# Position-dependent mass models and their nonlinear characterization

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## Abstract

We consider the specific models of Zhu-Kroemer and BenDaniel-Duke in a sech$^2$-mass background and point out interesting correspondences with the stationary 1-soliton and 2-soliton solutions of the KdV equation in a supersymmetric framework.

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In dealing with position dependent mass (PDM) models controlled by a sech$^2$-mass profile, we demonstrated [1] recently that, in the framework of a first-order intertwining relationship, such a mass environment generates an infinite sequence of bound states for the conventional free-particle problem. Noting that the intertwining relationships are naturally embedded in the formalism [2] of the so-called supersymmetric quantum mechanics (SUSYQM), we feel tempted to dig this issue a little deeper by choosing to examine the connections between the discrete eigenvalues of such a PDM quantum Hamiltonian (transformed appropriately so that a SUSY structure is evident) and the stationary soliton solutions of the Korteweg de Vries (KdV) equation that match with the mass function upto a constant of proportionality.

Let us begin with the standard time-independent representation of the PDM Schrödinger equation [3]

$$\left[ -\frac{d^2}{dx^2} + \frac{3}{4}\frac{M'^2}{M^2} - \frac{1}{2}\frac{M''}{M} + M(V_{eff} - \epsilon) \right]\psi = 0 \tag{1}$$

where $M(x)$ is the dimensionless equivalence of the mass function m(x) defined by $m(x) = m_0 M(x)$, and we have chosen units such that $\hbar = 2m_0 = 1$. The effective potential $V_{eff}$ contains, apart from the given $V(x)$, the real ambiguity parameters $\alpha$ and $\beta$ whose occurrences are typical in PDM settings :

$$V_{eff} = V(x) + \frac{1}{2}(\beta + 1)\frac{M''}{M^2} - \{\alpha(\alpha + \beta + 1) + \beta + 1\}\frac{M'^2}{M^3} \tag{2}$$

Suitable physical choices of $\alpha$ and $\beta$ have been reported in the literature [4-25] but of particular interest to us are the schemes of Zhu-Kroemer (ZK) [4] [$\alpha = -1/2, \beta = 0$] and BenDaniel-Duke (BDD) [5] [$\alpha = 0, \beta = -1$] which were shown [1] to be dual of each other for the free-particle case $V(x) = V_0$ that is independent of any choice of $M(x)$.

Substituting (2) in (1) and assuming for $V_0$ the form

$$V_0 = \epsilon - \lambda(\lambda + 1)q^2, \quad \lambda, q \in \mathbf{R} \tag{3}$$

we can recast (1) to the standard constant-mass Schrödinger equation namely

$$(-\frac{d^2}{dx^2} + u)\psi = 0 \tag{4}$$

with the energy level term missing. In (4), $u$ is given by

$$u = \left[\frac{3}{4} - \{\alpha(\alpha + \beta + 1) + \beta + 1\}\right]\frac{M'^2}{M^2} + \frac{1}{2}\beta\frac{M''}{M} - \lambda(\lambda + 1)Mq^2. \tag{5}$$

However, Eq. (4) can also be regarded as the linearized partner of the Riccati equation

$$u = v^2 + v' \tag{6}$$

upon putting $v = \frac{\psi'}{\psi}$. The latter is the Cole-Hopf transformation.

A nonlinear connection such as the one given by (6), also known as the Miura map, has an interesting implication. It transfers a solution of the modified KdV equation

$$v_t = 6v^2 v' - v''' \tag{7}$$

into a solution of the KdV equation

$$u_t = 6uu' - u''' \tag{8}$$

which is straightforward to check.

The KdV equation has a very rich internal structure [26-27]. In particular, it admits of a Lax representation $L_t = [B, L]$, where $L = -\partial^2 + u$ is a Schrödinger-like operator and B is given by $B = -4\partial_x^3 + 6u\partial_x + 3u'$. One can solve for L in the from $L(t) = S(t)L(0)S^{-1}(t)$ with $S_t = BS$. The related eigenvalue problem then implies that the spectrum of L is conserved and yields for the KdV an infinite chain of conserved charges.

