# $\mathcal{PT}$ symmetry of a conditionally exactly solvable potential

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**Abstract**

A conditionally exactly solvable potential, the supersymmetric partner of the harmonic oscillator is investigated in the $\mathcal{PT}$-symmetric setting. It is shown that a number of properties characterizing shape-invariant and Natanzon-class potentials generated by an imaginary coordinate shift $x - \mathrm{i}\epsilon$ also hold for this potential outside the Natanzon class.

## 1 Introduction

The concept of $\mathcal{PT}$ symmetry has generated much interest recently in one-dimensional quantum mechanical potential problems as many problems exhibiting $\mathcal{PT}$ symmetry, i.e. invariance under the simultaneous action of space ($\mathcal{P}$) and time ($\mathcal{T}$) inversion possessed real energy eigenvalues belonging to the discrete spectrum, although the correspondig Hamiltonians were not Hermitian [1]. Later it was found that in contrast with the first conjectures, $\mathcal{PT}$ symmetry is neither necessary, nor sufficient condition for having real discrete spectrum. More recently $\mathcal{PT}$ symmetry was recognized as a special case of $\eta$-pseudo-Hermiticity [2]: a Hamiltonian is $\eta$-pseudo-Hermitian if there exists a linear, Hermitian, invertible operator $\eta$, for which $H^\dagger = \eta H \eta^{-1}$ holds. In this context $\mathcal{PT}$ symmetry is $\mathcal{P}$-pseudo-Hermiticity for one-dimensional Hamiltonians of the type $H = p^2 + V(x)$, whereas conventional Hermiticity follows for $\eta = 1$. More

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recently the formalism of $\mathcal{PT}$ symmetry has been interpreted as the complex extension of quantum mechanics, by modifying it with the help of a dynamically constructed $\mathcal{C}$ operator, so that the inner product $\langle\psi_i|\mathcal{CP}|\psi_j\rangle$ leads to positive norm [3].

On the other hand, exactly solvable problems are of enormous importance in the understanding of physical systems, and this is also the case in $\mathcal{PT}$-symmetric quantum mechanics. The well-known textbook examples (e.g. the harmonic oscillator, Coulomb, Morse, Pöschl–Teller, Rosen–Morse, etc.) potentials belong to a rather narrow two- and three-parameter subset of the general six-parameter Natanzon potentials [4]. In particular, they are shape-invariant [5, 6] potentials having the property that a supersymmetric transformation eliminating their ground state does not modify the functional form of the potential, but only changes some parameters appearing in them. Supersymmetric quantum mechanics (SUSYQM) [7] has been a rather productive method of generating new exactly solvable potentials from known ones. The SUSY partner potentials generated this way have the same discrete energy spectrum, except perhaps a single level which is eliminated (the ground state), or is added to the spectrum (below the ground state). Applying a SUSYQM transformation to a general Natanzon (i.e. non-shape-invariant) potential results in a SUSY partner potential that is outside the Natanzon class, because in this case the bound-state wavefunctions can be written in terms of two (confluent) hypergeometric functions.

To widen the class of exactly solvable models of the Schrödinger equation, another concept used is conditional exact solvability (CES), rendering the potential exactly solvable only when the potential parameters appearing in them satisfy certain conditions. For example, the DKV potential was proven to be a Natanzon-class potential [8], and its CES nature stems from the fact that it has three potential terms, but only two free parameters. Another type of CES potentials has been identified in the SUSYQM construction: in this case the supersymmetric partner of shape-invariant potentials was constructed by inserting a new ground state below the original one, and it was found that this could be done at certain energies, which required setting a parameter to a numerical constant [9]. With this a CES potential outside the Natanzon class could be generated.

