

Reparametrizing the Polyakov–Nambu–Jona-Lasinio model

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The Polyakov–Nambu–Jona-Lasinio model has been quite successful in describing various qualitative features of observables for strongly interacting matter, that are measurable in heavy-ion collision experiments. The question still remains on the quantitative uncertainties in the model results. Such an estimation is possible only by contrasting these results with those obtained from first principles using the lattice QCD framework. Recently a variety of lattice QCD data were reported in the realistic continuum limit. Here we make a first attempt at reparametrizing the model so as to reproduce these lattice data. We find excellent quantitative agreement for the equation of state. Certain discrepancies in the charge and strangeness susceptibilities as well as baryon-charge correlation still remain. We discuss their causes and outline possible directions to remove them.

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I. INTRODUCTION

Thermodynamic properties of strongly interacting matter under extreme conditions is being actively studied theoretically as well as experimentally. A first principle approach is provided by the finite temperature formulation of quantum chromodynamics (QCD) on a space-time lattice. For light quarks these studies [1–11] indicate the possibility of a rapid crossover between the color confined and deconfined states. The chiral symmetry is also broken/restored spontaneously along with the confinement/deconfinement transition. For the physical case of two light quarks and a heavy strange quark, lattice QCD simulations for zero net conserved charges find this cross-over temperature to be in the range $150 \text{ MeV} < T_c < 160 \text{ MeV}$ as reported by the Hot-QCD [12, 13] and Wuppertal-Budapest [14] collaborations. A cross-over transition does not leave a singular boundary between two different phases. Nevertheless, near T_c various thermodynamic quantities exhibit a rapid change. Fluctuations of conserved charges are prominent quantities in this regard [15–17]. Lattice simulation results undoubtedly serve as a benchmark estimate over a large temperature window [18].

At the same time, it is also important to properly explore the QCD phase diagram to get a flavor of the physics at varying regimes of temperature and chemical potential. In fact an exciting question that has puzzled the community is whether there is any phase transition at non-zero baryon densities for strongly interacting matter. An interesting possibility associated with this issue is the existence of a critical end point somewhere on the phase diagram. Unfortunately in lattice QCD framework certain technical difficulties arise at the non-zero baryon chemical potentials. Various intelligent techniques exist to circumvent these difficulties to some extent [3, 6, 7, 19–26].

In this context various QCD inspired models are found to be useful in describing the aspects of strongly interacting matter at arbitrary temperature and chemical potentials. In the present article the various thermodynamic properties of strongly interacting matter are investigated within the framework of Polyakov loop enhanced Nambu–Jona-Lasinio (PNJL) model. One of the two key ingredients in the PNJL model is the Nambu–Jona-Lasinio (NJL) model [27–33]. This model includes the global symmetries of QCD in the fermionic sector, like the chiral symmetry, baryon number,

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electric charge, strange number symmetries etc. The multi-quark interactions in this model are responsible for the dynamical generation of mass, leading to spontaneous breaking of chiral symmetry. However, the gluon fields being integrated out, this model does not have an adequate mechanism of confinement, especially for non-zero temperatures. To this end the PNJL model [34–36] gives a sense of confinement by introduction of a temporal background gluon field along with its self interactions mimicking the pure glue effects. Thus by construction both chiral and deconfinement transitions are entwined within a single framework.

Interestingly a reasonable parametrization of the PNJL model could be achieved to obtain qualitatively similar results as in lattice QCD framework almost a decade ago [36–43]. Since then several studies were done to analyze the properties of this model as well as to improve the model step by step. Improvements in the model for inclusion of eight-quark interactions in the NJL part [44–47] necessary in order to stabilize the ground state, were introduced in [48, 50, 53]. In ref. [54], the first case study of the phase diagram in β -equilibrium has been reported using the PNJL model. In a recent work [55] the SU(3) color singlet ensemble of a quark-gluon gas has been shown to exhibit a $Z(3)$ symmetry and within stationary point approximation it becomes equivalent to the Polyakov loop ensemble. In ref. [56] it was shown that though in general a small amount of mass difference between the two light quarks does not affect the thermodynamics of the system much, it might have a significant effect on baryon-isospin correlations. Studies of various thermodynamic quantities and fluctuation and correlations of conserved charges incorporating finite volume effects have been reported in ref. [57, 58]. Also the first model study of the net charge fluctuations in terms of D-measure from the PNJL model [59] has been reported. In an interesting exercise, the validity of the fluctuation-dissipation theorem has been discussed in the context of the PNJL model [60]. As we know, viscous effects play pivotal role in the evolution of the hot and dense system. Study of these effects in terms of transport coefficients have been done in the NJL and PNJL model [61–68] and compared with hadron resonance gas studies [69–72]. In ref. [73–85], the authors have discussed behavioral pattern of different observables as extracted from the PNJL model. The QCD phase structure has also been investigated for imaginary chemical potentials in ref. [86–88] under PNJL model framework. Different interesting features of the Polyakov loop have instigated the development of different formalisms of the PNJL model [89–92]. Effects of consideration of gluon Polyakov loop have been discussed in ref. [93–95]. An important set of work being carried out recently are the improvements of the Polyakov loop potential by introducing the effects of back-reaction of the quarks, that are supposed to give a more realistic systematics of full QCD [96, 97]. We shall however restrict ourselves to the simplistic pure glue form of the Polyakov loop potential with the quark back-reaction essentially coming through the changes of the mean fields and model parameters.

Given that one of the most important application of the PNJL model would be to predict observables for non-zero baryon densities, it is important to at least reproduce observables for zero baryon densities where first principle results from lattice QCD are available. The qualitative agreement of results in the PNJL model with those available from lattice QCD has so far been quite satisfactory. The agreement seemed to be more convincing once the temperature dependent observables were plotted against T/T_c , where T_c in the model was not equal to that obtained on the lattice. However the lattice data used for these studies were at finite lattice spacings. Recently continuum extrapolations for a number of observables have been reported from lattice simulations. Therefore it is high time that one tries to set model parameters such as to reproduce the quantitative agreement of observables with the lattice results. In the present work we attempt to reset the PNJL model parameters to reproduce the T_c as well as the temperature dependence of pressure as obtained in the continuum limit of lattice QCD. Various other thermodynamic observables may then be obtained from appropriate derivatives of pressure and contrasted against the lattice QCD results. The parameters we shall modify are the ones for the Polyakov loop potential as the parameters of the NJL model are fixed at zero temperature and densities.

We organize the manuscript as follows. In the next section we describe the PNJL model focusing on the construction of the effective potential and the constraints on various parameters. In section III we detail the parameter fixing procedure. Thereafter we present some thermodynamic quantities in section IV and discuss the fluctuations and correlations of conserved charges in section V. In the final section we summarize and conclude.