Noting that the KdV is invariant under the set of transformations

$$t \rightarrow t', \quad x \rightarrow x' - 6ct', \quad u \rightarrow u' + c \tag{9}$$

where c is a constant, the energy levels $\mu_n$ can be introduced in (4):

$$(-\frac{d^2}{dx^2} + u)\psi = \mu_n \psi. \tag{10}$$

The manner of interplay between the PDM form (5) of $u$ for specific choices of the parameters $\alpha, \beta$ and the initial condition $u(x, 0) = u_0$ used as inputs to solve for the KdV (as is normally done in the inverse scattering problem) is our point of enquiry.

It can be proved that the discrete eigenvalues $\mu_n$ are time-independent. For this we have to express the KdV in the conserved form

$$u_t + (-3u^2 + u_{xx})_x = 0 \tag{11}$$

and substitute u from (10) into it. We obtain

$$(\mu_n)_t \psi^2 + (\psi\phi_x - \psi_x\phi)_x = 0 \tag{12}$$

where $\phi = \psi_t + \psi_{xxx} - 3(u + \mu)\psi_x$. On integrating (12) we find $(\mu_n)_t = 0$ where we have employed normalized $\psi$ and considered vanishing asymptotic conditions for

$\psi$ and its derivatives. The eigenvalues $\mu_n$ are determined using for the potential the initial value $u_0$ that corresponds to a stationary soliton solution of the the KdV equation.

In the context of (10), the Riccati equation (6) is transformed to

$$u = v^2 + v' + \mu \tag{13}$$

where v as a solution of the generalized MKdV equation

$$v_t = 6(v^2 + \mu)v' - v''' \tag{14}$$

ensures that u evolves according to the KdV equation.

For the 1-soliton and the 2-soliton solutions of the KdV, the corresponding starting solutions $u_0$ along with $\psi$, v and the eigenvalues $\mu_n$ (n=1, 2) are given by :

1-soliton : $u_0^{(1)} = -2q^2 sech^2 qx,$

$$\psi_1 = \frac{1}{\sqrt{2}} sech\, qx, \;\; v^{(1)} = -q \tanh qx, \;\; \mu_1 = -q^2 \tag{15}$$

2-soliton : $u_0^{(2)} = -6q^2 \text{sech}^2 \text{qx}$ ,

$$\psi_2^{(a)} = \frac{\sqrt{3}}{2} sech^2 qx, \;\; v_2^{(a)} = -2q \tanh qx, \;\; \mu_2^{(a)} = -4q^2$$

and

$$\psi_2^{(b)} = \frac{\sqrt{3}}{2} sech\, qx \tanh qx, \;\; v_2^{(b)} = q\frac{1 - 2\tanh^2 qx}{\tanh qx}, \;\; \mu_2^{(b)} = -q^2.$$

where note that for one discrete value of the Schrödinger equation (10), there exists a 1-soliton solution and vice-versa. Similarly for the 2-soliton case. Here the $\psi$'s are normalized.

The results in (15), which can also be extended to the N-soliton case, have been obtained by solving the eigenvalue problem for the Schrödinger equation (10). The solutions $u_0^{(1)}$ and $u_0^{(2)}$ act in (10) as the reflectionless potentials. The inverse scattering method, which exploits this reflectionless feature, determines the evolution of the scattering parameters. Subsequently the Gelfand-Levitan integral equation is solved to obtain the solution $u(x,t)$ of the KdV equation.

Turning now to the PDM induced u given by (5), we immediately recognize from (10) that for the choice of the mass function $M(x) = \text{sech}^2 \text{qx}$, the ZK scheme yields

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the 1-soliton results $u_0^{(1)}, \psi_1(\mu = -q^2)$ corresponding to $\lambda = 1, -2$ and the 2-soliton result $u_0^{(2)}, \psi_2^{(a)}(\mu = -4q^2)$ and $\psi_2^{(b)}(\mu = -q^2)$ corresponding to $\lambda = 2, -3$.

On the other hand, the BDD scheme is consistent with the form

$$u = q^2(1 - 2\text{sech}^2 qx), \quad \lambda = 1, -2 \tag{16}$$

for $\psi_1(\mu = 0)$ and

$$u = q^2(1 - 6\text{sech}^2 qx), \quad \lambda = 2, -3 \tag{17}$$

for both the sets $\psi_2^{(a)}(\mu = -3q^2)$ and $\psi_2^{(b)}(\mu = 0)$.