Constructing the $\mathcal{PT}$-symmetric version of exactly solvable potentials resulted in a number of interesting findings. It turned out, for example, that except for the Coulomb and Morse potentials all the shape-invariant potentials can be defined on a trajectory determined by the imaginary coordinate shift $x \to x - \mathrm{i}\epsilon$ [10, 11]. (The Coulomb [12] and Morse potentials [13] possess normalizable solutions only on some curved trajectories of the complex $x$ plane.) With the appropriate choice of the parameters, normalizable solutions with both real and complex energies could be generated in a straightforward way for all the remaining shape-invariant potentials [10, 11]. It also turned out that with the imaginary coordinate shift the singularities of real potentials (e.g. at the origin) could be cancelled, and as the result of this, the radial potentials could be extended to the whole $x$ axis formally, and new normalizable solutions appeared due to the less strict boundary conditions. Moreover, these potentials had *two* series of normalizable states, distinguished by the $q = \pm 1$ quasi-parity quantum number, which characterizes the bound states, but the potential itself does not depend on it [14]. This gave rise to *two* different ('fermionic') SUSY partners due to the presence of *two* nodeless solutions (with quasi-parity $q = 1$ and $q = -1$) to the original ('bosonic') potential [15, 16]. Furthermore, it was also proven that in case the original potential has unbroken $\mathcal{PT}$ symmetry, then the two partner potentials also have this property, but if the $\mathcal{PT}$ symmetry of the original potential is spontaneously broken, then the partner potentials cease to be $\mathcal{PT}$-symmetric. This finding proved valid for some shape-invariant potentials (e.g. the Scarf II potential [16]) and also for some Natanzon-class potentials (e.g. the generalized Ginocchio potential) [17].

All these results raise a number of questions concerning the properties under $\mathcal{PT}$ symmetry of various types of solvable potentials. A number of results seem to indicate that properties characterizing mainly $\mathcal{PT}$-symmetric shape-invariant potentials are actually, valid for more general potentials from the Natanzon class too. This is the case with the applicability of the imaginary coordinate shift, the presence of the quasi-parity quantum number and the behaviour of the SUSY partner potentials in the case of intact and spontaneously broken $\mathcal{PT}$ symmetry of the original potential. It is natural to ask whether these properties also characterize potentials even beyond the Natanzon class. A natural candidate for these studies is the SUSY partner of shape-invariant potentials generated by inserting a new ground state below the original one. The CES potential generated in this way is outside the Natanzon class, as we have discussed before [9]. However, all its main characteristics can be determined in terms of exact calculations, so we can hope to get answers to our questions formulated here.

## 2 A conditionally exactly solvable $\mathcal{PT}$-invariant potential

In this section, our aim is to construct CES potentials which are SUSY partners of the much studied $\mathcal{PT}$-symmetric harmonic oscillator [18]

$$V(x) = (x - i\epsilon)^2 + \frac{\alpha^2 - \frac{1}{4}}{(x - i\epsilon)^2} \ . \tag{1}$$

Shifting the co-ordinate from $x$ to $z = x - i\epsilon$, removes the singularities on the real line, and extends the potential from the half line to the full line. The eigenvalues and eigenfunctions of (1) are well known [18]

$$\psi_{nq} = N_{nq} e^{-z^2/2} z^{-q\alpha+1/2} L_n^{(-q\alpha)}\left(z^2\right) \ , \tag{2}$$

$$E_{nq} = 4n + 2 - 2q\alpha \ , \tag{3}$$

where $L_n^{(\sigma)}(z^2)$ are the associated Laguerre polynomials [19] and $q = \pm 1$ is the quasi-parity [14].

We wish to find Hamiltonians which are isospectral to (1), with the possible exception of the ground state. For this purpose we define two intertwining operators $A_{(q)}$ and $B_{(q)}$

$$A_{(q)} = \frac{\mathrm{d}}{\mathrm{d}x} + W_q(x) \tag{4}$$

and

$$B_{(q)} = -\frac{\mathrm{d}}{\mathrm{d}x} + W_q(x) \ , \tag{5}$$

where $W_q(x)$ (the so-called superpotential in conventional Hermitian quantum mechanics), is, in general, a complex-valued function. It is easy to observe that if $\psi_q^+(x)$ is an eigenfunction of $H_+^{(q)}$ with eigenvalue $E_q$, then $\psi_q^-(x) = B\psi_q^+(x)$ is an eigenfunction of $H_-^{(q)}$ with the same eigenvalue $E_q$, where the partner Hamiltonians $H_\pm$ are given by

$$H_+^{(q)} = A_{(q)}B_{(q)} - \beta_q = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+^{(q)}(x) - \beta_q \tag{6}$$

$$H_-^{(q)} = B_{(q)}A_{(q)} - \beta_q = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-^{(q)}(x) - \beta_q \tag{7}$$

with

$$V_\pm^{(q)}(x) = W_{(q)}^2(x) \pm W'_{(q)}(x) \ . \tag{8}$$

It is worth mentioning here that $\beta_q$ can be $q$-dependent. In fact, as we shall see later, $\beta_q$ turns out to be the $q$-dependent factorization energy. The interesting point to note is that unlike in conventional Hermitian quantum mechanics, $A_q$ and $B_q$ are not mutually adjoint operators. They may be related by a linear, invertible, Hermitian operator $\eta$ to form mutually pseudo-adjoint pair [20]. The role of the quasi-parity quantum number ($q = \pm 1$), is quite important as it gives rise to a doublet set of isospectral partners for the original potential.

