

# Pseudo-Hermitian versus Hermitian position-dependent-mass Hamiltonians in a perturbative framework

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## Abstract

We formulate a systematic algorithm for constructing a whole class of Hermitian position-dependent-mass Hamiltonians which, to lowest order of perturbation theory, allow a description in terms of  $\mathcal{PT}$ -symmetric Hamiltonians. The method is applied to the Hermitian analogue of the  $\mathcal{PT}$ -symmetric cubic anharmonic oscillator. A new example is provided by a Hamiltonian (approximately) equivalent to a  $\mathcal{PT}$ -symmetric extension of the one-parameter trigonometric Pöschl-Teller potential.

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Pseudo-Hermitian Hamiltonians and their subclass of  $\mathcal{PT}$ -symmetric ones have aroused a great deal of interest since it was observed that some of them may have a real, positive spectrum [1]. Pseudo-Hermiticity of  $H$  with respect to a positive-definite (Hermitian and invertible) operator  $\eta_+$ , i.e.,

$$H^\dagger = \eta_+ H \eta_+^{-1} \quad (1)$$

has been identified as one of the necessary and sufficient conditions for this situation to occur [2]. Any Hamiltonian endowed with such a property is then equivalent to a Hermitian one

$$h = \rho H \rho^{-1} \quad (2)$$

where the similarity transformation is implemented by  $\rho = \sqrt{\eta_+}$ . Further, to any observable  $o$  and to any wavefunction  $\psi(x) = \langle x|\psi\rangle$  in the Hermitian theory described by  $h$ , one can associate an operator  $O = \rho^{-1} o \rho$  and a wavefunction  $\Psi(x) = \langle x|\rho\psi\rangle$  in the (physical) pseudo-Hermitian theory, respectively.

Recently Jones [3] and, independently, Mostafazadeh [4] constructed the Hermitian analogue  $h$ , as well as the pseudo-Hermitian position and momentum operators  $X = \rho^{-1} x \rho$ ,  $P = \rho^{-1} p \rho$ , for the  $\mathcal{PT}$ -symmetric cubic anharmonic oscillator  $H = \frac{1}{2}(p^2 + x^2) + i\epsilon x^3$  (with  $\epsilon \in \mathbb{R}$ ). The latter, which has been shown both numerically [1] and mathematically [5] to have a real, positive and discrete spectrum, can only be treated in perturbation theory [6]. A very interesting outcome of [3] and [4] is that to lowest order such a system describes an ordinary quartic anharmonic oscillator with real and positive coupling constants but a position-dependent mass (PDM). As revealed by a more recent study of Bender *et al* [7], this Hermitian PDM theory is however difficult to work out because it leads to divergent Feynman graphs, which must be regulated to obtain the correct answer, whereas the corresponding non-Hermitian  $\mathcal{PT}$ -symmetric theory is completely free from such difficulties.

At this stage, it is worth mentioning that Hermitian PDM Hamiltonians are attracting a lot of attention due to their relevance in describing the physics of many microstructures of current interest, such as compositionally graded crystals (see [8] and references quoted therein). Several classes of physically-interesting solvable non-Hermitian potentials have also been generated [9, 10, 11] in a PDM background by employing various techniques,

such as the point canonical transformations or Lie algebraic methods, or using ideas from supersymmetric quantum mechanics. In particular, constructions of  $\mathcal{PT}$ -symmetric potentials have been carried out for different choices of mass functions. These include the  $\mathcal{PT}$ -symmetric Scarf potential [9] and the  $\mathcal{PT}$ -symmetric oscillator model [10]. Even the PDM version of the complex Morse potential [12], which is known to be pseudo-Hermitian [13], has been obtained [10].

