

\mathcal{PT} -supersymmetric partner of a short-range square well

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Abstract

In a box of size L , a spatially antisymmetric square-well potential of a purely imaginary strength ig and size $l < L$ is interpreted as an initial element of the SUSY hierarchy of solvable Hamiltonians, the energies of which are all real for $g < g_c(l)$. The first partner potential is constructed in closed form and discussed.

1 Introduction

The technically slightly complicated but quantum-mechanically straightforward solution of the one-dimensional, \mathcal{PT} -symmetric Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi_n(x) = E_n\psi_n(x), \quad n = 0, 1, 2, \dots, \quad (1)$$

with the Dirichlet boundary conditions $\psi(\pm L) = 0$ and with a purely imaginary $V(x)$ may be found elsewhere [1, 2, 3]. Here, such a model with real spectrum and

$$V(x) = V^{(+)}(x) = \begin{cases} 0 & \text{for } L > |x| > l, \\ ig \operatorname{sign} x, & g > 0, \text{ for } |x| \leq l, \end{cases} \quad (2)$$

will be considered factorized and complemented by another similar model,

$$-\frac{d^2}{dx^2} + V^{(+)}(x) - D_0 = \bar{\mathcal{A}}\mathcal{A} \equiv H^{(+)}, \quad \mathcal{A}\bar{\mathcal{A}} = -\frac{d^2}{dx^2} + V^{(-)}(x) - D_0 \equiv H^{(-)}, \quad (3)$$

where the well-known operators and identities

$$\mathcal{A} = \frac{d}{dx} + W(x), \quad \bar{\mathcal{A}} = -\frac{d}{dx} + W(x), \quad V^{(\pm)} - D_0 = W^2 \mp W' \quad (4)$$

are employed. We denote the wave functions of $H^{(+)}$ (resp. $H^{(-)}$) by symbols $\psi_n^{(+)}(x)$ (resp. $\psi_n^{(-)}(x)$), $n = 0, 1, 2, \dots$, and we assume the so-called unbroken-supersymmetry condition $\mathcal{A}\psi_0^{(+)}(x) = 0$ of Witten's supersymmetric quantum mechanics (SUSYQM) [4] (hence $D_0 = E_0^{(+)}$ hereabove). As long as the application of such a formalism to non-Hermitian operators is always subject to caution, we believe that both the construction and some unusual properties of the partner potential $V^{(-)}(x)$ deserve an explicit description.

2 The \mathcal{PT} -symmetric SUSY partner potential $V^{(-)}(x)$

The purpose of the present section is to construct and study the SUSY partner $H^{(-)}$ of the square-well Hamiltonian $H^{(+)}$ in the physically-relevant unbroken \mathcal{PT} -symmetry regime, corresponding to $g < g_c(l)$ of ref. [1].

2.1 Determination of the parameters

Let us denote the four regions $-L < x < -l$, $-l < x < 0$, $0 < x < l$, $l < x < L$ by $L2$, $L1$, $R1$, $R2$, respectively, and write for $V^{(+)}$, defined in (2), $V_{L2}^{(+)}(x) = 0$, $V_{L1}^{(+)}(x) = -ig$, $V_{R1}^{(+)}(x) = ig$, $V_{R2}^{(+)}(x) = 0$. Setting $D_0 = E_0^{(+)}$, where

$$E_n^{(+)} = k_n^2 = t_n^2 - s_n^2, \quad \kappa_n = s_n + it_n, \quad g = 2s_n t_n, \quad (5)$$

for $n = 0, 1, 2, \dots$, we obtain for the superpotential and the partner potential the respective formulae

$$W(x) = \begin{cases} W_{L2}(x) = k_0 \tan[k_0(x + x_{L2})] \\ W_{L1}(x) = -\kappa_0^* \tanh[\kappa_0^*(x + x_{L1})] \\ W_{R1}(x) = -\kappa_0 \tanh[\kappa_0(x - x_{R1})] \\ W_{R2}(x) = k_0 \tan[k_0(x - x_{R2})] \end{cases} \quad (6)$$

and

$$V^{(-)}(x) = \begin{cases} V_{L2}^{(-)}(x) = 2k_0^2 \sec^2[k_0(x + x_{L2})] \\ V_{L1}^{(-)}(x) = -2\kappa_0^{*2} \operatorname{sech}^2[\kappa_0^*(x + x_{L1})] - ig \\ V_{R1}^{(-)}(x) = -2\kappa_0^2 \operatorname{sech}^2[\kappa_0(x - x_{R1})] + ig \\ V_{R2}^{(-)}(x) = 2k_0^2 \sec^2[k_0(x - x_{R2})] \end{cases}. \quad (7)$$