II. PNJL MODEL

PNJL model was initialized with a Polyakov loop effective potential being added to the NJL model [34–36]. While the chiral properties are taken care of by the NJL part, the Polyakov loop explains the deconfinement physics. Extensive studies have been carried out using PNJL model with 2 and 2+1 flavors [36–38, 40, 48, 98–102]. Here, we consider 2+1 flavor PNJL model taking up to six and eight quark interaction terms as in [48, 98]. The thermodynamic potential in this case reads as,

$$\begin{aligned}
\Omega(\Phi, \bar{\Phi}, \sigma_f, T, \mu) &= 2g_S \sum_{f=u,d,s} \sigma_f^2 - \frac{g_D}{2} \sigma_u \sigma_d \sigma_s + 3\frac{g_1}{2} (\sum_f \sigma_f^2)^2 + 3g_2 \sum_f \sigma_f^4 - 6 \sum_f \int_0^\infty \frac{d^3p}{(2\pi)^3} E_f \Theta(\Lambda - |\vec{p}|) \\
&- 2T \sum_f \int_0^\infty \frac{d^3p}{(2\pi)^3} \ln \left[1 + 3 \left(\Phi + \bar{\Phi} e^{-(E_f - \mu_f)/T} \right) e^{-(E_f - \mu_f)/T} + e^{-3(E_f - \mu_f)/T} \right] \\
&- 2T \sum_f \int_0^\infty \frac{d^3p}{(2\pi)^3} \ln \left[1 + 3 \left(\bar{\Phi} + \Phi e^{-(E_f + \mu_f)/T} \right) e^{-(E_f + \mu_f)/T} + e^{-3(E_f + \mu_f)/T} \right] \\
&+ \mathcal{U}'(\Phi, \bar{\Phi}, T)
\end{aligned} \tag{1}$$

The fields $\sigma_f = \langle \bar{\psi}_f \psi_f \rangle$ correspond to the two light flavor ($f = u, d$) condensates and the strange ($f = s$) quark condensate respectively. There is a four quark coupling term with coefficient g_S , a six quark coupling term breaking the axial U(1) symmetry explicitly with a coefficient g_D , and eight quark coupling terms with coefficients g_1 and g_2 necessary to sustain a stable minima in the NJL Lagrangian. The corresponding quasiparticle energy for a given flavor f is $E_f = \sqrt{p^2 + M_f^2}$, with the dynamically generated constituent quark masses given by,

$$M_f = m_f - 2g_S \sigma_f + \frac{g_D}{2} \sigma_{f+1} \sigma_{f+2} - 2g_1 \sigma_f (\sigma_u^2 + \sigma_d^2 + \sigma_s^2) - 4g_2 \sigma_f^3 \tag{2}$$

In the above, if $\sigma_f = \sigma_u$, then $\sigma_{f+1} = \sigma_d$ and $\sigma_{f+2} = \sigma_s$, and so on in a clockwise manner. The finite range integral gives the zero point energy. The different parameters as obtained from [48] are given in Table I.

Interaction	m_u (MeV)	m_s (MeV)	Λ (MeV)	$g_S \Lambda^2$	$g_D \Lambda^5$	$g_1 \times 10^{-21}$ (MeV ⁻⁸)	$g_2 \times 10^{-22}$ (MeV ⁻⁸)
6-quark	5.5	134.758	631.357	3.664	74.636	0.0	0.0
8-quark	5.5	183.468	637.720	2.914	75.968	2.193	-5.890

TABLE I: Parameters in the NJL model

The finite temperature and chemical potential contributions of the constituent quarks are given by the next two terms. Note that these are basically coming from the fermion determinant in the NJL model modified due to the presence of the fields corresponding to the traces of Polyakov loop and its conjugate given by $\Phi = \frac{Tr_\tau L}{N_c}$ and $\bar{\Phi} = \frac{Tr_\tau L^\dagger}{N_c}$ respectively. Here $L(\vec{x}) = \mathcal{P}exp \left[i \int_0^{1/T} d\tau A_4(\vec{x}, \tau) \right]$ is the Polyakov loop, and A_4 is the temporal component of background gluon field.

The effective potential that describes the self interaction of the Φ and $\bar{\Phi}$ fields are given by \mathcal{U}' . Various forms of the potential exist in the literature (see e.g. [39, 40, 75, 103, 104]). We shall use the form prescribed in [40] which reads as,

$$\frac{\mathcal{U}'(\Phi, \bar{\Phi}, T)}{T^4} = \frac{\mathcal{U}(\Phi, \bar{\Phi}, T)}{T^4} - \kappa \ln[J(\Phi, \bar{\Phi})]. \tag{3}$$

Here $\mathcal{U}(\Phi, \bar{\Phi}, T)$ is a Landau-Ginzburg type potential commensurate with the global Z(3) symmetry of the Polyakov loop [36]. $J(\Phi, \bar{\Phi})$ is the Jacobian of transformation from the Polyakov loop to its traces, and κ is a dimensionless parameter which is determined phenomenologically. The effective potential is chosen to be of the form,

$$\frac{\mathcal{U}(\Phi, \bar{\Phi}, T)}{T^4} = -\frac{b_2(T)}{2} \bar{\Phi} \Phi - \frac{b_3}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4} (\bar{\Phi} \Phi)^2 \tag{4}$$

The coefficient $b_2(T)$ is chosen to have a temperature dependence of the form,

$$b_2(T) = a_0 + a_1 \exp\left(-a_2 \frac{T}{T_0}\right) \frac{T_0}{T}, \tag{5}$$

and b_3 and b_4 are chosen to be constants. In the next section we discuss the methodology for fixing these parameters.

III. FIXING POLYAKOV LOOP POTENTIAL PARAMETERS

The Polyakov loop fields are expected to approach unity for large temperatures. Therefore for an effective model of pure glue theory the minimization of the potential would be obtained for $\lim_{T \rightarrow \infty} \Phi = 1$. Also the pressure should be that of the massless free gluon gas. Using these two conditions one may obtain b_3 and b_4 in terms of $b_2(T \rightarrow \infty) = a_0$. The parameters a_1 , a_2 and T_0 and κ may thereafter be fixed phenomenologically by requiring that the crossover temperature comes around $T_c \sim 160\text{MeV}$, along with the pressure to agree with the lattice QCD results for various temperatures.

Interaction	T_0 (MeV)	a_0	a_1	a_2	b_3	b_4	κ
6-quark	175	6.75	-9.0	0.25	0.805	7.555	0.1
8-quark	175	6.75	-9.8	0.26	0.805	7.555	0.1

TABLE II: Parameters for the Polyakov loop potential.

We first fixed the parameter values of a_0 , T_0 and κ . Then b_3 and b_4 were obtained in terms of a_0 . Thereafter a_1 and a_2 were adjusted to get the best combination for the crossover temperature T_c and the pressure vs temperature plot to agree with continuum limit obtained from lattice QCD computations. The set of parameters thus obtained is given in Table II.