In view of all these considerations, it may prove interesting to see under which conditions a Hermitian PDM Hamiltonian may be approximately equivalent to a non-Hermitian  $\mathcal{PT}$ -symmetric one, which, according to the experience gained in [7], would presumably be easier to handle. In the spirit of [3] and [4], this is tantamount to determining those  $\mathcal{PT}$ -symmetric Hamiltonians,

$$H = H_0 + \varepsilon H_1 \quad H_0 = \frac{p^2}{2m_0} + V^{(r)}(x) \quad H_1 = iV^{(i)}(x) \quad (3)$$

with  $\varepsilon \in \mathbb{R}$ ,  $V^{(r)}(x) = V^{(r)}(-x) \in \mathbb{R}$ ,  $V^{(i)}(x) = -V^{(i)}(-x) \in \mathbb{R}$  and configuration space  $\mathbb{R}$  (or a subset of it), that have a Hermitian counterpart

$$h = H_0 + \varepsilon^2 h^{(2)} + \varepsilon^4 h^{(4)} + \dots \quad (4)$$

which to lowest order in  $\varepsilon$  reduces to some PDM Hamiltonian, i.e.,

$$H_0 + \varepsilon^2 h^{(2)} = p \frac{1}{2m(x)} p + V_{\text{eff}}(x) \quad (5)$$

with  $1/m(x) = (1/m_0)[1 + \varepsilon^2 M^{(2)}(x)]$ ,  $V_{\text{eff}}(x) = V^{(r)}(x) + \varepsilon^2 V_{\text{eff}}^{(2)}(x)$  and  $M^{(2)}(x), V_{\text{eff}}^{(2)}(x) \in \mathbb{R}$ . It should be noted that the right-hand side of (4) only contains even powers of  $\varepsilon$  because the coefficients of odd powers have been shown to vanish [3, 4], while the right-hand side of (5) is the most general expression of Hermitian PDM Hamiltonians [8]. The latter is written in terms of an effective potential  $V_{\text{eff}}(x)$  including some mass terms depending on two ambiguity parameters, which take the noncommutativity of the momentum and PDM operators into account [14].

It proves convenient to introduce dimensionless quantities defined by

$$\begin{aligned} x &= \ell^{-1} x & p &= \ell \hbar^{-1} p \\ H &= \nu^{-1} H = H_0 + \varepsilon H_1 & H_0 &= \frac{1}{2} p^2 + V^{(r)}(x) & H_1(x) &= iV^{(i)}(x) \\ h &= \nu^{-1} h = H_0 + \varepsilon^2 h^{(2)} = \frac{1}{2} p [1 + \varepsilon^2 M^{(2)}(x)] p + V^{(r)}(x) + \varepsilon^2 V_{\text{eff}}^{(2)}(x) \end{aligned} \quad (6)$$

in terms of some length and energy scales,  $\ell$  and  $\nu = \hbar^2/(m_0\ell^2)$ . Note that in (3), (4) and (5),  $\varepsilon$  is also dimensionless, as well as  $M^{(2)}(x)$ .

In [3] and [4] (see also [6]), it has been shown that for the positive-definite metric operator  $\eta_+$ , one may take

$$\eta_+ = e^{-Q(x,p)} \quad Q(x,p) = \varepsilon Q_1(x,p) + \varepsilon^3 Q_3(x,p) + \dots \quad (7)$$

where every  $Q_j(x,p)$ ,  $j = 1, 3, \dots$ , is such that  $Q_j(x,p) = Q_j^\dagger(x,p) = Q_j(-x,p) = -Q_j(x,-p)$ . Then to lowest order in  $\varepsilon$ , equations (1) and (2) lead to the two conditions

$$[H_0, Q_1] = -2H_1 \quad \frac{1}{4}[H_1, Q_1] = h^{(2)} \quad (8)$$

which in the case of (3) and (5) amount to

$$\left[ \frac{1}{2}p^2 + V^{(r)}(x), Q_1 \right] = -2iV^{(i)}(x) \quad (9)$$

$$\frac{i}{4}[V^{(i)}(x), Q_1] = \frac{1}{2}pM^{(2)}(x)p + V_{\text{eff}}^{(2)}(x). \quad (10)$$

For  $Q_1$ , let us choose a general ansatz somewhat different from those previously considered:

$$Q_1 = \sum_{k=0}^{\infty} \{R_k(x), p^{2k+1}\} \quad R_k(x) = R_k(-x). \quad (11)$$

