

Phase transitions in Ising model on a Euclidean network

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A one dimensional network on which there are long range bonds at lattice distances $l > 1$ with the probability $P(l) \propto l^{-\delta}$ has been taken under consideration. We investigate the critical behavior of the Ising model on such a network where spins interact with these extra neighbours apart from their nearest neighbours for $0 \leq \delta < 2$. It is observed that there is a finite temperature phase transition in the entire range. For $0 \leq \delta < 1$, finite size scaling behaviour of various quantities are consistent with mean field exponents while for $1 \leq \delta \leq 2$, the exponents depend on δ . The results are discussed in the context of earlier observations on the topology of the underlying network.

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I. INTRODUCTION

During the last few years there has been a lot of activity in the study of networks once it was realised that networks of diverse nature exhibit some common features in their underlying structure. The Watts-Strogatz (WS) model [1] was proposed to simulate the small-world feature of real networks (i.e., the property that the networks have a small diameter as well as a large clustering coefficient). In this model, the nodes are placed on a ring and each node has connection to a finite number of nearest neighbours initially. The nearest neighbour links are then rewired with a rewiring probability p_r to form random long range links. This model displays phase transitions from a regular to a small world to a random network by varying the single parameter p_r representing disorder. The regular to small world network transition was shown to take place at $p_r \rightarrow 0$. In a slight variation of the WS model, the addition type network was considered in which random long range bonds are added with a probability p , keeping the nearest neighbour links undisturbed. Phase transition to a small world network was again observed for $p \rightarrow 0$ [2, 3].

Phase transitions in networks can also be driven by factors other than the above kind of disorder. In many real world networks, the linking probability is dictated by factors like Euclidean distances separating them, aging etc. It is possible to achieve phase transitions in the network by tuning the parameters governing such factors. These networks are indeed not just theoretical concepts. Many real world networks like the Internet at the router level, transport networks, power grid network and even collaboration networks are indeed described on a Euclidean space in which the geographical locations of the nodes play an important role in the construction of the network [4, 5, 6, 7]. Similarly, aging factor is important in social networks and citation network [3, 8, 9] where it

is found that linking with older nodes are generally less probable.

In the theoretical modelling of Euclidean networks [10], it is usually assumed that two nodes separated by a Euclidean distance l are connected with the probability $P(l)$ which follows a power law variation, i.e.,

$$P(l) \sim l^{-\delta}. \quad (1)$$

In one dimension, where the nodes are placed on a ring, the typical networks generated for different values of δ and the corresponding adjacency matrix are shown graphically in Fig. 1. Growing networks on Euclidean spaces where the linking probability is modified with such a probability factor have also been considered recently [11]. In a real situation, however, $P(l)$ may not have such a well-defined behaviour [5, 7].

A variation of $P(l)$ in the above form was first studied by Kleinberg [12] on a two dimensional plane with the aim to find out how navigation in the network depends on the parameter δ . Later, a number of other properties of networks (both static and dynamic) with such a probabilistic attachment have been studied [10, 13, 14, 15, 16, 17].

In the present study, we have reconsidered a static one dimensional Euclidean network (OEN) with connection probability at distance l given by eq (1). We are interested in the phase transitions which can be achieved by varying the parameter δ . Although this issue has been addressed earlier [13, 14, 15, 16], there remain some questions about the exact nature of phase transitions in the region $0 < \delta < 2$. We have tried to resolve the problems by taking a different approach here. In the earlier studies, the topological features like the diameter, shortest paths and the clustering coefficient of the network had been studied. Here we have considered Ising spins on the nodes of such a network. The critical behaviour of this system is expected to reflect the nature of the network at different values of δ indirectly. That the Ising model undergoes phase transitions on the addition type WS network is an established fact [18, 19, 20]. It has also been shown that in the small world phase the addition type WS has a mean field nature [21, 22].

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In section II, we have reviewed the available results of the Euclidean networks and also justified the necessity of the present investigations. In section III, studies of Ising models on networks have been briefly reviewed. Our results of the study of the Ising model on a Euclidean network is given in section IV. Finally, in section V, summary and conclusions are given.