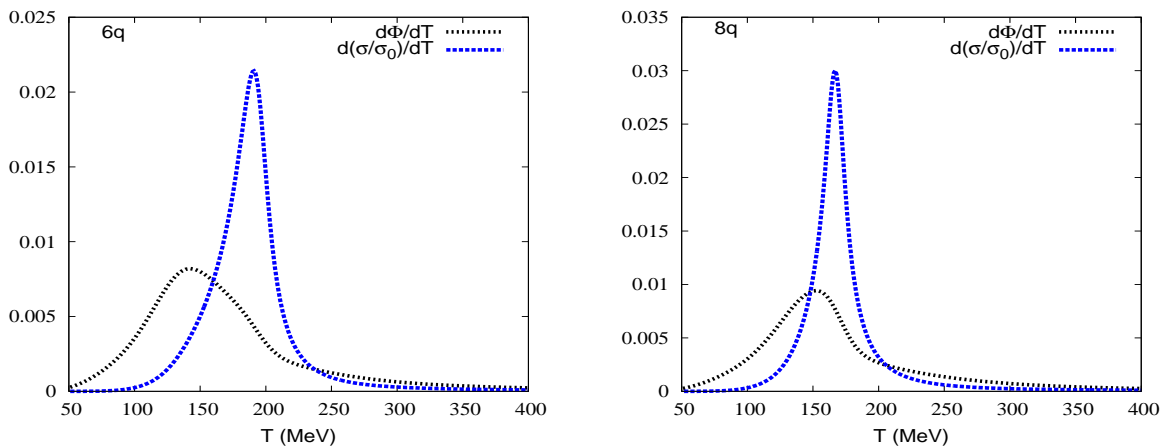


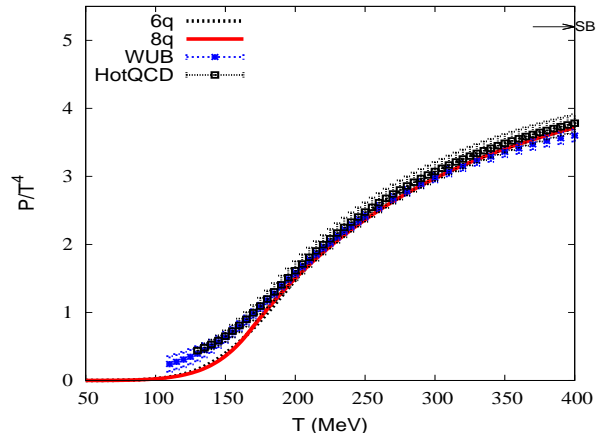
FIG. 1: (color online) The temperature derivatives of σ_f for light flavors and Polyakov loop fields for 6-quark (left) and 8-quark (right) interactions. σ_0 denotes the value of σ_f at $T = 0$.

The deconfinement temperature obtained in lattice QCD with physical quark masses from the fluctuation of the Polyakov loop is much higher than the chiral transition temperature [8, 10]. However the deconfinement temperature as measured from the peak of the entropy of a static quark is found to be consistent with the chiral transition temperature [105]. In our model framework we consider the temperature derivatives of the mean fields and locate their peaks to obtain the transition temperature. The temperature derivative of the Polyakov loop is closely related to the definition of temperature derivative of the static quark free energy that gives its entropy as defined in [105]. The plots for $d\sigma_f/dT$ for light flavors and $d\Phi/dT$ are shown in Fig. 1. The corresponding T_c was obtained from the average of the two peak positions. It was observed that the modification of parameter values did not produce any appreciable reduction of the T_c from what we have obtained. This means that with only the adjustments of parameters of the Polyakov loop potential T_c cannot be reduced further. In fact there is a drastic reduction in T_c for 6-quark interactions here compared to our earlier parametrization reported in [48]. The reduction is quite small for the 8-quark interaction. The resulting values of T_c are listed in Table III.

In Fig. 2 we show scaled pressure, as a function of temperature. The scaled pressure grows from close to zero at small temperatures and reaches almost 75% of the Stefan-Boltzmann (SB) limit commensurate with present day continuum lattice data [13, 14]. This is in sharp contrast to the earlier results in which the scaled pressure was shown to grow to almost 90% of the SB limit [48], commensurate with finite lattice spacing data available at that time [49]. Thus by refixing the parameters of the Polyakov loop potential we have been able to achieve both a crossover temperature of $T_c \sim 160$ MeV as well as quantitative agreement of temperature variation of pressure with the lattice QCD continuum estimation. For temperatures below T_c the model results do differ slightly from the lattice data.

Interaction	Peak position of $d\Phi/dT$ (MeV)	Peak position of $d\sigma/dT$ (MeV)	T_c (MeV)
6-quark	142	191	166.5
8-quark	158	167	162.5

TABLE III: Location of crossover temperature

FIG. 2: (color online) Variation of pressure scaled with T^4 as function of temperature. The continuum extrapolated dataset of HotQCD [13] and Wuppertal-Budapest (WUB) [14] collaborations are shown.

We note that though the lattice data by the Hot-QCD and Wuppertal-Budapest group agree within error bars for the lower values of temperature there is about a standard deviation of difference for the higher temperature ranges. We simply adjusted the parameters so that in the PNJL model the pressure goes through values from one of them chosen randomly - in this case the Hot-QCD data. We also note that there is almost no difference between pressure vs temperature plot of the 6-quark and 8-quark interaction versions of the PNJL model by construction.

IV. THERMODYNAMICS

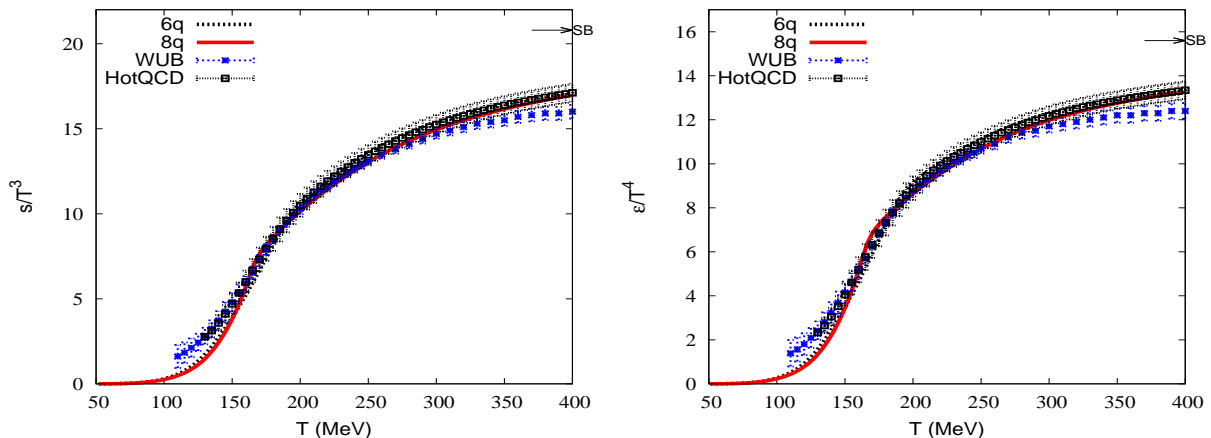


FIG. 3: (color online) The scaled entropy (left) and scaled energy density (right) as functions of temperature. The continuum extrapolated dataset of HotQCD and Wuppertal-Budapest (WUB) collaborations are taken respectively from [13] and [14].

