# OPERATIONAL REPRESENTATIONS FOR THE CLASSICAL HERMITE POLYNOMIALS

by

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DMS–535–IR February 1990

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1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 33A65; Secondary 47E05.

*Key words and phrases*. Classical orthogonal polynomials, Hermite polynomials, differential operators.

In this paper the authors aim at presenting a systematic analysis of the various known (or seemingly new) operational representations for the classical Hermite polynomials. The inter-relationships between many of these results, and the potential for their applications in other related areas of interest, are also indicated.

## 1. INTRODUCTION

The orthogonal polynomials $H_n(x)$ and $He_n(x)$, defined by

$$H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} (2x)^{n-2k} = 2^{n/2} He_n(x\sqrt{2}), \tag{1}$$

are both known in the literature as the Hermite polynomials. In the present paper we shall study this important member of the family of the classical orthogonal polynomials (*cf.* [26, p. 29]; see also [16, p. 189]) in the relatively more popular notation $H_n(x)$; the reader will have no difficulty in restating each result in terms of the alternative notation $He_n(x)$, which is used widely in statistics (see, *e.g.*, [20, p. 267]).

For the classical Hermite polynomials, the most frequently encountered operational representation is the so-called Rodrigues formula:

$$H_n(x) = (-1)^n \exp(x^2) D^n\{\exp(-x^2)\} \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{2}$$

where, and throughout this note, $D \equiv d/dx$, and $\mathbb{N} = \{1,2,3,\cdots\}$.

The usefulness of the operational representation (2) in the study of the Hermite polynomials cannot be overemphasized. Nevertheless, with a view to giving a remarkably elegant proof of the well-known linearization relation:

$$H_m(x)H_n(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\ 2^k\ H_{m+n-2k}(x), \tag{3}$$

which was derived markedly differently by Feldheim [11] and Watson [27], Burchnall [2] developed (from (2) *and* the Leibniz rule) the operational formula:

$$(D-2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x)\ D^k, \tag{4}$$

which readily yields the operational representation:

$$H_n(x) = (2x-D)^n \cdot 1. \tag{5}$$

(See also Rota *et al*. [18, p. 723] and Rota [17, p. 45].)

In view of Burchnall's novel application of the operational representation (5) in proving (3), and motivated by a more recent demonstration by Subramanian ([23],[24]) unveiling the beauty of (5) by using it to rederive several important properties of the Hermite polynomials, we aim at presenting a systematic analysis of the various other known (or seemingly new) operational representations for the Hermite polynomials. We also indicate the inter-relationships between many of these results and the potential for their applications in other related areas of interest.

## 2. FURTHER OPERATIONAL FORMULAS AND OPERATIONAL REPRESENTATIONS

We begin by recalling the following operational formula analogous to (4):

$$\prod_{j=1}^{n} (xD-2x^2-n+j) = x^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x)\ D^k, \tag{6}$$

which was given by Chatterjea [4]. In fact, Chatterjea [4, p. 185] proved by mathematical induction that

$$x^{-n} \prod_{j=1}^{n} (xD-2x^2-n+j) \equiv (D-2x)^n, \tag{7}$$

which obviously yields (6) as an immediate consequence of Burchnall's result (4).

Replacing $j$ on the left-hand side of (6) by $j-1$, and then reversing the order of the product, we obtain the operational formula:

$$\prod_{j=0}^{n-1} (xD-2x^2-j) = x^n \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} H_{n-k}(x)\ D^k, \tag{8}$$

which was derived differently in another paper by Chatterjea [5, p. 381, Equation (2.10)].

Next we set $x = \sqrt{t}$ in the Rodrigues formula (2), so that

$$D = 2\sqrt{t}\,\frac{d}{dt} \qquad (x = \sqrt{t}), \tag{9}$$

and we find from (2) rather trivially that

$$H_n(\sqrt{x}) = (-2)^n\ e^x(\sqrt{x}\ D)^n\ \{e^{-x}\}. \tag{10}$$

Now recall that the associated differential operator $\delta = xD$ satisfies the interesting relationships (*cf.*, *e.g.*, [22, p. 310]):

$$x^n\ D^n = \prod_{j=1}^{n} (\delta-j+1) = \prod_{j=1}^{n} (\delta-n+j) \tag{11}$$

and

$$f(\delta)\{\exp(g(x))h(x)\} = \exp(g(x))f(\delta+xg'(x))\{h(x)\}. \tag{12}$$

In particular, if $g(x) = \ell n\ x^a$, (12) gives

$$f(\delta)\{x^a\ h(x)\} = x^a\ f(\delta+a)\{h(x)\}, \tag{13}$$

whence

$$x^{na} \prod_{j=1}^{n} f(\delta+(j-1)a) = (x^a\ f(\delta))^n. \tag{14}$$

In terms of the Pochhammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \forall\ n \in \mathbb{N}, \end{cases} \tag{15}$$

a special case of the operator relationship (14) when $\alpha = -\frac{1}{2}$ and $f(\delta) = \delta$ may be rewritten as

$$x^{-n/2}(-2x\ D)_n = (-2)^n\ (\sqrt{x}\ D)^n \qquad (n \in \mathbb{N}_0), \tag{16}$$

whose validity for $n = 0$ is obvious. Thus the operational representation (10) can be put in its *equivalent* form:

$$H_n(\sqrt{x}) = x^{-n/2}\ e^{x}(-2x\ D)_n\{e^{-x}\}. \tag{17}$$

The operational representations (10) and (17) were proven by Chatterjea [6] by making use of the following immediate consequence of (6) and (8):