To construct isospectral, non-shape-invariant partners of the $\mathcal{PT}$-symmetric oscillator, we assume the following ansatz for $W_{(q)}(x)$ :

$$W_{(q)}(x) = (x - i\epsilon) + \frac{\lambda}{(x - i\epsilon)} + \sum_{k=1}^{N} \frac{2g_k(x - i\epsilon)}{1 + g_k(x - i\epsilon)^2} \quad , \quad g_k \geq 0 \ , \tag{9}$$

which reduces to the superpotential of the $\mathcal{PT}$-symmetric oscillator for $g_1 = g_2 = g_3 = \cdots = g_N = 0$. Thus the supersymmetric techniques applied here are different from the standard ways when the ground state is eliminated. Rather this is a reverse procedure in a sense, in which $V_+$ is a simple potential and we construct $V_-$ which has one more state. Further, in this work we restrict ourselves to the case $N = 1$. The partner potentials then assume the form

$$V_+^{(q)}(x) \ = \ (x - i\epsilon)^2 \ + \ \frac{\lambda\,(\lambda - 1)}{(x - i\epsilon)^2} \ + \ \frac{4g\lambda + 2g - 4}{1 + g\,(x - i\epsilon)^2} \ + \ 2\lambda \ + \ 5 \ , \tag{10}$$

$$V_-^{(q)}(x) \ = \ (x - i\epsilon)^2 \ + \ \frac{\lambda\,(\lambda + 1)}{(x - i\epsilon)^2} \ - \ \frac{4g\lambda - 2g - 4}{1 + g\,(x - i\epsilon)^2} \ + \ \frac{8g^2(x - i\epsilon)^2}{\{1 + g(x - i\epsilon)^2\}^2} \ + \ 2\lambda \ + \ 3 \ . \tag{11}$$

Since the functional form of $V_+^{(q)}(x)$ should be of the same form as that of $V(x)$ in (1), simple algebra shows that

$$\lambda = -q\alpha + \frac{1}{2} \ . \tag{12}$$

This is an example of CES problem, as exact solvability occurs only when the potential parameter $g$ assumes the specific value

$$g = \frac{1}{-q\alpha + 1} \tag{13}$$

reducing the partners to

$$\begin{aligned} v_+(x) \ &\equiv \ V_+^{(q)}(x) - \beta_q \\ &= \ (x - i\epsilon)^2 + \frac{\alpha^2 - \frac{1}{4}}{(x - i\epsilon)^2} + 6 \ , \end{aligned} \tag{14}$$

$$\begin{aligned} v_-^{(q)}(x) \ &\equiv \ V_-^{(q)}(x) - \beta_q \\ &= \ (x - i\epsilon)^2 + \frac{\alpha^2 - 2q\alpha + \frac{3}{4}}{(x - i\epsilon)^2} - \frac{4}{1 - q\alpha + (x - i\epsilon)^2} + \frac{8(x - i\epsilon)^2}{[(-q\alpha + 1) + (x - i\epsilon)^2]^2} + 4 \ . \end{aligned} \tag{15}$$

Note that after the substitution of $g$ from (13) $v_+(x)$ is independent from $q$, while $v_-^{(q)}(x)$ is not. Equation (15) is an example of a CES potential, for the particular value of $g$ given in (13). Moreover, it has more terms than its partner $v_+(x)$, and hence cannot have the same functional form. Thus we obtain a non-shape-invariant isospectral partner of the $\mathcal{PT}$-symmetric oscillator. In the above $\beta_q$ stands for

$$\beta_q = -2q\alpha \ . \tag{16}$$

Thus to each $v_+(x)$, there exist two non-shape-invariant, isospectral partners $v_-^{(q)}(x)$, as shown above. The interesting feature observed here is that there are no singularities on the real line, and hence the CES potentials so constructed are defined on the full line $(-\infty, +\infty)$.
The possible zero-energy eigenfunctions of $H_\pm^{(q)} + \beta_q$ are of the form