The various thermodynamic quantities can now be obtained from corresponding derivatives of pressure that arise from the respective thermodynamic relations. From the first order derivative of pressure with respect to temperature, one can obtain the entropy density $s = \frac{\partial P}{\partial T}$ and energy density $\epsilon = T^2 \frac{\partial(P/T)}{\partial T} = T \frac{\partial P}{\partial T} - P$. These are plotted in Fig. 3.

They are also contrasted with recent Hot-QCD and Wuppertal-Budapest continuum results. We find that the results of PNJL model satisfactorily reproduce lattice data quantitatively. Here again the difference between the two sets of lattice QCD data at high temperatures are evident, and our results align well with the Hot-QCD data by construction. For $T < T_c$ the PNJL results deviate from lattice QCD data by a small amount similar to that observed for pressure.

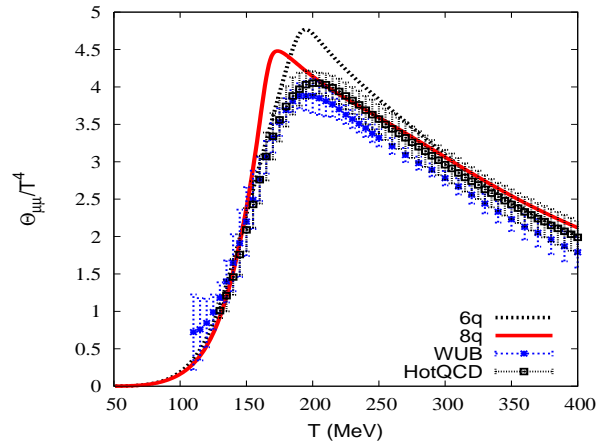


FIG. 4: (color online) Trace of energy momentum tensor as a function of temperature. The continuum extrapolated dataset of HotQCD and Wuppertal-Budapest (WUB) collaborations are taken respectively from [13] and [14].

Given that the equation of state in the PNJL model agrees well with the lattice data we now consider other observables. The energy-momentum tensor $\Theta_{\mu\mu} = \epsilon - 3P$ obtained in the PNJL model has a small difference with the lattice data near T_c as shown in Fig. 4. In fact there is a similar small difference between the 6-quark and 8-quark versions of the PNJL model. But the overall agreement over the full range of temperature is quite satisfactory. Comparing to earlier estimates based on finite lattice spacings it may be noted that the quantitative value of the height of the peak here has reduced to almost half of what was reported in [48].

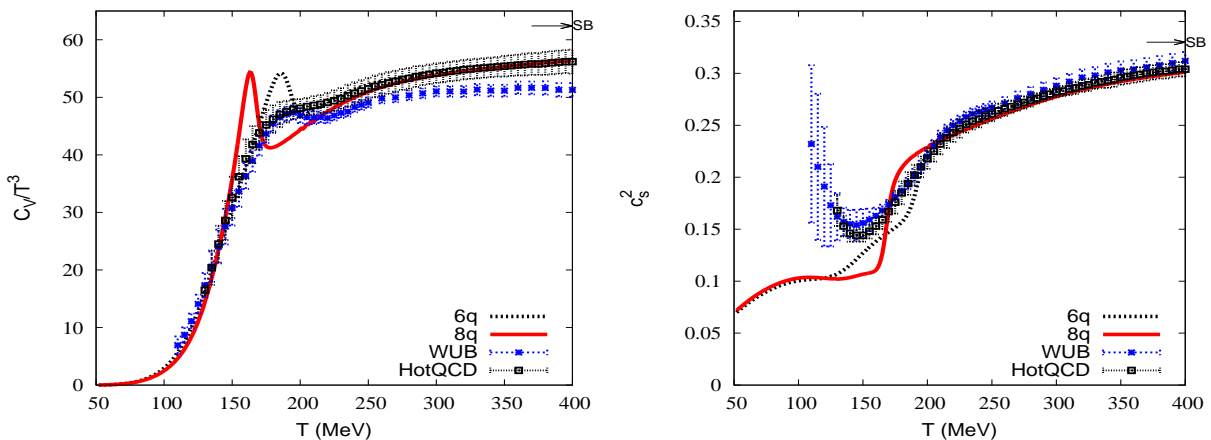


FIG. 5: (color online) The specific heat (left) and squared speed of sound (right) as functions of temperature. The continuum extrapolated dataset of HotQCD and Wuppertal-Budapest (WUB) collaborations are taken respectively from [13] and [14].

From the second order derivative of pressure with respect to temperature we obtain two important quantities namely the specific heat at constant volume $C_V = \frac{\partial \epsilon}{\partial T} = T \frac{\partial^2 P}{\partial T^2}$, and the squared speed of sound $c_s^2 = \frac{\partial P}{\partial \epsilon} = \frac{s}{C_V}$. These are shown in Fig. 5. We find the specific heat obtained in PNJL model to agree well with the lattice QCD results except near the crossover region. In this region, C_V/T^3 obtained from the PNJL model shows a small peak, but the lattice results are completely smooth. Though the lattice results do not show any peak there is a definite indication of a hump near the critical region. The differences between $\Theta_{\mu\mu}$ and C_V obtained in the PNJL model and those on the lattice indicate that the crossover in the model is somewhat sharper than that on the lattice. However the size of the peak obtained here is substantially reduced compared to what was obtained with the earlier parametrizations [50], and remains below the SB limit..

The temperature variation of the speed of sound is shown in Fig. 5. One expects that at very low temperatures the speed of sound would be small as the pressure of the system is negligible and hadrons are massive. With increase in temperature the speed of sound will increase. However with increasing temperature the hadron resonances with higher and higher masses would be excited and the speed of sound would not reach the SB limit. In fact it may even start decreasing with temperature [51]. After the crossover the degrees of freedom change from hadronic to partonic and therefore speed of sound may again increase. The minimum of the speed of sound known as the softest point may be a crucial indicator of the transition to be observed in heavy-ion collisions [52]. Such a minimum in the temperature variation of speed of sound is visible in the lattice QCD data as shown in Fig. 5, but is clearly absent in the PNJL model results. We note that the PNJL model results are consistent with the lattice data above T_c . The disagreement ensues in the phase where hadronic degrees of freedom are dominant. The PNJL model in the present form do not encapsulate the hadronic excitations effectively which has resulted in this discrepancy. We shall address proper extensions of the model elsewhere.