Here x_{L2} , x_{L1} , x_{R1} and x_{R2} denote four integration constants. We choose

$$x_{L2} = L + \frac{\pi}{2k_0}, \quad x_{R2} = L - \frac{\pi}{2k_0} \quad (8)$$

to ensure that $V_{L2}^{(-)}$ and $V_{R2}^{(-)}$ blow up at the end points $x = -L$ and $x = L$. This is in tune with [5]. We thus get

$$V_{L2}^{(-)}(x) = 2k_0^2 \csc^2[k_0(x + L)], \quad V_{R2}^{(-)}(x) = 2k_0^2 \csc^2[k_0(x - L)]. \quad (9)$$

Observe that for the superpotential, $W_{L2}(x)$ and $W_{R2}(x)$ also blow up at these points:

$$W_{L2}(x) = -k_0 \cot[k_0(x + L)], \quad W_{R2}(x) = -k_0 \cot[k_0(x - L)]. \quad (10)$$

The ground-state wavefunction of $H^{(+)}$ is given by [1]

$$\psi_{0R2}^{(+)}(x) = \psi_{0L2}^{(+)*}(-x) = A_0^{(+)} \sin[k_0(L - x)], \quad (11)$$

$$\psi_{0R1}^{(+)}(x) = \psi_{0L1}^{(+)*}(-x) = B_0^{(+)} \cosh(\kappa_0 x) + i \frac{C_0^{(+)}}{\kappa_0 l} \sinh(\kappa_0 x), \quad (12)$$

where $A_0^{(+)}$, $B_0^{(+)}$, $C_0^{(+)}$ are three constants, $B_0^{(+)}$, $C_0^{(+)}$ are real and

$$A_0^{(+)} = B_0^{(+)} \frac{\kappa_0 \csc[k_0(L - l)] \operatorname{csch}(\kappa_0 l)}{k_0 \cot[k_0(L - l)] + \kappa_0 \coth(\kappa_0 l)}, \quad (13)$$

$$C_0^{(+)} = i\kappa_0 l B_0^{(+)} \frac{k_0 \cot[k_0(L - l)] \coth(\kappa_0 l) + \kappa_0}{k_0 \cot[k_0(L - l)] + \kappa_0 \coth(\kappa_0 l)}, \quad (14)$$

as a result of the matching conditions on $\psi_0^{(+)}(x)$ and its derivative at $x = 0$ and $x = \pm l$. It turns out that the unbroken-SUSY condition is automatically satisfied in the regions $R2$ and $L2$ due to the choice made for the integration constants x_{R2} , x_{L2} in (8). In the region $R1$, we find a condition fixing the value of x_{R1} ,

$$\tanh(\kappa_0 x_{R1}) = -\frac{iC_0^{(+)}}{\kappa_0 l B_0^{(+)}} = \frac{k_0 \cot[k_0(L - l)] \coth(\kappa_0 l) + \kappa_0}{k_0 \cot[k_0(L - l)] + \kappa_0 \coth(\kappa_0 l)}. \quad (15)$$

A similar relation applies in $L1$, thus leading to the result

$$x_{L1} = x_{R1}^*. \quad (16)$$

Note that in contrast with the real integration constants x_{R2} , x_{L2} , the constants x_{R1} and x_{L1} are complex. Separating both sides of equation (15) into a real and an imaginary part, we obtain the two equations

$$\frac{\sinh X \cosh X}{\cosh^2 X \cos^2 Y + \sinh^2 X \sin^2 Y} = \frac{N^r}{D}, \quad (17)$$

$$\frac{\sin Y \cos Y}{\cosh^2 X \cos^2 Y + \sinh^2 X \sin^2 Y} = \frac{N^i}{D}, \quad (18)$$

where we have used the decompositions $\kappa_0 = s_0 + it_0$, $x_{R1} = x_{R1}^r + ix_{R1}^i$, $\kappa_0 x_{R1} = X + iY$, implying that

$$X = s_0 x_{R1}^r - t_0 x_{R1}^i, \quad Y = t_0 x_{R1}^r + s_0 x_{R1}^i, \quad (19)$$

and we have defined

$$N^r = \{-s_0^2 \cos[2k_0(L-l)] + t_0^2\} \sinh(2s_0 l) + k_0 s_0 \sin[2k_0(L-l)] \cosh(2s_0 l), \quad (20)$$

$$N^i = \{s_0^2 - t_0^2 \cos[2k_0(L-l)]\} \sin(2t_0 l) - k_0 t_0 \sin[2k_0(L-l)] \cos(2t_0 l), \quad (21)$$

$$D = \{-s_0^2 \cos[2k_0(L-l)] + t_0^2\} \cosh(2s_0 l) + \{s_0^2 - t_0^2 \cos[2k_0(L-l)]\} \cos(2t_0 l) \\ + k_0 \sin[2k_0(L-l)] [s_0 \sinh(2s_0 l) + t_0 \sin(2t_0 l)]. \quad (22)$$

