{aligned} & \pi\ell d\ell \Sigma_{n_1=0}^{n} \{^nC_{n_1}\} n_1 [y - \pi\ell^2/2]^{n_1-1}(1-y)^{n_2} \\ = \ & \pi\ell d\ell (n) [1 - \pi\ell^2/2]^{n-1}. \end{aligned}$$

Here the weight factor from each partitioning of n into n_1 and n_2 has been taken into account. The above probability after integration over y from ℓ to 1 and summed over all n from 0 to ∞ gives the net contribution to $D(\ell)$ for all $y > \ell$:

$$\begin{aligned} D_{DMGN}(y > \ell) &= \pi\ell[1-\ell]\Sigma_{n=0}^{\infty} n[1 - \pi\ell^2/2]^{n-1} \\ &= \frac{4}{\pi\ell^3}(1-\ell). \end{aligned}$$

Similarly the probability that the $(n+1)$-th point is at a specific height $y < \ell$ is

$$\begin{aligned} & 2\ell sin^{-1}\frac{y}{\ell} d\ell \Sigma_{n_1=0}^{n} \{^nC_{n_1}\}(n_1) \\ & \times [y - (\ell^2 sin^{-1}\frac{y}{\ell} + y\sqrt{\ell^2 - y^2})]^{n_1-1}(1-y)^{n_2} \\ = \ & 2\ell sin^{-1}\frac{y}{\ell} n [1 - (\ell^2 sin^{-1}\frac{y}{\ell} + y\sqrt{\ell^2 - y^2})]^{n-1} d\ell. \end{aligned}$$

As before a similar sum over n and integration over y from $y = 0$ to $y = \ell$ in the above expression gives the following contribution to $D(\ell)$ for all $y < \ell$:

$$\begin{aligned} D_{DMGN}(y < \ell) &= \frac{A}{\ell^2} \\ \text{where} \quad A &= \int_0^1 \frac{2sin^{-1}z}{sin^{-1}z + z\sqrt{1-z^2}} dz. \end{aligned}$$

Hence the total distribution is given by:

$$D_{DMGN}(\ell) = \frac{4}{\pi\ell^3}(1 - (1 - \frac{\pi}{4}A)\ell). \tag{4}$$

For the DSFN also one has:

$$\begin{aligned} D_{DSFN}(y > \ell) &= C\int_{\ell}^{1} \frac{\pi\ell}{y} dy \\ \text{and} \quad D_{DSFN}(y < \ell) &= C\int_0^{\ell} \frac{2sin^{-1}(y/\ell)}{y} dy \end{aligned}$$

where C is a constant. The total distribution is therefore given by

$$\begin{aligned} D_{DSFN}(\ell) &= C(B\ell - \ell\ln(\ell)) \\ \text{where} \quad B &= 2\int_0^1 \frac{sin^{-1}z}{z} dz. \end{aligned} \tag{5}$$

From our numerical calculations we estimate $C \approx 1.59$ and $B \approx 1.86$. For any non-zero α, the corresponding distribution can be obtained by simply multiplying the above expression by ℓ^{α}.

The clustering co-efficient $C(N)$ of a network of N nodes measures the the local correlations among the links of the network. More precisely it measures the probability that two neighbours of an arbitrary node are also neighbours. If the i-th node has the degree k_i and there are e_i links among the k_i neighbours of i then the clustering co-efficient of the site i is: $C_i = 2e_i/[k_i(k_i-1)]$ whereas the clustering co-efficient of the whole network is: $C(N) = \langle C_i \rangle$. For a number of networks it has been observed that the clustering co-efficient decreases with N like $C(N) \sim N^{-\beta_N}$ as the network size N increases. Also one can define a clustering co-efficient $C(k)$ averaged over the subset of nodes of degree k on the network. It has been also observed that $C(k) \sim k^{-\beta_k}$ for some networks. We estimated these exponents for DSFN and found that $\beta_N \approx 0.23$ whereas $C(k)$ has a logarithmic modulation like $C(k) \sim [k(lnk)]^{-\beta_k}$ with $\beta_k \approx 0.64$ (Fig. 3).

Finally we study the variation of the degree distribution $P(k)$ with the parameter α. For finite negative values of α the distribution fits very well to a stretched exponential form: $P(k) \sim \exp(-ck^{\beta(\alpha)})$ where $\beta(\alpha)$ is expected to reach to one as $\alpha \to -\infty$ and to zero as $\alpha \to \alpha_c$. Our numerical estimates for β have been plotted in Fig. 4 with α and this data fits very well to form $\beta(\alpha) = a(-\alpha)^{\nu} - b$ where the constants are estimated to be $a \approx 0.47$, $\nu \approx 0.51$ and $b \approx 0.43$. This implies that the stretched exponential form continues to be valid till $\alpha = \alpha_c$ where $\beta = 0$ and beyond that the degree distribution is a power law. Though from the values of a, ν and b, α_c is estimated to be -0.85 we believe $\alpha_c = -1$ is more plausible. Also our numerical results indicate that for all $\alpha > \alpha_c$ the degree distribution exponent γ maintains its value of $\alpha = 0$.

To summarize, we studied the directed version of the Barabasi-Albert scale-free network grown on a two-dimensional vertical plane. Our numerical results on degree as well as branch size distributions indicate that this network belongs to a new universality class compared to its undirected version. A competition between the degree k of the nodes and a link length dependent factor ℓ^{α} in the attachment probability is seen to control the network behaviour. In the limit $\alpha \to -\infty$ one gets the directed minimally growing network with exponentially decaying degree distribution. However for finite negative values of α stretched exponential distributions are observed. The link length distribution is calculated analytically for all values of α.

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