To interpret the above results, a few remarks on SUSY are in order [28]. We first of all verify that not only (13) but also $u = v^2 - v' + \mu$ carries a solution of the generalized KdV (14) into a solution of the KdV.

Denoting

$$V^{(\pm)} \equiv u^\pm - \mu = v^2 \mp v' \tag{18}$$

we notice that the combination $V^{(\pm)}$ can be identified as the usual partner potentials of SUSYQM.

To examine the role of $V^{(\pm)}$ in the present context, let there be a Hamiltonian $H_1$ with potential $V_1$ that is asymptotically vanishing and having a set of n discrete eigenvalues $\mu_1, \mu_2, ......, \mu_n$. If we define $V^+ = V_1 - \mu_{n+1}$ then, in unbroken SUSY, we at once know that the spectra of $V^{(+)}$ and those of $V^{(-)}$ are one to one except that the latter has an additional $\mu = 0$ state. In other words, the eigenvalues of $V^{(-)}$ are $\mu_1 - \mu_{n+1}, \mu_2 - \mu_{n+1}, ....., \mu_n - \mu_{n+1}$ and 0. This means that the Hamiltonian $H_2$ with potential $V_2$ defined by $V_2 = V^{(-)} + \mu_{n+1}$ has (n+1) discrete eigenvalues $\mu_1, \mu_2, ....., \mu_n, \mu_{n+1}$.

Let us apply the above ideas to the simple case of $V_1 = 0$ and generate the corresponding potential $V_2$ with a single bound state with $\mu = -q^2$ [29]. We have $V^{(+)} = v^2 - v' = q^2 > 0$: in other words, $V^{(+)}$ has no bound state at all. Solving we get $v = -q\tanh qx$ (i.e. the 1-soliton result) which in turn gives $V^{(-)} = q^2(1-2\text{sech}^2 qx)$ that supports a zero energy ($\mu = 0$) bound state $\psi_0 \sim sech qx$ :

$$H_-\psi_0 = \psi_0'' + (v^2 + v')\psi_0 \tag{19}$$

Thus $V_2 = -2q^2\text{sech}^2 qx$ has a single bound state.

We immediately recognize $V_2$ and $V^{(-)}$ to be the PDM potential u for the ZK and BDD schemes respectively corresponding to the 1-soliton case. The same is true for the 2-soliton results with $v$ matching with the 2-soliton solutions and $V^{(-)}$

emerging similar to (17).

One-dimensional supersymmetric approach to PDM quantum systems has been explored before in PDM scenarios. The partner potentails were found to obey [22] the same PDM dependence but in different potentials. The approach of this work is however different in spirit from such a viewpoint in that we have sought to establish a link between a hierarchy of reflectionless potentials ( corresponding to the stationary soliton solutions of the KdV ) with an arbitrary bound state spectrum and those of SUSY in PDM models for suitable values of the ambiguity parameters. Our starting potential pertaining to the free-particle case $V(x) = V_0$ can be made to coincide with $V^{(+)}$ by choosing, for example, $\epsilon = 3q^2$ in the 1-soliton case and $\epsilon = 12q^2$ in the 2-soliton case.

Finally, we can extend our treatment to other special cases of the effective potential $V_{eff}$ namely those of the Bastard [30] and Li and Kuhn (redistributed) [31] Hamiltonians. For the 1-soliton result of (15), $u$ for the Bastard scheme is $u = -q^2(1 + 3\text{sech}^2 qx)$ $(\mu = -2q^2)$ while for the 2-soliton results given by (16), $u$ turns out to be $-q^2(1 + 6\text{sech}^2 qx)$ both for $\psi_2^{(a)}$ and $\psi_2^{(b)}$, with an associated $\mu$-value of $\mu = -5q^2$ and $\mu = -2q^2$ respectively. However, in the Bastard model $\lambda$ is non-integral. A non-integral $\lambda$ also emerges in the Li-Kuhn scheme where we find $u = -2q^2\text{sech}^2 qx(\mu = -q^2)$ for the 1-soliton solution and $u = -6q^2\text{sech}^2 qx$ for both the 2-soliton solutions of $\psi_2^{(a)}(\mu = -4q^2)$ and $\psi_2^{(b)}(\mu = -q^2)$.

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