$$\psi_{0q}^{(\pm)}(x) = N_{0q} \ \exp\left( \pm \int^x \ W_{(q)}(t)\mathrm{d}t \right) \ , \tag{17}$$

where $N_q$ is the normalization constant. Since $\psi_{0q}^{(-)}(x)$ given by

$$\psi_{0q}^{-}(x) \;=\; N_{0q}\frac{1}{1+g(x-i\epsilon)^2}e^{-\frac{(x-i\epsilon)^2}{2}}(x-i\epsilon)^{-q\alpha+\frac{1}{2}} \tag{18}$$

is normalizable, the situation may be compared to that of unbroken supersymmetry. So the eigenfunctions $\psi_{nq}^{(\pm)}$ and eigenvalues $E_{nq}$ of the partner Hamiltonians $H_{\pm}^{(q)}$ are related by $(n = 0, 1, 2, \ldots)$

$$E_{0q}^{-} = 0, \qquad E_{(n+1)q}^{-} = E_{nq}^{+} > 0 \;, \tag{19}$$

$$\psi_{nq}^{+} = (E_{(n+1)q}^{-})^{-1/2} A_{(q)} \psi_{(n+1)q}^{(-)} \;, \tag{20}$$

$$\psi_{(n+1)q}^{-} = (E_{nq}^{+})^{-1/2} B_{(q)} \psi_{nq}^{(+)} \;. \tag{21}$$

Since the eigenfunctions and eigenvalues for the Hamiltonian $H_{(q)}^{+}$ are well known, the wave functions of the partner Hamiltonian $H_{(q)}^{-}$ are calculated to be

$$\begin{aligned}\psi_{(n+1)q}^{-} \;&=\; N_{nq}(E_{(n+1)q}^{+})^{-1/2}\; \exp\left(-\tfrac{z^2}{2}\right)\; z^{-q\alpha+\frac{1}{2}} \\ &\times\; \left[\left(2z + \tfrac{2gz}{1+gz^2} - \tfrac{2(n+1)}{z}\right) L_{n+1}^{(-q\alpha)}\left(z^2\right) \;-\; \tfrac{2(n-q\alpha+1)}{z}\; L_{n}^{(-q\alpha)}\left(z^2\right)\right] \;.\end{aligned} \tag{22}$$

From the structure of (15) and (22) it is evident that $v_{-}^{(q)}(x)$ belongs to a class of potentials which is beyond the shape-invariant as well as the Natanzon class potentials. However, in analogy with the shape-invariant and Natanzon potentials, the states are characterized by the quasi-parity $q$, giving rise to two SUSY isospectral partners for the original potential.

Now let us analyse the conditions for having real and complex energy eigenvalues. As we shall see the role played by the potential parameter $\alpha$ is very crucial in this regard.

# 3 $\mathcal{PT}$ symmetry of the SUSY partner

(i) $\alpha$ is real : The $\mathcal{PT}$ symmetry of the original potential $v_{+}(x)$ is unbroken.

The parameter $g$ turns out to be real in this case, and the two partner potentials $v_{-}^{(q)}(x)$ (with $q = \pm 1$) are also $\mathcal{PT}$-invariant, as can be seen from the behaviour of their real and imaginary components, which are even and odd functions of $x$, respectively. Figure 1 shows the imaginary parts of the original potential together with its two partners (corresponding to $q = \pm 1$) for $\alpha = 0.3$ and $\epsilon = 0.5$ while figure 2 shows the real parts of the same potentials for identical parameter values.

(ii) $\alpha$ is pure imaginary : $\mathcal{PT}$ symmetry spontaneously broken in the original potential $v_{+}(x)$.