Equations (17) and (18), when solved numerically, furnish the values of both the parameters x_{R1}^r and x_{R1}^i . One may also observe that the resulting superpotential $W(x) = -W^*(-x)$ and partner potential $V^{(-)}(x) = V^{(-)*}(-x)$ are \mathcal{PT} -antisymmetric and \mathcal{PT} -symmetric, respectively.

2.2 Eigenfunctions in the partner potential

On exploiting the SUSY intertwining relations, the eigenfunctions $\psi_n^{(-)}(x)$, $n = 0, 1, 2, \dots$, of $H^{(-)}$ can be obtained by acting with \mathcal{A} on $\psi_{n+1}^{(+)}(x)$, subject to the preservation of the

boundary and continuity conditions

$$\psi_{nL2}^{(-)}(-L) = 0, \quad \psi_{nR2}^{(-)}(L) = 0, \quad (23)$$

$$\psi_{nL2}^{(-)}(-l) = \psi_{nL1}^{(-)}(-l), \quad \partial_x \psi_{nL2}^{(-)}(-l) = \partial_x \psi_{nL1}^{(-)}(-l), \quad (24)$$

$$\psi_{nL1}^{(-)}(0) = \psi_{nR1}^{(-)}(0), \quad \partial_x \psi_{nL1}^{(-)}(0) = \partial_x \psi_{nR1}^{(-)}(0), \quad (25)$$

$$\psi_{nR1}^{(-)}(l) = \psi_{nR2}^{(-)}(l), \quad \partial_x \psi_{nR1}^{(-)}(l) = \partial_x \psi_{nR2}^{(-)}(l). \quad (26)$$

Application of \mathcal{A} leads to the forms

$$\begin{aligned} \psi_{nL2}^{(-)}(x) &= C_{nL2}^{(-)} A_{n+1}^{(+)*} \sin[k_{n+1}(L+x)] \\ &\quad \times \{k_{n+1} \cot[k_{n+1}(L+x)] - k_0 \cot[k_0(L+x)]\}, \end{aligned} \quad (27)$$

$$\begin{aligned} \psi_{nL1}^{(-)}(x) &= C_{nL1}^{(-)} B_{n+1}^{(+)} \sinh(\kappa_{n+1}^* x) \{ \kappa_{n+1}^* - \kappa_0^* \tanh[\kappa_0^*(x+x_{R1}^*)] \coth(\kappa_{n+1}^* x) \} \\ &\quad + C_{nL1}^{(-)} \frac{iC_{n+1}^{(+)}}{\kappa_{n+1}^* l} \sinh(\kappa_{n+1}^* x) \\ &\quad \times \{ \kappa_{n+1}^* \coth(\kappa_{n+1}^* x) - \kappa_0^* \tanh[\kappa_0^*(x+x_{R1}^*)] \}, \end{aligned} \quad (28)$$

$$\begin{aligned} \psi_{nR1}^{(-)}(x) &= C_{nR1}^{(-)} B_{n+1}^{(+)} \sinh(\kappa_{n+1} x) \{ \kappa_{n+1} - \kappa_0 \tanh[\kappa_0(x-x_{R1})] \coth(\kappa_{n+1} x) \} \\ &\quad + C_{nR1}^{(-)} \frac{iC_{n+1}^{(+)}}{\kappa_{n+1} l} \sinh(\kappa_{n+1} x) \\ &\quad \times \{ \kappa_{n+1} \coth(\kappa_{n+1} x) - \kappa_0 \tanh[\kappa_0(x-x_{R1})] \}, \end{aligned} \quad (29)$$

$$\begin{aligned} \psi_{nR2}^{(-)}(x) &= C_{nR2}^{(-)} A_{n+1}^{(+)} \sin[k_{n+1}(L-x)] \\ &\quad \times \{-k_{n+1} \cot[k_{n+1}(L-x)] + k_0 \cot[k_0(L-x)]\}, \end{aligned} \quad (30)$$

where $C_{nL2}^{(-)}$, $C_{nL1}^{(-)}$, $C_{nR1}^{(-)}$, $C_{nR2}^{(-)}$ denote some complex constants and equation (16) has been used. Boundary conditions (23) are satisfied. It remains to impose the continuity conditions (24) – (26).