II. NETWORK ON EUCLIDEAN SPACE

The one dimensional Euclidean network (OEN), with the connection probability given by eq (1), is equivalent to the addition type WS model at $\delta = 0$ when addition of long range links can take place at any length scale. The choice of the number of long range neighbours in this OEN should be such that it allows a phase transition at $\delta = 0$. In the addition type WS model, this transition for a system of N nodes is achieved for $p = 1/N$ (implying $p \rightarrow 0$ in the thermodynamics limit). Connection probability $p = 1/N$ implies a total number of N long range edges in the system or the existence of one long range bond per node on an average. Therefore in order to obtain a phase transition in the OEN, it is sufficient to keep N long range bonds. For large values of δ the nodes in the network have short range connections only and therefore it behaves as a regular network [13, 14, 15]. Thus there should exist at least one phase transition in this network.

In all, there could be four kinds of behaviour of this network: (a) Regular: when the network is like a short ranged one dimensional system (b) finite dimensional: when the network behaves as a system with effective dimensionality greater than one but still short ranged (c) Small world and (d) Random. A few studies in which these phase transitions have been investigated have not been able to give a concrete or unique answer. The network behaves as a regular network above $\delta = 2$; this is obtained in all the earlier studies. The interesting region is $0 \leq \delta \leq 2$ as here the network no longer behaves like a one-dimensional system. In [13], the behaviour of the OEN was studied by releasing a random walker on the network. The results indicated that the walker has the same behaviour as on a small world network for all $\delta < 2$.

To detect whether a network has small world behaviour, one can calculate the shortest distances separating two nodes, take the average and analyse its behaviour with the system size. The largest of these shortest paths is called the diameter of the network. The average shortest path and diameter of a network are expected to have the same scaling behaviour. Sen and Chakrabarti [14] studied the diameters of the OEN while the shortest paths were calculated by Moukarzel and de Menezes [15] giving contradictory results. In [14], it was found that the diameter behaved as $\ln N$ on chains of size N for all $\delta < 2$ and hence it was concluded that small world behaviour occurs for $0 \leq \delta \leq 2$ while in [15], it was argued that small world behaviour occurs only for $\delta < 1$. In the

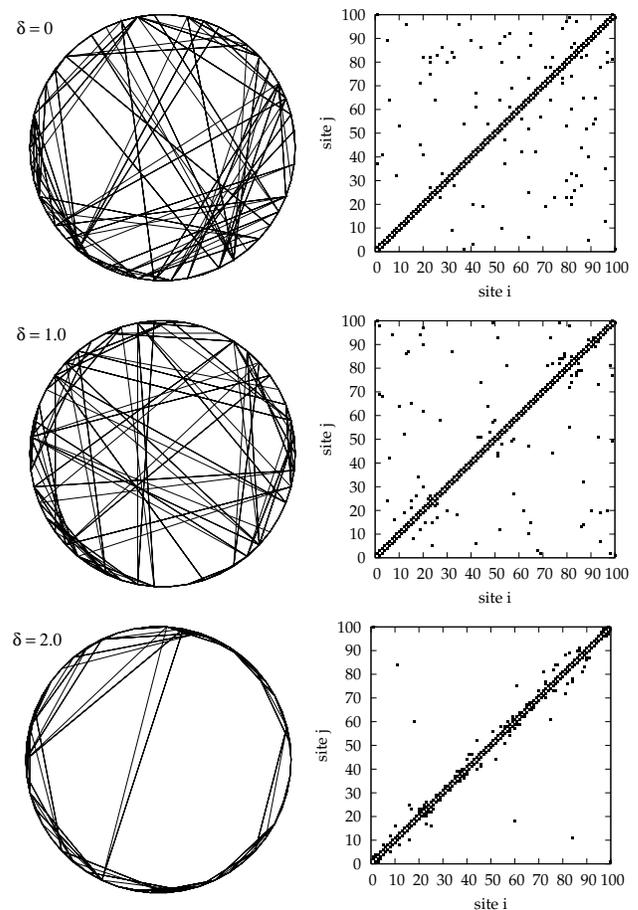


FIG. 1: Structures of the networks (left panel) and the corresponding adjacency diagrams (right panel) for $\delta = 0, 1.0, 2.0$ (top to bottom) in the model for a system of size $N = 100$.

region $1 < \delta < 2$, according to [15], the shortest distances scale as N^θ with the value of the shortest path exponent θ ($0 < \theta < 1/2$) depending on δ . The system sizes considered in [15] were much larger compared to those in [14] and the discrepancy of the results could be accounted for by this fact. It must also be mentioned here the above results are for $pN = 1$, results for other values of p have also been considered in these studies.