Let $\alpha = ia$, where $a$ is real. This choice of $\alpha$ renders $g$ to be complex. The $\mathcal{PT}$-invariant $v_{+}(x)$ sector has an attractive, but non-singular core

$$v_{+}(x) = (x-i\epsilon)^2 - \frac{a^2+\frac{1}{4}}{(x-i\epsilon)^2} + 6 \tag{23}$$

with complex conjugate pairs of energies

$$E_{nq}^{+} = 4n + 8 - i2qa \;. \tag{24}$$

Though there still exist two values of $g$ and consequently of $v_-^{(q)}(x)$, the partners given by

$$v_-^{(q)}(x) = (x - i\epsilon)^2 + \frac{-a^2 - 2iqa + \frac{3}{4}}{(x - i\epsilon)^2} - \frac{4}{1 - iqa + (x - i\epsilon)^2} + \frac{8(x - i\epsilon)^2}{\left[1 - iqa + (x - i\epsilon)^2\right]^2} + 4 \qquad (25)$$

are no longer $\mathcal{PT}$-invariant. Figure 3 shows the imaginary parts of the original potential together with its two partners (corresponding to $q = \pm 1$) for $\alpha = 0.3i$ and $\epsilon = 0.5$ while figure 4 shows the real parts of the same potentials for identical parameter values.

As illustrative examples we calculate the ground state and the first excited state wave functions for this case:

$$\psi_{0q}^- = C_{0q}(1 - iqa)\; e^{-\frac{z^2}{2}}\; \frac{z^{-iqa+\frac{1}{2}}}{1 - iqa + z^2}\;, \qquad (26)$$

$$\psi_{1q}^- = C_{1q}\; e^{-\frac{z^2}{2}}\; z^{-iqa+\frac{1}{2}}\; \left[\left(2z + \frac{2gz}{1 + gz^2} - \frac{2}{z}\right)\; \left(1 - iqa - z^2\right) \; - \; \frac{2(1 - iqa)}{z}\right]\;. \qquad (27)$$

where $C_{0q}$ and $C_{1q}$ are some normalization factors. It is clearly seen that the eigenfunctions are no longer $\mathcal{PT}$-invariant, rather the $\mathcal{PT}$ operation transforms $\psi_{nq_+}$ and $\psi_{nq_-}^-$ ($q_\pm = \pm 1$) into each other. It is worthwhile to note that this relation also holds between $v_-^{(q)}$ and $v_-^{(-q)}$ in the case of imaginary $\alpha$.

## 4 Summary and conclusions

We analysed a conditionally exactly solvable potential, the supersymmetric partner of the $\mathcal{PT}$-symmetric harmonic oscillator, which is outside the Natanzon potential class by construction. Our motivation was to investigate whether some typical features originally found for the $\mathcal{PT}$-symmetric version of most shape-invariant potentials and later proved also for some non-shape-invariant Natanzon-class potentials (the generalized Ginocchio potential) generated by an imaginary coordinate shift $x - \mathrm{i}\epsilon$ remain valid for this kind of potential too. These features included the presence of the quasi-parity quantum number $q = \pm 1$, the "sudden" realization of the spontaneous breakdown of $\mathcal{PT}$ symmetry (i.e. the simultaneous disappearance of real energy eigenvalues and their re-emergence as complex conjugated pairs at a certain value of a parameter) and the finding that the spontaneous breakdown of the original potential implies the manifest breakdown of the $\mathcal{PT}$ symmetry of its two supersymmetric partner. Our study confirmed that all these features are valid in this case too, so they characterize a much wider potential class than originally thought. It seems that these features appear for all the $\mathcal{PT}$-symmetric potentials that are generated by an imaginary coordinate shift. Certainly they are absent in shape-invariant [12, 13] and non-shape-invariant Natanzon-class [21] potentials defined on curved trajectories of the complex $x$ plane. Further work is needed for the detailed analysis of these differences.


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Figure Captions

Fig 1. Imaginary parts of the original potential (solid line), the partners (dashed line for $q = 1$, dotted line for $q = -1$) for unbroken $\mathcal{PT}$ symmetry for $\alpha = 0.3, \epsilon = 0.5$.

Fig 2. Real parts of the original potential (solid line), the partners (dashed line for $q = 1$, dotted line for $q = -1$) for unbroken $\mathcal{PT}$ symmetry for $\alpha = 0.3, \epsilon = 0.5$.

Fig 3. Imaginary parts of the original potential (solid line), the partners (dashed line for $q = 1$, dotted line for $q = -1$) for spontaneously broken $\mathcal{PT}$ symmetry for $\alpha = 0.3i, \epsilon = 0.5$.

Fig 4. Real parts of the original potential (solid line), the partners (dashed line for $q = 1$, dotted line for $q = -1$) for unbroken $\mathcal{PT}$ symmetry for $\alpha = 0.3i, \epsilon = 0.5$.

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