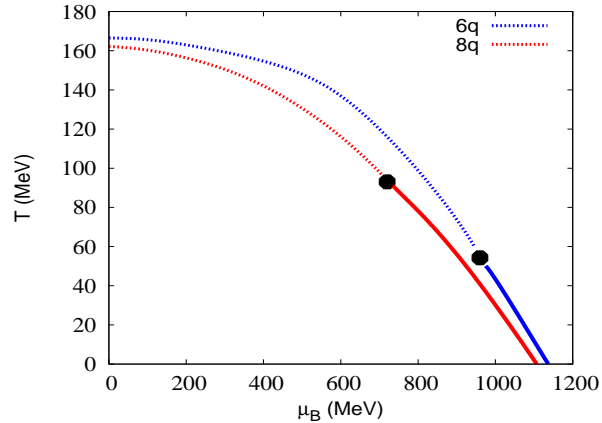


FIG. 6: (color online) Phase diagram for 2+1 flavor PNJL with 6 and 8 quark interactions

The Phase Diagram :

Exploration of the phase diagram of strongly interacting matter is one of the major goals of the heavy-ion collision experiments. The currently running Beam Energy Scan experiments at the RHIC facility [106], and the upcoming Compressed Baryonic Matter experiment at the FAIR facility [107] and the experiments at the NICA facility [108] are specifically designed for this purpose.

The phase diagram in the $T - \mu_B$ plane for strongly interacting matter is being investigated theoretically for quite some time [109]. While there is a crossover of hadronic phase to partonic phase along the T direction as suggested by lattice QCD studies, the transition along the μ_B direction is expected to be of first order from the various effective model analysis. The first order line is expected to bend towards the T axis starting from some finite μ_B and end at a critical end point (CEP). This will have some value of temperature T_E and chemical potential μ_{BE} .

A direct location of the CEP in lattice QCD is spoilt due the appearance of complex weight factors for non-zero μ_B in the Monte Carlo simulations. Several techniques exist that can circumvent this difficulty to a limited extent. Using a reweighing technique the location of CEP was estimated first in [115]. Calculations in the imaginary chemical potential shows conflicting results of existence of CEP depending on the version of lattice fermions chosen [111, 112]. Radius of convergence analysis for the Taylor series expansion of pressure may also lead to an estimate of the CEP [5, 21, 23, 113, 114, 117]. However a conclusive estimate of the CEP does not seem to have been reached. The present spread in the location of CEP is in the range $0.95T_c < T_E < 0.99T_c$ and $1.5T_c < \mu_{BE} < 2.5T_c$.

Interaction	T_E (MeV)	T_E/T_c	μ_{BE} (MeV)	μ_{BE}/T_c
6-quark	54.3	0.326	960	5.77
8-quark	93.0	0.572	720	4.43

TABLE IV: Location of critical end point

We have plotted the possible phase diagram in the PNJL model in Fig. 6 considering both 6-quark and 8-quark interactions. The parameter values are held at those obtained along the temperature axis. We have used the inflection

points i.e. the temperature derivative of the chiral condensate as well as that of the Polyakov loop and considered their average as the estimate of the transition temperature for a given chemical potential. For a first order transition however we located the point of discontinuous jump of the field values themselves. At the critical end point the discontinuity vanishes and the derivative is sharply diverging.

The location of the critical end point is presented in Table IV. The values are expectedly quite different from those obtained by us earlier with different set of parameter values [48]. Given that the T_c itself has been decreased by more than 25 MeV here for the 6-quark interaction, the T_E has reduced by about 40 MeV. For the 8-quark interaction the T_c value is reduced here by about 6 MeV, which has resulted in reducing the corresponding T_E by about 25 MeV. The μ_{BE} values are quite large and differ within 30 MeV for both the interaction models. The estimates of the location of CEP obtained from the lattice QCD simulations with various limitations as summarized in [18], are still significantly different from our model estimates.

V. FLUCTUATIONS OF CONSERVED CHARGES

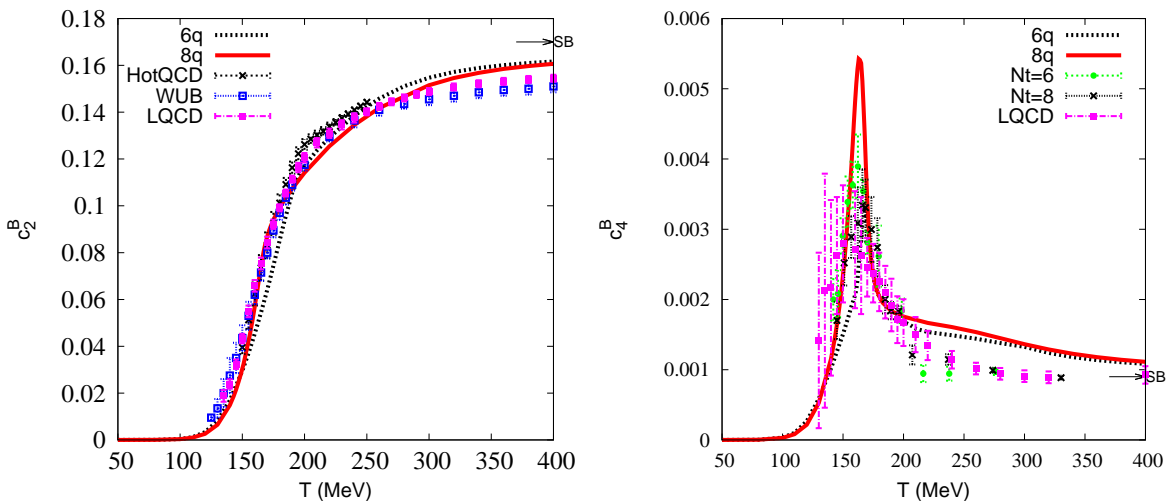


FIG. 7: (color online) Variation of c_2^B and c_4^B as functions of temperature. The continuum extrapolated dataset for c_2^B of HotQCD and Wuppertal-Budapest (WUB) collaborations are taken respectively from [118] and [130], as well as the data from [134] denoted as LQCD. For c_4^B HotQCD data for $N_t=6$ and 8 [135] and the continuum data from [134] are considered.