Random and small world phases can be distinguished by the clustering coefficient which remains finite in the latter. The estimate of the clustering coefficients of the present network [16] showed that below $\delta = 1$ it vanishes. This was considered to be a signature of the network being random for $\delta < 1$ and based on the findings of [14], it was concluded that the region $0 \leq \delta < 2$ was equally divided into a random (for $\delta < 1$) and a small world (for $\delta > 1$) phase. But this is by no means foolproof as in this particular network the clustering coefficient is bound to be small close to $\delta = 0$ as the number of nearest neighbour is small when δ is small. (In the original WS model, the clustering coefficient remained high for finite values of $p_r \neq 1$ only when the number of nearest neighbours

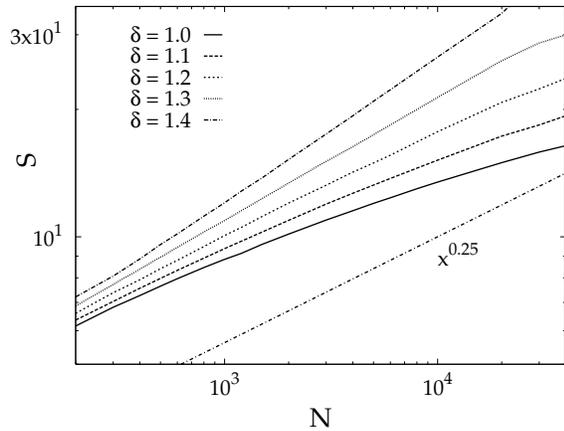


FIG. 2: Behavior of shortest path S with system size N .

was at least four to begin with.) Thus it may not indicate a random to small world phase transition at $\delta = 1$ as conjectured in [16], it could, in principle, correspond to the small world to a finite dimensional network as found in [15]. That there could be two transitions, one at $\delta = 1$ and the second at $\delta = 2$ was also supported by simple scaling arguments for the average bond lengths [16].

It has been mentioned in the beginning of this section that the the total number of random long range bonds per site has been kept equal to 1 on an average such that $p = 1/N$. This is also consistent with the fact that such a choice will keep the probability normalised. However, in an analytical approach there will be a problem at $\delta = 1$ where such a normalisation is not possible. This issue has been discussed in [13] and does not cause a problem in numerical calculations.

Navigation or searching on small world networks are known to give rise to shortest paths which do *not* behave as $\ln(N)$ but rather have sublinear variations with N except for special points: this was first detected in [12] for two dimensional lattices and later confirmed for the WS model [23]. In the present one dimensional Euclidean model this was again demonstrated in [17].

The more recent results [15, 16] indicate that there is indeed a phase transition at $\delta = 1$ and that the network behaves as a small world only for $\delta < 1$. In the present work we want to confirm this using a completely different approach. If the network's behaviour is that of a small world it should be reflected in the critical exponents of the Ising model which would assume mean field values. On the other hand, if it behaves like a finite dimensional system with effective dimension greater than one, a phase transition will be observed with critical exponents assuming values different from the mean field ones (note that the lower critical dimension of the Ising model is 1).

Another interesting point which can be studied by considering the Ising model on this network is the relation between the shortest path exponent θ and the effective dimension of the network for $1 < \delta < 2$. If one interprets

θ as the inverse of some effective dimension, d_{eff} , then, according to [15], for some values of $\delta > 1$, d_{eff} is actually greater than 4. In fact we have used the burning algorithm [24] to find out independently the shortest paths between *all* pairs of nodes in network of size N and found out that indeed θ has values below 0.25 upto $\delta \simeq 1.2$ (Fig. 2). Our estimates of θ are slightly larger than those obtained in [15], which may be due to the smaller system size considered, but still we find $\theta < 0.25$ over a considerable range of values of $\delta > 1$. The question that naturally arises is whether the Ising model shows mean field behaviour here as the effective dimension is higher than the upper critical dimension, i.e., does it have a mean field behaviour even though the network is not a small world?

III. ISING MODELS ON NETWORKS

Ising models with long range neighbours have been considered in different contexts in many earlier studies. It is well known that the short ranged Ising model on a one dimensional lattice does not show any phase transition while interaction with all the neighbours gives rise to a mean field like phase transition. The occurrence of the random long ranged bonds, also enables a phase transition [18, 19, 21, 22, 25, 26, 27].