By expressing  $p$  as  $-id/dx$  and using the commutation relation

$$\left[ \frac{d^k}{dx^k}, f(x) \right] = \sum_{l=0}^{k-1} \binom{k}{l} \frac{d^{k-l}f(x)}{dx^{k-l}} \frac{d^l}{dx^l} \quad (12)$$

$Q_1$  can be written in normal form, i.e., with all functions of  $x$  on the left of the differential operators, as

$$Q_1 = -i \sum_{k=0}^{\infty} S_k(x) \frac{d^k}{dx^k} \quad (13)$$

where

$$\begin{aligned} S_{2k} &= \sum_{l=k}^{\infty} (-1)^l \binom{2l+1}{2k} \frac{d^{2l-2k+1}R_l}{dx^{2l-2k+1}} \\ S_{2k+1} &= \sum_{l=k}^{\infty} (1 + \delta_{l,k}) (-1)^l \binom{2l+1}{2k+1} \frac{d^{2l-2k}R_l}{dx^{2l-2k}} \end{aligned} \quad (14)$$

for  $k = 0, 1, 2, \dots$

On inserting (13) in (9) and (10) and employing (12) again, we find after some straightforward calculations that equation (9) is equivalent to the conditions

$$\frac{1}{2} \frac{d^2 S_0}{dx^2} + \sum_{l=1}^{\infty} S_l \frac{d^l V^{(r)}}{dx^l} = -2V^{(i)} \quad (15)$$

$$\frac{dS_{k-1}}{dx} + \frac{1}{2} \frac{d^2 S_k}{dx^2} + \sum_{l=k+1}^{\infty} \binom{l}{k} S_l \frac{d^{l-k} V^{(r)}}{dx^{l-k}} = 0 \quad k = 1, 2, \dots \quad (16)$$

while equation (10) leads to

$$\sum_{l=1}^{\infty} S_l \frac{d^l V^{(i)}}{dx^l} = -4V_{\text{eff}}^{(2)} \quad (17)$$

$$\sum_{l=2}^{\infty} \binom{l}{1} S_l \frac{d^{l-1} V^{(i)}}{dx^{l-1}} = 2 \frac{dM^{(2)}}{dx} \quad (18)$$

$$\sum_{l=3}^{\infty} \binom{l}{2} S_l \frac{d^{l-2} V^{(i)}}{dx^{l-2}} = 2M^{(2)} \quad (19)$$

$$\sum_{l=k+1}^{\infty} \binom{l}{k} S_l \frac{d^{l-k} V^{(i)}}{dx^{l-k}} = 0 \quad k = 3, 4, \dots \quad (20)$$

To be able to solve the general equations (15)–(20), it is appropriate to make some simplifying assumption. Inspired by the example of the  $\mathcal{PT}$ -symmetric cubic anharmonic oscillator considered in [3, 4], where  $Q_1$  only contains linear and cubic powers of  $p$ , let us assume that  $R_k(x) = 0$ ,  $k = 2, 3, \dots$ , in equation (11). It then follows from (14) that only the first four functions  $S_k$  in the expansion (13) may be nonvanishing and that they are given in terms of  $R_0$ ,  $R_1$ , and their derivatives by  $S_0 = R'_0 - R'''_1$ ,  $S_1 = 2R_0 - 3R''_1$ ,  $S_2 = -3R'_1$  and  $S_3 = -2R_1$ .

Let us first solve equations (15) and (16). In the latter,  $k$  is now restricted to  $k \leq 4$ . For  $k = 4$ , we obtain that  $S_3$  must be a constant, this implying that

$$R_1(x) = c_1. \quad (21)$$

Hence the remaining nonvanishing  $S_k$ 's are

$$S_0 = R'_0 \quad S_1 = 2R_0 \quad S_3 = -2c_1. \quad (22)$$

From equation (16) with  $k = 2$ , we get

$$R_0(\mathbf{x}) = 3c_1 V^{(r)}(\mathbf{x}) + c_0 \quad (23)$$

where  $c_0$  is another integration constant, while the equations with  $k = 1$  or  $k = 3$  are automatically satisfied. Equation (15) then provides us with a condition on  $V^{(i)}$ ,

$$V^{(i)}(\mathbf{x}) = \frac{1}{4}c_1 V^{(r)'''}(\mathbf{x}) - [3c_1 V^{(r)}(\mathbf{x}) + c_0]V^{(r)' }(\mathbf{x}). \quad (24)$$