Fluctuations and correlations of conserved charges are considered important for their role in determining the state of strongly interacting matter at high temperatures and densities [15, 50, 53, 119, 120]. They may also be useful as signatures of a possible phase transition or crossover [16, 59, 121–129]. The pressure of the system at a given temperature and arbitrary chemical potentials may be expanded as a Taylor series around zero chemical potentials. The coefficients of this series are directly related via fluctuation dissipation theorem [60], to the fluctuations and correlations at various orders. The basic globally conserved quantities in the strong interactions are the various flavors considered. These are related to the experimentally observed charges of baryon number B , electric charge Q and strangeness S . The diagonal Taylor coefficients $c_n^X(T)$ ($X = B, Q, S$) of n^{th} order in an expansion of the scaled pressure $P(T, \mu_B, \mu_Q, \mu_S)/T^4$ may be written in terms of the fluctuations $\chi_n^X(T)$ of the corresponding order as,

$$c_n^X(T) = \frac{1}{n!} \frac{\partial^n (P/T^4)}{\partial (\frac{\mu_X}{T})^n} = T^{n-4} \chi_n^X(T) \quad (6)$$

where the expansion is carried out around $\mu_B = 0 = \mu_Q = \mu_S$. The off-diagonal coefficients $c_{n,m}^{X,Y}(T)$ ($X, Y = B, Q, S$; $X \neq Y$) in the $(m+n)^{\text{th}}$ order in the Taylor expansion are related to the correlations between the conserved charges $\chi_{n,m}^{X,Y}(T)$ as,

$$c_{m,n}^{X,Y} = \frac{1}{m!n!} \frac{\partial^{m+n} (P/T^4)}{(\partial (\frac{\mu_X}{T})^m)(\partial (\frac{\mu_Y}{T})^n)} = T^{m+n-4} \chi_{n,m}^{X,Y}(T) \quad (7)$$

Various fluctuations and correlations of the conserved charges have been measured in the lattice QCD framework either in the continuum limit [11, 118, 130–134] or for small lattice spacings, which are expected to be not far from the continuum limit [135]. Here we present a comparative study of these quantities with the present parametrization of the PNJL model. The quantities were obtained in the model by a suitable Taylor series fitting as has been discussed in detail in [37].

In Fig. 7 the variation of the baryon number susceptibilities c_2^B and c_4^B are shown as functions of temperature. While c_2^B mimics the behavior of an order parameter, c_4^B acts as its fluctuation. Apart from the qualitative similarity with the lattice QCD data, the quantitative agreement is encouraging. The second order susceptibility c_2^B seems to be impressively close to the lattice data except for a small difference beyond $T \sim 300$ MeV. Also the difference between the results for the 6-quark and the 8-quark interactions are quite small. For the fourth order susceptibility c_4^B similar difference remains between the lattice and model results at the higher temperature region.

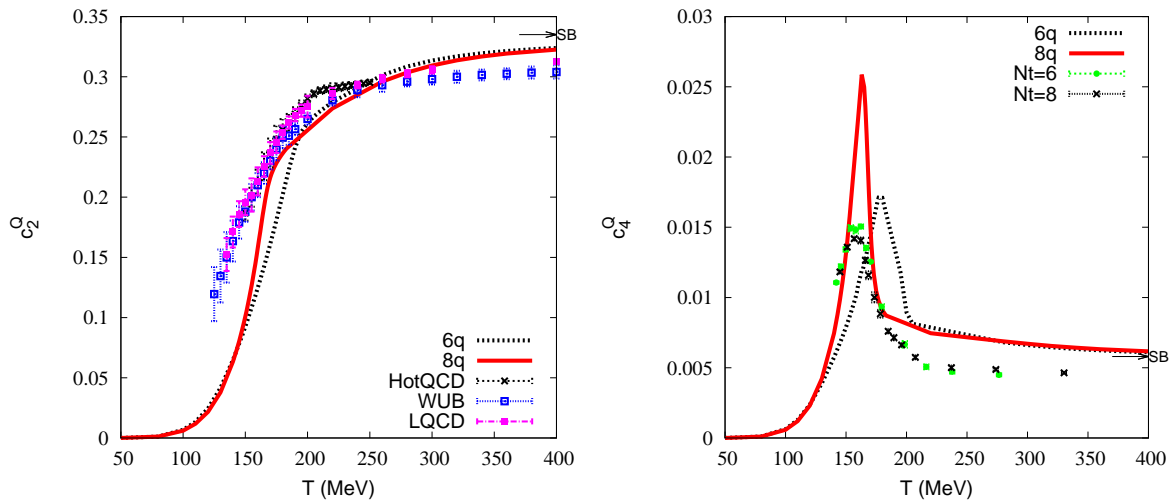


FIG. 8: (color online) Variation of c_2^Q and c_4^Q as functions of temperature. The continuum extrapolated dataset for c_2^Q of HotQCD and Wuppertal-Budapest (WUB) collaborations are taken respectively from [118] and [130], as well as the data from [134] denoted as LQCD. For c_4^Q HotQCD data for $N_t=6$ and 8 [135].

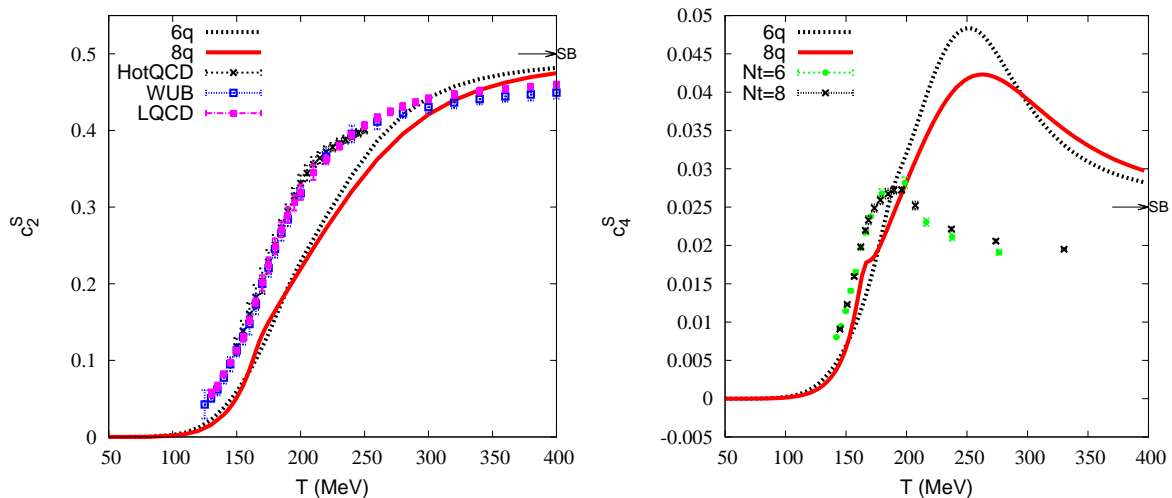


FIG. 9: (color online) Variation of c_2^S and c_4^S as functions of temperature. The continuum extrapolated dataset for c_2^S of HotQCD and Wuppertal-Budapest (WUB) collaborations are taken respectively from [118] and [130], as well as the data from [134] denoted as LQCD. For c_4^S HotQCD data for $N_t=6$ and 8 [135].