In an analytical study of the Ising model on a WS type network, it was assumed that there are two nearest neighbours and additional long range neighbours occur randomly with probability $p = \rho/N$ [19] where ρ is a parameter. Phase transition occurs here for $\rho \geq 1$ implying that the addition of one long range bond to each node, (on an average) could enhance a phase transition. In the numerical studies also, the addition of one random long range neighbour per site enables a phase transition [25]. The fact that the Ising model on these networks show a phase transition even when the additional long range connections are finite in number (i.e., $O(N)$) compared to that of the the infinite range model (where there are $O(N^2)$ number of bonds) is an evidence of the key role played by the randomness in these networks [26].

Another kind of long range Ising model has been considered in which the interactions decrease with the distance as a power law, i.e, the Hamiltonian is given by [28]

$$H = -\sum_{ij} J_{ij} S_i S_j, \quad (2)$$

where $J_{ij} \sim \frac{1}{r_{ij}^\mu}$; r_{ij} is the distance separating the spins at site i and j . In the fully connected model, phase transitions occur at the values $\mu = 3/2$ and $\mu = 2$; for $\mu < 3/2$, the system is meanfield like while for $3/2 < \mu < 2$ it is like a finite dimensional lattice and for $\mu > 2$, it behaves like a one dimensional system with no phase transition. On a small world network, the interactions exist randomly among neighbours, maintaining the form of J_{ij} as above. However, there is no phase transition displayed by this model [29] except at $\mu = 0$ where it coincides with the WS model.

As mentioned earlier, the critical behaviour of Ising model on a WS addition type network is mean field type. The intriguing feature is the validity of the finite size scaling analysis in the mean field regime. Moreover, it has been observed [25] that the data collapse in the finite size scaling analysis can be achieved when $|T - T_c|$, the deviation from the critical temperature is scaled by the factor $N^{-1/\bar{\nu}}$ in the scaling argument with $\bar{\nu} = 2$. Interpreting $\bar{\nu}$ as νd , where ν is the critical exponent for the correlation function and d the effective dimensionality of the system, this could imply that one has effectively a system with $d = 4$ with the mean field value of $\nu = 1/2$. This effective dimension is identical to the upper critical dimension of the Ising model.

Phase transition of the Ising model has also been observed in scale-free networks [30, 31] where the transition temperature diverges with the system size. Several other aspects of Ising models on networks have been considered recently, for example the self-averaging property of Ising model on networks[26], quenching dynamics [32] etc., which are not directly related to the content of the present paper.

IV. ISING MODEL ON EUCLIDEAN NETWORK

We have considered an Ising model on a finite chain of length N with periodic boundary condition. The nodes are assigned position coordinates $1, 2, \dots, N$ along the chain. Each node is connected to its two nearest neighbours. In order to generate the long range bonds, two nodes are selected randomly. If $l > 1$ is the distance separating them, they are connected with probability $P(l)$ as given by eq (1). The process is repeated $N/2$ times generating N long range bonds (each bond is counted twice so that there are N long range bonds in the system). This ensures that there is one long range neighbour for each site on an average. As has been mentioned in the previous section, this is sufficient to achieve a phase transition. For each realisation of the network, the dynamical evolution from a uniform state (all spins up) was allowed following a Metropolis algorithm for different temperatures T .

We have computed the following quantities on this network after it reaches equilibrium:

- *Magnetisation per spin* $m = \sum_i S_i / N$
- *Binder cumulant* $U = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}$.
- *Susceptibility per spin* is calculated from the fluctuation of the order parameter:

$$\chi = \frac{N}{K_B T} [\langle M^2 \rangle - \langle M \rangle^2]$$

where M is the total magnetisation, T is the temperature and K_B the Boltzmann's constant.