Let us next turn ourselves to equations (17)–(20). It is easy to see that only equations (17) and (19) impose some new conditions, namely

$$M^{(2)}(\mathbf{x}) = -3c_1 V^{(i)' }(\mathbf{x}) \quad V_{\text{eff}}^{(2)}(\mathbf{x}) = \frac{1}{2}\{-[3c_1 V^{(r)}(\mathbf{x}) + c_0]V^{(i)' }(\mathbf{x}) + c_1 V^{(i)'''}(\mathbf{x})\} \quad (25)$$

where  $V^{(i)}(\mathbf{x})$  must be expressed in terms of  $V^{(r)}(\mathbf{x})$  through equation (24). This completes the solution of equations (9) and (10).

It is then straightforward to go back to  $x$ ,  $p$  and unscaled operators. This leads to the conclusion that there exists a whole class of Hermitian PDM Hamiltonians, which to lowest order of perturbation theory allow an equivalent  $\mathcal{PT}$ -symmetric description and might therefore be easier to deal with than generic ones. The various members of the class are distinguished by the choice of the zeroth-order part  $V^{(r)}(x)$  of the effective potential  $V_{\text{eff}}^{(2)}(x)$  and that of two integration constants  $c_0$ ,  $c_1$ . The lowest-order corrections to the mass term  $M^{(2)}(x)$  and to the effective potential in the PDM equation, as well as the imaginary part  $V^{(i)}(x)$  of the corresponding  $\mathcal{PT}$ -symmetric potential, are indeed entirely fixed by such a choice.

The classical Hamiltonians  $H_c(x_c, p_c)$  corresponding to the members of this class can be obtained by replacing  $x$  and  $p$  in  $h$  by the classical variables  $x_c$  and  $p_c$  and evaluating the resulting expressions in the limit  $\hbar \rightarrow 0$  (assuming this limit exists), i.e.,  $H_c(x_c, p_c) = \lim_{\hbar \rightarrow 0} h(x_c, p_c)$ .

The  $\eta_+$ -pseudo-Hermitian position and momentum operators  $X$  and  $P$ , as well as the physical wavefunctions  $\Psi(x)$ , can be calculated in the same way as  $h$ . To second order in  $\varepsilon$ , the pseudo-Hermitian operators are given by

$$O = o - \frac{1}{2}\varepsilon[o, Q_1] + \frac{1}{8}\varepsilon^2[[o, Q_1], Q_1] \quad o = x \text{ or } p. \quad (26)$$

For the dimensionless operators, we find

$$\begin{aligned} [x, Q_1] &= i \sum_{k=0}^{\infty} (k+1) S_{k+1} \frac{d^k}{dX^k} & [[x, Q_1], Q_1] &= \sum_{k=0}^{\infty} T_k \frac{d^k}{dX^k} \\ [p, Q_1] &= - \sum_{k=0}^{\infty} \frac{dS_k}{dX} \frac{d^k}{dX^k} & [[p, Q_1], Q_1] &= i \sum_{k=0}^{\infty} U_k \frac{d^k}{dX^k} \end{aligned} \quad (27)$$

where  $T_k$  and  $U_k$  are defined by

$$T_k = \sum_{l=0}^k \sum_{m=k-l+1}^{\infty} T_k^{(l,m)} \quad U_k = \sum_{l=0}^k \sum_{m=k-l+1}^{\infty} U_k^{(l,m)} \quad (28)$$

with

$$\begin{aligned} T_k^{(l,m)} &= \binom{m}{k-l} \left[ (m+1) S_{m+1} \frac{d^{l+m-k} S_l}{dX^{l+m-k}} - (l+1) S_m \frac{d^{l+m-k} S_{l+1}}{dX^{l+m-k}} \right] \\ U_k^{(l,m)} &= \binom{m}{k-l} \left[ \frac{dS_m}{dX} \frac{d^{l+m-k} S_l}{dX^{l+m-k}} - S_m \frac{d^{l+m-k+1} S_l}{dX^{l+m-k+1}} \right]. \end{aligned} \quad (29)$$