The variation of the charge susceptibilities with temperature are shown in Fig. 8. The qualitative as well as quantitative comparison between the two interaction models and the lattice QCD data are quite similar to that

discussed for the baryon number susceptibilities for $T > T_c$. However we now find significant difference between PNJL and lattice results for c_2^Q below the crossover temperature T_c . The lattice data is much larger than the model results. This seems to be expected from our earlier discussions of discrepancies in speed of sound. In the charge sector the dominant contributors are the light hadrons, and these excitations are effectively absent in the present form of the PNJL model. Therefore though the baryon fluctuations are well accounted for by the constituent quarks, proper considerations of other hadronic degrees of freedom below T_c is crucial to obtain the charge fluctuations.

The temperature variation of the strangeness susceptibilities c_2^S and c_4^S are shown in Fig. 9. Here also the quantitative results of c_2^S are found to be different between the model and lattice QCD data up to T_c . Proper inclusion of the light strange hadrons would be crucial in describing this region of temperature [136]. Above T_c the agreement is again much better. However for c_4^S there is a large difference between the PNJL model results and lattice data for $T > T_c$. The maxima obtained in the model is much larger, wider, as well as shifted towards higher temperatures as compared to the lattice data. As discussed by some of us earlier in Ref. [50] this is due to the melting of the strange quark condensate at higher temperatures in the PNJL model. This is possibly an artefact of constraining the NJL model parameters to be fixed at values obtained at zero temperature and chemical potentials. It would be important to investigate the necessary changes in the quark interactions in the NJL Lagrangian, but is beyond the scope of the present work.

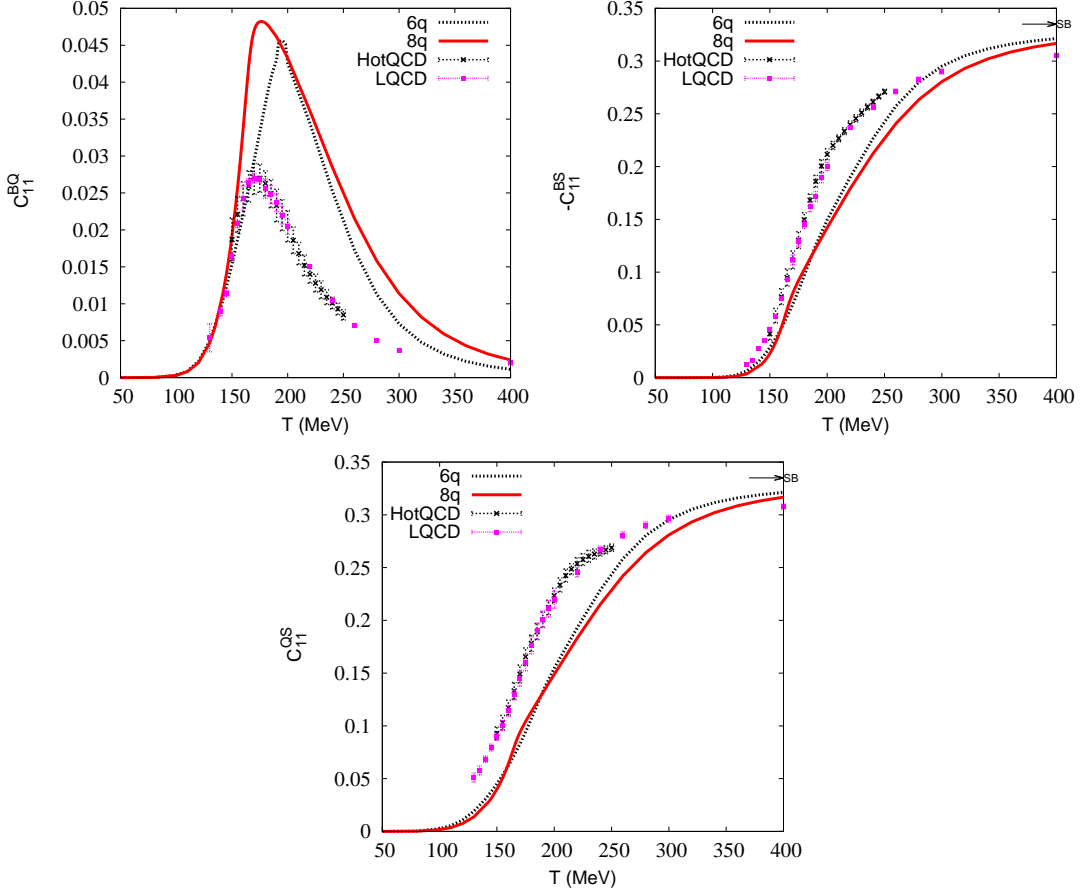


FIG. 10: (color online) Leading order correlation coefficients as a function of temperature. HotQCD continuum dataset have been taken from [118]. Lattice data from [134] have denoted as LQCD.

We now discuss the leading order correlations between the conserved charges. These are shown as functions of temperature in Fig. 10. The baryon number to electric charge (BQ) correlation c_{11}^{BQ} shows a hump around the crossover region and vanishes for both low and high temperatures. In the hadronic phase the baryon and electric charge are correlated because baryons have positive electric charge and anti-baryons have negative electric charge. However their masses being large, the correlations come out to be insignificant. With increasing temperature however the correlation becomes non-zero. On the other hand in the partonic phase, for the 2+1 flavor theory, there are three quarks with equal baryon number but electric charge of *down* and *strange* quarks are together opposite of that of the *up* quark, implying that in this phase the BQ correlation is zero. Thus we get the temperature variation of BQ

correlation as shown in Fig. 10. We note that the BQ correlation in the PNJL model is larger than that obtained in the lattice QCD data.

The baryon number to strangeness (BS) correlation c_{11}^{BS} as well as the electric charge to strangeness (QS) correlation c_{11}^{QS} show a order parameter like behavior. This is because at low temperatures they are suppressed due to large hadronic masses, and eventually increases with increase in temperatures. For these two correlations we note that PNJL model results are significantly lower than the lattice QCD data. This is similar to the behavior of the second order strangeness susceptibility c_2^s , which should be as we discuss below. For these correlators we find the lattice results to be larger than the PNJL results.