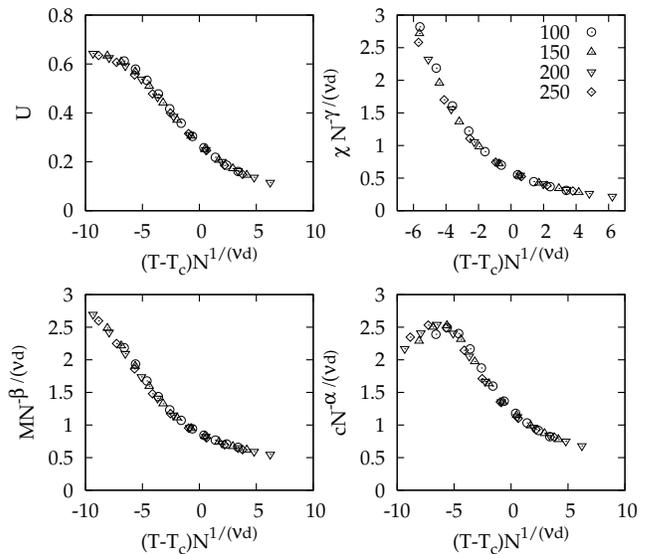


FIG. 3: Data collapse for different system sizes $N = 100, 150, 200, 250$ for U, χ, M and c , for $\delta = 0.6$ ($\bar{\nu} = \nu d$).

- *Specific heat per spin* is calculated from the fluctuation of the energy of the system as

$$c = \frac{N}{(K_B T)^2} [\langle E^2 \rangle - \langle E \rangle^2]$$

where E is the energy of the system.

A. Results

From the intersection of U for different N in the plot of U vs T we estimate T_c and a data collapse is obtained by plotting U vs $(T - T_c)N^{1/\bar{\nu}}$. With this value of $\bar{\nu}$, one can now estimate the exponents β, γ and α , the critical exponents for the order parameter m , susceptibility χ and specific heat c respectively using finite size scaling. Figures 3 and 4 show the data collapse for these quantities for $\delta = 0.6$ and $\delta = 1.4$. Similar results have been obtained for other values of δ .

Transition temperature

As expected we find a transition temperature T_c which decreases with δ (Fig. 5). T_c varies very slowly in the region $0 < \delta < 0.5$ and much faster for higher values of δ .

Critical exponents

From the finite size scaling analysis, we obtain the critical exponents $\bar{\nu}, \beta, \gamma$ and α (Figs 6,7). We find that $\bar{\nu}$ is equal to 2 for the entire range $0 \leq \delta < 1$. However, for $\delta > 1$ we find that $\bar{\nu}$ decreases and then rises again.

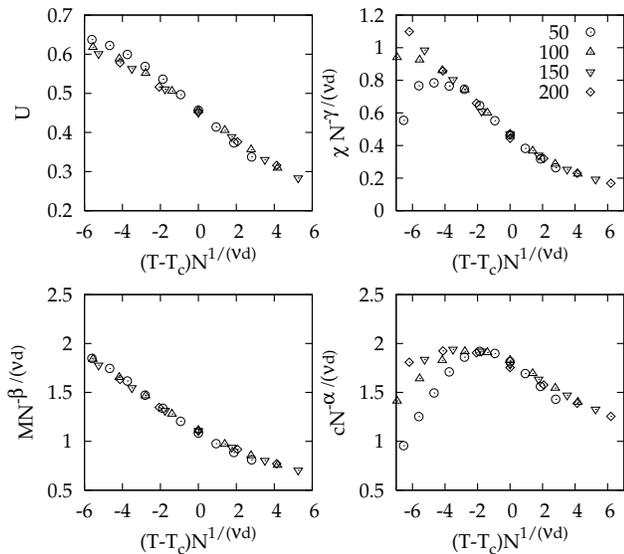


FIG. 4: Data collapse for different system sizes $N = 50, 100, 150, 200$ for U , χ , M and c , for $\delta = 1.4$ ($\bar{\nu} = \nu d$).

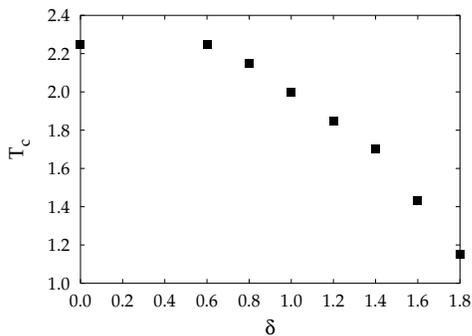


FIG. 5: Variation of critical temperature T_c with δ . Error in this and the next two figures are of the order of the size of the data points.

The magnetisation exponent β is equal to 0.5 for $\delta < 1$, while for $\delta \geq 1$, it decreases with δ . The susceptibility exponent γ is close to 1 for $\delta < 1$ and increases with δ beyond $\delta = 1$. The specific heat exponent α remains constant at a value close to 0.1 for $\delta < 1$ and appears to deviate from this value as δ is made larger than 1. It becomes small as δ approaches 2.