Similarly, the physical wavefunctions can be expressed as

$$\Psi(x) = \psi(x) - \frac{\varepsilon}{2} \langle x | Q_1 | \psi \rangle + \frac{\varepsilon^2}{8} \langle x | Q_1^2 | \psi \rangle \quad (30)$$

where  $Q_1$  is given by (13) and

$$Q_1^2 = - \sum_{k=0}^{\infty} W_k(x) \frac{d^k}{dX^k} \quad (31)$$

with

$$W_k = \sum_{l=0}^k \sum_{m=k-l}^{\infty} W_k^{(l,m)} \quad W_k^{(l,m)} = \binom{m}{k-l} S_m \frac{d^{l+m-k} S_l}{dX^{l+m-k}}. \quad (32)$$

With the simplifying assumption (22) and taking equations (23) and (26)–(32) into account, we obtain

$$\begin{aligned} X &= x - i\varepsilon(3c_1 V^{(r)} + c_0 + 3c_1 p^2) + \frac{3}{4} \varepsilon^2 c_1 [-c_1(6V^{(r)} V^{(r)'} + V^{(r)''''}) - 2c_0 V^{(r)'} \\ &\quad - 6ic_1 V^{(r)''} p + 6c_1 V^{(r)'} p^2] \end{aligned} \quad (33)$$

$$\begin{aligned} P &= p + \frac{3}{2} \varepsilon c_1 (V^{(r)''} + 2iV^{(r)'} p) + \frac{3}{4} i \varepsilon^2 c_1 \{ c_1 (3V^{(r)'} V^{(r)''} - 3V^{(r)} V^{(r)''''} + V^{(r)''''}) \\ &\quad - c_0 V^{(r)''''} + i [c_1 (6V^{(r)2} - 6V^{(r)} V^{(r)''} + 5V^{(r)''''}) - 2c_0 V^{(r)''}] p - 9c_1 V^{(r)''} p^2 \\ &\quad - 6ic_1 V^{(r)''} p^3 \} \end{aligned} \quad (34)$$

and

$$\begin{aligned}
\Psi(x) = & \psi(x) + \frac{1}{2}i\epsilon \left[ 3c_1 V^{(r)'} + 2(3c_1 V^{(r)} + c_0) \frac{d}{dx} - 2c_1 \frac{d^3}{dx^3} \right] \\
& - \frac{\epsilon^2}{8} \left\{ 3c_1 [c_1 (3V^{(r)2} + 6V^{(r)}V^{(r)''} - 2V^{(r)''''}) + 2c_0 V^{(r)''}] \right. \\
& + 6c_1 [c_1 (12V^{(r)}V^{(r)'} - 5V^{(r)''''}) + 4c_0 V^{(r)'}] \frac{d}{dx} + 2[9c_1^2 (2V^{(r)2} - 3V^{(r)''}) \\
& + 12c_0 c_1 V^{(r)} + 2c_0^2] \frac{d^2}{dx^2} - 48c_1^2 V^{(r)'} \frac{d^3}{dx^3} - 8c_1 (3c_1 V^{(r)} + c_0) \frac{d^4}{dx^4} \\
& \left. + 4c_1^2 \frac{d^6}{dx^6} \right\}. \tag{35}
\end{aligned}$$