The matching of the regions $L1$ and $R1$ at $x = 0$ leads to two conditions, which are compatible because the two constraints

$$\kappa_0 \tanh(\kappa_0 x_{R1}) = -\kappa_0^* \tanh(\kappa_0^* x_{R1}^*), \quad (31)$$

$$\kappa_{n+1}^{*2} - \kappa_{n+1}^2 = \kappa_0^{*2} - \kappa_0^2 = -2ig, \quad (32)$$

are satisfied owing to (15) and (5), respectively. It therefore remains a single condition

$$C_{nR1}^{(-)} = C_{nL1}^{(-)}. \quad (33)$$

For the matching between $R1$ and $R2$ at $x = l$, a similar situation happens due this time to the two constraints

$$\kappa_0 \tanh[\kappa_0(l - x_{R1})] = -k_0 \cot[k_0(L - l)], \quad (34)$$

$$\kappa_{n+1}^2 - \kappa_0^2 = k_0^2 - k_{n+1}^2. \quad (35)$$

The resulting condition reads

$$C_{nR1}^{(-)} = C_{nR2}^{(-)}. \quad (36)$$

Since a result similar to (36) applies at the interface between regions $L2$ and $L1$, we conclude that the partner potential eigenfunctions are given by equations (27) – (30) with

$$C_{nL2}^{(-)} = C_{nL1}^{(-)} = C_{nR1}^{(-)} = C_{nR2}^{(-)} \equiv C_n^{(-)}. \quad (37)$$

Such eigenfunctions are \mathcal{PT} -symmetric provided we choose $C_n^{(-)}$ imaginary:

$$C_n^{(-)*} = -C_n^{(-)}. \quad (38)$$

3 Discontinuities in the partner potential $V^{(-)}(x)$

In subsection 2.1, we have constructed the SUSY partner $V^{(-)}(x)$ of a piece-wise potential with three discontinuities at $x^{(i)} = -l, 0$ and l , where $i = 1, 2, 3$. We may now ask the following question: does the former have the same discontinuities as the latter or could the discontinuity number decrease? We plan to prove here that the second alternative can be ruled out.

For such a purpose, let us determine the jump (if any) of the partner potential at $x^{(i)}$, $\Delta V^{(-)}(x^{(i)}) \equiv \lim_{x \rightarrow x_+^{(i)}} V^{(-)}(x) - \lim_{x \rightarrow x_-^{(i)}} V^{(-)}(x)$. A simple calculation leads to $\Delta V^{(-)}(0) = -2\kappa_0^2 \operatorname{sech}^2(\kappa_0 x_{R1}) + ig - [-2\kappa_0^{*2} \operatorname{sech}^2(\kappa_0^* x_{R1}^*) - ig] = -2ig$, where use has been made of (31) and (32). Similarly, from equations (5) and (34) it follows that $\Delta V^{(-)}(\pm l) = ig$.

This confirms that $V^{(-)}(x)$ has the same discontinuities as $V^{(+)}(x)$. However, when we compare the jumps of the former with those of the latter resulting from definition (2), we find

$$\Delta V^{(-)}(x^{(i)}) = -\Delta V^{(+)}(x^{(i)}), \quad i = 1, 2, 3. \quad (39)$$

Such a behaviour can be traced back to the superpotential, which turns out to be a continuous function of x on $(-L, +L)$, in contrast with its derivative, which is discontinuous at $x^{(i)}$, $i = 1, 2, 3$. The third relation in (4) then immediately leads to (39).

4 Conclusion

Under the simplest assumption of unbroken SUSY, we have shown that for the weakly non-Hermitian square well with three discontinuities at $x = -l, 0$ and l , the SUSY partners $H^{(\pm)}$ are both non-Hermitian and \mathcal{PT} -symmetric. Moreover, the partner potential $V^{(-)}(x)$ has the same three discontinuities as $V^{(+)}(x)$.

It should be noted that in the two limiting cases $l \rightarrow 0$ and $l \rightarrow L$, our results give back those relative to the real square well [8] and to the \mathcal{PT} -symmetric square well with a single discontinuity [5], respectively.

It is conjectured that as for the strongly non-Hermitian square well with a single discontinuity at $x = 0$ [3], a charge-conjugation operator \mathcal{C} [6] may be constructed in a specific form differing from the unit operator mostly in a finite-dimensional subspace of the Hilbert space [7]. This is one of the most important merits of all the square-well models with $L < \infty$. It seems to open a new inspiration for a direct physical applicability of non-Hermitian models whenever their spectrum remains real.

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