The behaviour of all the quantities are consistent with the fact that for $\delta < 1$, there is a mean field behaviour of the Ising model (consistent with the available solution for the random long-range bond small world network [27]) and for $\delta > 1$, its behaviour is like a finite dimensional lattice with effective dimension greater than one.

As has been argued for $\delta = 0$, here also one can interpret $\bar{\nu}$ as νd , where d is the effective dimension, such that it satisfies $\bar{\nu} = 2 - \alpha$. Since below $d = 4$ α is nonzero, $\bar{\nu}$ decreases from 2 at $\delta > 1$. Theoretically, the behaviour of α is non-monotonic as a function of dimensionality d as d varies from 4 to 1; α is zero for both $d = 4$ and $d = 2$

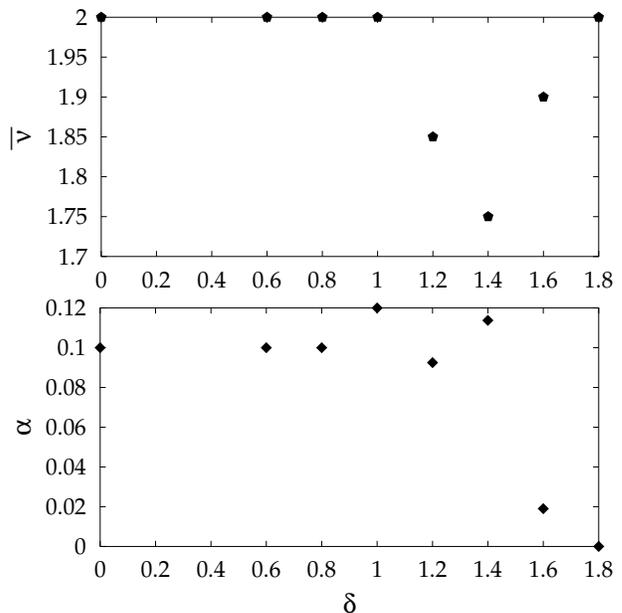


FIG. 6: Variation of $\bar{\nu}$ and specific heat exponent α with δ .

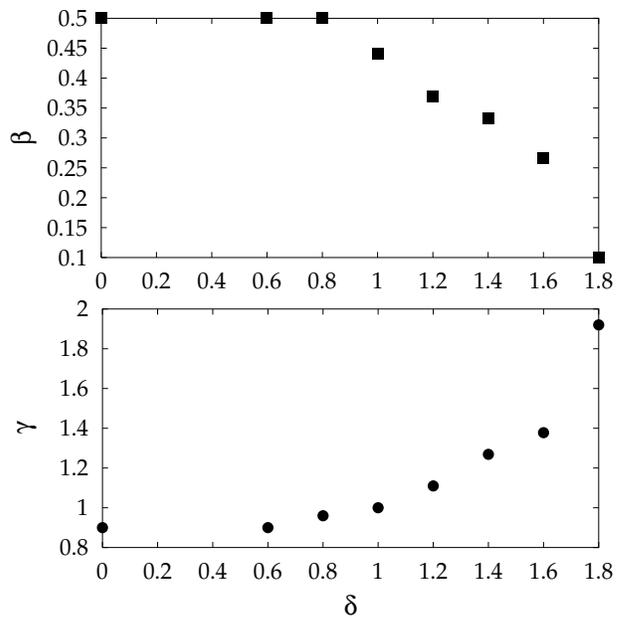


FIG. 7: Variation of magnetisation exponent β and susceptibility exponent γ with δ .

and this behaviour is reflected in the fact that $\bar{\nu}$ again rises to 2 at around $\delta = 1.8$. However, our result for α does not show this non-monotonic behaviour very accurately, the reason possibly being that the magnitude of α is small ($O(0.1)$) even when it is non-zero and is difficult to estimate accurately in a numerical study.

Mean field value for γ is 1.0 which comes out to be slightly lower in our estimate and also the value of α is higher than the corresponding 0, but the scaling relation

$\alpha + 2\beta + \gamma = 2$ is obeyed with a high degree of accuracy. Although there are deviations of α and γ from the mean field values, the fact remains that they show no variation over the region $0 \leq \delta < 1$, indicating that the critical behaviour for $\delta < 1$ is identical to that for $\delta = 0$, which is known to be mean field like.