It is easy to check that, as expected, the Hermitian PDM quartic anharmonic oscillator of [3, 4] belongs to the class of Hermitian PDM Hamiltonians with an approximate  $\mathcal{PT}$ -symmetric counterpart. On setting  $V^{(r)}(x) = \frac{1}{2}\mathcal{M}^2 x^2$ ,  $c_0 = 0$  and  $c_1 = -2/(3\mathcal{M}^4)$  in equation (24), where the dimensionless quantities are defined as in equations (17)–(20) of [4], we indeed obtain  $V^{(i)}(x) = x^3$ , so that  $V^{(i)}(x) = x^3$ . Furthermore, from equations (25), (33) and (34), we obtain  $m(x) = m_0[1 + 6(\epsilon^2/\mu^4)x^2]^{-1}$ ,  $V_{\text{eff}}^{(2)}(x) = (3m_0\mu^2 x^4 - 4\hbar^2)/(2m_0\mu^4)$ ,  $X = x + i(\epsilon/\mu^4)(\mu^2 x^2 + 2p^2/m_0) + (\epsilon^2/\mu^6)(-\mu^2 x^3 - 2i\hbar p/m_0 + 2xp^2/m_0)$  and  $P = p - i(\epsilon/\mu^2)(2xp - i\hbar) + (\epsilon^2/\mu^6)(2p^3/m_0 - \mu^2 x^2 p + i\hbar\mu^2 x)$ , which after some reordering agree with [3, 4], as does the classical Hamiltonian. Similarly, equation (35) gives rise to equation (65) of [4].

A new example is provided by selecting for  $V^{(r)}(x)$  a one-parameter trigonometric Pöschl-Teller potential [15]

$$V^{(r)}(x) = V_0 \sec^2 kx \quad V_0 = \frac{\hbar^2 k^2}{2m^2} \lambda(\lambda - 1) \quad \lambda > 2 \tag{36}$$

on the interval  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ . On setting  $\ell = k^{-1}$  and  $\nu = \hbar^2 k^2/m_0$  for the length and energy scales, respectively, we obtain the dimensionless quantities  $x = kx$ ,  $p = p/(\hbar k)$  and  $V^{(r)}(x) = \frac{1}{2}\lambda(\lambda - 1) \sec^2 x$  with  $\lambda(\lambda - 1) = 2V_0/\nu$ .

The choice  $c_0 = -c_1 = \frac{1}{3}$  in (24) leads to

$$V^{(i)}(x) = \frac{1}{2}(\lambda + 1)\lambda(\lambda - 1)(\lambda - 2) \sec^4 x \tan x = \frac{2}{\nu^2} V_0 (V_0 - \nu) \sec^4 x \tan x. \tag{37}$$



This means that the corresponding  $\mathcal{PT}$ -symmetric Hamiltonian may be written as

$$H = \frac{p^2}{2m_0} + V_0 \sec^2 kx + i\epsilon \sec^4 kx \tan kx \quad (38)$$

where  $\epsilon$  has the dimension of an energy and is given in terms of the dimensionless  $\varepsilon$  by  $\epsilon = 2\varepsilon V_0(V_0 - \nu)/\nu$ .

To second order in  $\epsilon$ , such a non-Hermitian Hamiltonian is equivalent to a Hermitian PDM one, given by equation (5), where

$$m(x) = m_0 \left( 1 + \frac{\epsilon^2}{2V_0(V_0 - \nu)} \sec^4 kx (5 \sec^2 kx - 4) \right)^{-1} \quad (39)$$

and

$$V_{\text{eff}}(x) = V_0 \sec^2 kx + \frac{\epsilon^2}{4V_0(V_0 - \nu)} \sec^4 kx [5(V_0 - 14\nu) \sec^4 kx - (4V_0 - 85\nu) \sec^2 kx - 20\nu]. \quad (40)$$

The corresponding  $\eta_+$ -pseudo-Hermitian position and momentum operators can be expressed as

$$\begin{aligned} X &= x - i \frac{\epsilon}{2kV_0(V_0 - \nu)} \left( -V_0 \sec^2 kx + \frac{\nu}{3} - \frac{p^2}{m_0} \right) - \frac{\epsilon^2}{4kV_0(V_0 - \nu)^2} \sec^2 kx \\ &\quad \times \left\{ [(V_0 + 2\nu) \sec^2 kx - \nu] \tan kx + i \sqrt{\frac{\nu}{m_0}} (3 \sec^2 kx - 2)p - \tan kx \frac{p^2}{m_0} \right\} \quad (41) \\ P &= p - \frac{\epsilon}{2(V_0 - \nu)} \sec^2 kx [\sqrt{m_0\nu} (3 \sec^2 kx - 2) + 2i \tan kx p] - i \frac{\epsilon^2}{4V_0(V_0 - \nu)^2} \sec^2 kx \\ &\quad \times \left\{ \sqrt{m_0\nu} [3V_0 \sec^4 kx - 2\nu(30 \sec^4 kx - 19 \sec^2 kx + 1)] + i[V_0 \sec^4 kx \right. \\ &\quad \left. - \nu(50 \sec^4 kx - 49 \sec^2 kx + 6)]p + 6 \sqrt{\frac{\nu}{m_0}} (3 \sec^2 kx - 1) \tan kx p^2 \right. \\ &\quad \left. + \frac{i}{m_0} (3 \sec^2 kx - 2)p^3 \right\}. \quad (42) \end{aligned}$$