Now it seems strange that the correlators at the same order have opposite behavior for c_{11}^{BQ} versus c_{11}^{BS} and c_{11}^{QS} , when PNJL model is compared to the lattice QCD data. Let us try to argue how this could naturally arise. For that we first express the correlators in terms of the fluctuations and correlations in terms of the flavor basis. The relations are given as,

$$c_{11}^{BQ} = \frac{1}{9} (c_2^u - c_2^s + c_{11}^{ud} - c_{11}^{us}), \quad (8)$$

$$c_{11}^{BS} = \frac{1}{3} (-c_2^s - 2c_{11}^{us}), \quad (9)$$

$$c_{11}^{QS} = \frac{1}{3} (c_2^s - c_{11}^{us}), \quad (10)$$

where $c_2^u = c_2^d$ and c_2^s are the second order flavor susceptibilities and c_{11}^{ud} and c_{11}^{us} are the second order flavor correlations. We note that if we consider the flavor correlators to be numerically much smaller than the flavor susceptibilities one may again describe the observed behavior of the correlators in Fig. 10. The up flavor diagonal and the off-diagonal susceptibilities are presented in Fig. 11. The strange flavor diagonal susceptibility is identical to the strangeness diagonal susceptibility as shown in Fig. 9. While c_{11}^{BS} and c_{11}^{QS} will inherit the order parameter like behavior of c_2^s , c_{11}^{BQ} will vary depending on the difference between c_2^u and c_2^s . This may explain the higher value obtained in the PNJL model with respect to lattice QCD data. To see this we note that in [56] some of us discussed the variation of the baryon number to isospin (BI) correlation $c_{11}^{BI} = \frac{1}{6} (c_2^u - c_2^d)$ with different current masses for the *up* and *down* quarks in a 2 flavor system. For identical light quark masses, c_{11}^{BI} should be zero, but it becomes non-zero when the current masses are different. It was further discussed that value of c_{11}^{BI} is proportional to this mass difference and for small quark masses it has a consistent scaling with the amount of mass splitting. Here for c_{11}^{BQ} a similar situation arises due to the large strange quark current mass difference with that of the light quarks.

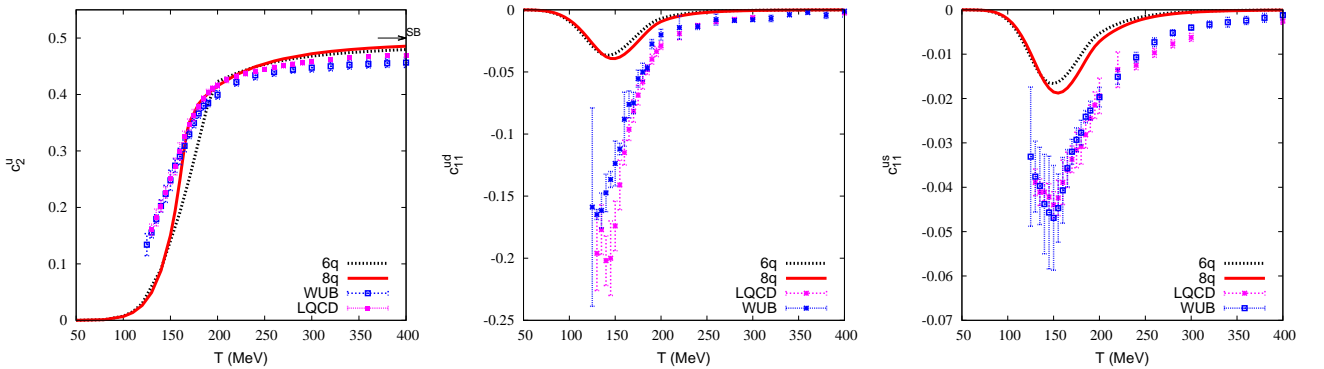


FIG. 11: (color online) Leading order fluctuations and correlations of quark flavors as a function of temperature. Lattice continuum data for WUB are from [130] and LQCD are from [134].

For the PNJL model we have considered the current quark masses as given in Table I. For the lattice QCD data the bare quark mass in physical units are found to have an average value of $m_s = 81\text{MeV}$ (with a spread of 2 MeV), for the temperature range of the data as obtained from Table XII of Ref. [12]. This difference in the bare masses may account for the difference in BQ correlation between PNJL model and the lattice QCD results. A detailed study in this direction will be presented elsewhere.

The strange quark mass being smaller for the lattice data it is highly conceivable that the second order susceptibilities are higher on the lattice exactly as observed in the behavior of c_{11}^{BS} and c_{11}^{QS} . This would also partially be responsible for the large difference of c_2^s obtained in the PNJL model and on the lattice. A proper reparametrization of the NJL

model with lower current mass for the strange quark may therefore bridge the gap in the various susceptibilities and correlations related to the strangeness sector and will be addressed elsewhere. It should also be noted that a further suppression to the BQ correlation in the lattice data is due to a significant contribution from the ud correlation. The flavor correlations in the PNJL model are quite suppressed compared to the continuum lattice data, which is probably due to the lack of proper considerations of the hadronic degrees of freedom.

VI. CONCLUSIONS

QCD in the non-perturbative domain is best realized with lattice QCD simulations which are however very costly. Simpler model approaches are efficient in the extraction of the quantities of interest at arbitrary values of external parameters like temperature, chemical potential etc. which however needs to have reliability validated quantitatively. In this work we discussed how far the PNJL model is suitable in describing the thermodynamic properties of strongly interacting matter. Recently, lattice QCD simulations have been extrapolated to the continuum limit and almost physical quark masses, obtaining a variety of interesting information for a wide range of temperature. Therefore it seemed timely that a reparametrization of the PNJL model be made to check if it can satisfactorily predict various measured observables on the lattice.

An important observation in the continuum extrapolated lattice results is that the pressure of strongly interacting matter is significantly below that of ideal gas of quarks and gluons even at reasonably large temperatures. This implies that the gluon mediated interactions must be strong even though the degrees of freedom may have changed from hadronic to partonic ones. So we chose to reparametrize the Polyakov loop self interactions in the PNJL model which is supposed to mimic the gluonic effects. The NJL model parameters were set from hadronic properties at zero temperature and chemical potentials.

We found excellent agreement of the equation of state in the PNJL model with that of lattice QCD data in a wide range of temperatures. The specific heat has a small peak in the model near the crossover in the model. Though not a prominent peak but a hump is surely present in the lattice QCD data. The speed of sound agrees with lattice data except for $T < T_c$.

The second and fourth order susceptibilities of the baryon number were again found to be in reasonable quantitative agreement with the lattice data. For the electric charge susceptibilities we found some disagreement for $T < T_c$. The disagreement in this region for speed of sound as well as susceptibilities could possibly be due to absence of light hadrons in the present formulation of the PNJL model.

Significant disagreement was observed for baryon-charge, baryon-strangeness and charge-strangeness correlations. The values were more in the PNJL model for the baryon-charge correlation and opposite for the other correlators. We argued that this could possibly due to the difference in the bare strange quark masses used in the PNJL model and the lattice formulations. With this argument the opposing discrepancies in the correlators could also be explained. This could also be partially responsible for the discrepancies in the strangeness susceptibilities. The most significant disagreement is observed for the fourth order susceptibility of strangeness for $T > T_c$. The slow melting of the strange quark condensate seems to be a major cause for this discrepancy.

Thus even though the quantitative agreement of a variety of observables in the PNJL model with the lattice QCD data was found to be encouraging, certain differences still remain. A proper consideration of hadronic excitations and reparametrization of the NJL part seems necessary. We would like to address these issues elsewhere.

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