B. Effective dimension

Our results confirm that the network behaves as a finite dimensional system in the region $1 < \delta < 2$. However, the finite size scaling analysis of various quantities in the Ising model does not allow the estimation of d and ν independently. It may be expected that the behaviour of the average shortest path should give us some idea about the effective dimension. However, as we argue below, it is not possible. The estimate of the exact average shortest path is consistent with the mean field picture for $\delta < 1$ as it shows a logarithmic increase with N (small world effect). As mentioned before, for $\delta > 1$, it has a power law increase N^θ and we have a region where $\theta < 0.25$. So if one interpretes θ as a inverse dimension, the dimensionality is greater than 4 here. Even then, the Ising model does not show mean field behaviour here. From this we conclude that although there is an effective dimension coming out from the topological behaviour of the network, it does not exactly act as the spatial dimension as far as physics of Ising model is concerned.

TABLE I: Table showing the effective dimension and comparative values of νd from ϵ expansion (ee) and $\bar{\nu}$ from finite size scaling (fss) analyses.

δ	ν_{eff}	d_{eff}	νd (ee)	$\bar{\nu}$ (fss)
1.2	0.56	3.34	1.85	1.85
1.4	0.59	2.90	1.72	1.75
1.6	0.61	2.74	1.66	1.9
1.8	0.72	1.36	0.98	2.0

We adopt a different approach to estimate ν and d which works well for δ close to 1. The behaviour of the critical exponents is fully consistent with the fact that the effective dimensionality d decreases as δ is made larger than 1 (e.g., the value of γ increases while that of β decreases). In order to find out d , we use the epsilon expansions of α , β and γ to first order, each of which gives an independent estimate of $\epsilon = 4 - d$:

$$\begin{aligned}\epsilon &= 3(1 - 2\beta) \\ \epsilon &= 6(\gamma - 1) \\ \epsilon &= 6\alpha.\end{aligned}$$

Ideally, all three estimates should give similar value of ϵ , but here we have used the expansions upto linear terms only and the estimates may not agree very well (especially for the estimate from α and to some extent, that from γ). It is not very convenient to use the expansions

to higher degrees also. To take care of this fact, we take $\langle \epsilon \rangle$, the average of the above three estimates of ϵ , and use it to find out the value of ν from the equation

$$\nu = \frac{1}{2} \left(1 + \frac{\langle \epsilon \rangle}{6} \right). \quad (3)$$

Thus an estimate of νd is obtained which can be compared to $\bar{\nu}$ obtained from the finite size scaling. We see that when $\delta - 1$ is small, the agreement is nice (Table 1). Naturally, when δ is considerably away from 1, the effective dimensionality decreases and the epsilon expansions are not much useful here, particularly for α and γ . This is reflected in the mismatch of the estimates from epsilon expansion and the finite size scaling analysis for $\delta = 1.6$ and $\delta = 1.8$.

V. SUMMARY AND CONCLUSIONS

We have investigated the behaviour of the Ising model on a one dimensional network in which there are links to random long range neighbours existing with a probability which varies inversely with the distance along the original chain following eq (1). The effective dimensionality of the network deviates from 1 continuously as δ is made smaller than 2 and a finite temperature phase transition is observed. The transition temperature increases as δ is made smaller and the critical exponents vary with δ as the dimensionality changes. The results show that there is a mean field behaviour for $\delta < 1$ and a finite dimensional behaviour for $1 < \delta < 2$. In the mean field regime, the finite size scaling analysis works with an effective dimensionality four, the upper critical dimension of the Ising model. The present study confirms the conclusion made in [15] that the small world behaviour of the underlying network exists for $\delta < 1$ only.

Our results also show that the effective dimensionality cannot be simply extracted from the behaviour of the shortest paths on this network although the latter shows a power law behaviour with the number of nodes. We have used an alternative method to estimate the effective dimensionality which works quite well for δ close to 1.

The present study was conducted to address certain issues related to the topology of a network but the study of phase transition of Ising model on networks has its own inherent interest as the Ising model has got wide applications in many interdisciplinary areas. For example, the study and nature of phases in such models gives us important insight into social opinion dynamics [33] in closely knit populations, where the networks look very similar to the models discussed here, and opinions being modelled as states of spin vectors at each site.

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