Similar results can be found for physical wavefunctions. For lack of space, let us only mention the result in dimensionless variable obtained for the function  $\psi(x) = \cos^\lambda(x)$  (corresponding to the ground state of the real potential (36)):

$$\begin{aligned} \Psi(x) &= \cos^\lambda(x) \left\{ 1 + \frac{i}{6} \varepsilon (\lambda + 1) \lambda (\lambda - 1) (\sec^2 x + 2) \tan x - \frac{\varepsilon^2}{72} (\lambda + 1) \lambda (\lambda - 1) \right. \\ &\quad \times [(\lambda - 4)(\lambda - 2)(\lambda + 15) \sec^6 x + 3(\lambda - 2)(\lambda^2 - 4\lambda + 15) \sec^4 x \\ &\quad \left. - 4(\lambda + 1) \lambda (\lambda - 1) \right\}. \quad (43) \end{aligned}$$

In the classical limit,  $\nu$  goes to zero. To get a nonvanishing limit for  $V_0$ , we must therefore assume that  $\lambda$  goes to infinity as  $\hbar^{-1}$  (this implying, in particular, that  $\lambda$  becomes negligibly small compared with  $\lambda^2$ ). To second order in  $\epsilon$ , the classical Hamiltonian corresponding to (38) is obtained as

$$H_c = \frac{p_c^2}{2m(x_c)} + V_0 \sec^2 kx_c + \frac{\epsilon^2}{4V_0} \sec^6 kx_c (5 \sec^2 kx_c - 4) \quad (44)$$

where

$$m_c(x_c) = m_0 \left( 1 - \frac{\epsilon^2}{2V_0^2} \sec^4 kx_c (5 \sec^2 kx_c - 4) \right) \quad (45)$$

while the classical  $\eta_+$ -pseudo-Hermitian variables  $X_c, P_c$  are

$$\begin{aligned} X_c &= x_c + i \frac{\epsilon}{2kV_0^2} \left( V_0 \sec^2 kx_c + \frac{p_c^2}{m_0} \right) - \frac{\epsilon^2}{4kV_0^3} \sec^2 kx_c \\ &\quad \times \left( V_0 \sec^2 kx_c - \frac{p_c^2}{m_0} \right) \tan kx_c \end{aligned} \quad (46)$$

$$\begin{aligned} P_c &= p_c - i \frac{\epsilon}{V_0} \sec^2 kx_c \tan kx_c p_c + \frac{\epsilon^2}{4V_0^3} \sec^2 kx_c \\ &\quad \times \left[ V_0 \sec^4 kx_c + (3 \sec^2 kx_c - 2) \frac{p_c^2}{m_0} \right] p_c. \end{aligned} \quad (47)$$

It is worth noting that in contrast with what happens for the  $\mathcal{PT}$ -symmetric cubic anharmonic oscillator, the operators  $X$  and  $P$  involve  $\hbar$  even after rewriting them in a symmetrized form. As a consequence, the  $\eta_+$ -pseudo-Hermitian quantization of the classical Hamiltonian (44) is far from trivial. This illustrates the importance of the factor-ordering problem in pseudo-Hermitian quantum mechanics.

In conclusion, the generalization of the works in [3] and [4] that we have proposed here contributes to exploring further the relationships between  $\mathcal{PT}$ -symmetric and Hermitian PDM Hamiltonians started there and continued in [7, 16]. Moreover, it suggests the interest of performing detailed calculations for some new  $\mathcal{PT}$ -symmetric systems, such as the one defined in (38).

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