$$H_n(x) = (-x)^{-n} \prod_{j=1}^{n} (xD-2x^2-n+j) \cdot 1 \tag{18}$$

or, equivalently,

$$H_n(x) = (-x)^{-n} \prod_{j=1}^{n} (xD-2x^2-j+1) \cdot 1. \tag{19}$$

This last result (19) was obtained by Al-Salam [1, p. 383] who also exhibited its straightforward connection with the Rodrigues formula (2). Indeed, by the shift rule (12) with

$$g(x) = x^2 \quad \text{and} \quad h(x) = \exp(-x^2), \tag{20}$$

the second member of (19) is precisely

$$(-x)^{-n} \exp(x^2) \prod_{j=1}^{n} (xD-j+1)\{\exp(-x^2)\}$$

$$= (-x)^{-n} \exp(x^2)\; x^n D^n\{\exp(-x^2)\}$$

$$= (-1)^n \exp(x^2)\; D^n\{\exp(-x^2)\}, \tag{21}$$

where we have also applied the operator relationship (11). Conversely, if we reverse these steps, using (11) first and the shift rule (12) later, we can exhibit (18) or (19) as a straightforward consequence of the Rodrigues formula (2). In case we make use of (11) alone, the Rodrigues formula (2) will reduce, in just one step, to the alternative form (*cf.* [8]):

$$H_n(x) = (-x)^{-n} \exp(x^2) \prod_{j=1}^{n} (xD-n+j)\{\exp(-x^2)\} \tag{22}$$

or, equivalently,

$$H_n(x) = (-x)^{-n} \exp(x^2) \prod_{j=1}^{n} (xD-j+1)\{\exp(-x^2)\}, \tag{23}$$

which incidentally is already implied by the very first member of (21).

Operational representations for the Hermite polynomials can also be derived by exploiting their connection with the classical Laguerre polynomials:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} . \tag{24}$$

Numerous interesting operational representations for the Laguerre polynomials have appeared in the literature (see, for details, an earlier work by Srivastava and Chatterjea [19]). In particular, Carlitz [3] gave the operational representation:

$$L_n^{(\alpha)}(x) = \frac{1}{n!} \prod_{j=1}^{n} (xD-x+\alpha+j) \cdot 1, \tag{25}$$

which, in view of the shift rule (12) with

$$g(x) = x \quad \text{and} \quad h(x) = e^{-x}, \tag{26}$$

can be rewritten in the form:

$$L_n^{(\alpha)}(x) = \frac{e^x}{n!} \prod_{j=1}^{n} (xD+\alpha+j)\{e^{-x}\} \tag{27}$$

or, equivalently,

$$L_n^{(\alpha)}(x) = \frac{e^x}{n!} (xD+\alpha+1)_n\{e^{-x}\}, \tag{28}$$

where we have simply used the definition (15).

Since[1] (*cf.*, *e.g.*, [21, p. 75, Equation 1.8(27)])

$$H_{2n+\epsilon}(x) = (-1)^n \, 2^{2n+\epsilon} \, n! \; x^\epsilon \, L_n^{(\epsilon-\frac{1}{2})}(x^2) \qquad (\epsilon = 0 \quad \text{or} \quad 1), \tag{29}$$

we find from Carlitz's result (25) and its variations (27) and (28) that

$$H_{2n+\epsilon}(\sqrt{x}) = (-1)^n \, 2^{2n+\epsilon} \, x^{\epsilon/2} \prod_{j=1}^{n} (xD-x+j+\epsilon-\tfrac{1}{2}) \cdot 1 \tag{30}$$

$$(\epsilon = 0 \quad \text{or} \quad 1),$$

[1]Formula (29) appears in [21, p. 75, Equation 1.8(27)] with a misprint (see also [20, p. 55, Equation II.8(33)]).

$$H_{2n+\epsilon}(\sqrt{x}) = (-1)^n \, 2^{2n+\epsilon} \, x^{\epsilon/2} \, e^x \prod_{j=1}^{n} (xD+j+\epsilon-\tfrac{1}{2})\{e^{-x}\} \tag{31}$$

$$(\epsilon = 0 \text{ or } 1),$$

and

$$H_{2n+\epsilon}(\sqrt{x}) = (-1)^n \, 2^{2n+\epsilon} \, x^{\epsilon/2} \, e^x \, (xD+\epsilon+\tfrac{1}{2})_n \, \{e^{-x}\} \tag{32}$$

$$(\epsilon = 0 \text{ or } 1).$$

The last two operational representations (31) and (32) may be compared with the earlier results (10) and (17). Formula (30), on the other hand, is comparable with the known results (18) and (19).

## 3. APPLICATIONS

Consider the following homogeneous linear differential equation[2] of order $n$:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} x^k \, D^{n-k} \, y = 0, \tag{33}$$

which incidentally is *not* of the Cauchy-Euler type. Klamkin [14] observed that the differential equation (33) can be reduced to a linear equation with constant coefficients by means of the operational formula:

$$(i\sqrt{2})^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} x^k \, D^{n-k} \, y$$

$$= \exp(\tfrac{1}{2}x^2) \, H_n\left[\frac{iD}{\sqrt{2}}\right]\{y \exp(-\tfrac{1}{2}x^2)\}, \tag{34}$$

[2]Two special cases of the differential equation (33) when $n = 3$ and $n = 4$ are reported in Kamke [13, p. 511, Entry 3.21; p. 529, Entry 4.13].

involving Hermite polynomials. In fact, as already pointed out by Chatterjea [7], the operational formula (34) is equivalent to the identity:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} H_k(x)\ D^{n-k}\ y$$

$$= \exp(x^2)\ D^n\{y\ \exp(-x^2)\}, \tag{35}$$

which, in view of (2) and (5), is Burchnall's result (4).

Next we turn to some recent applications of other operational representations of the Hermite polynomials. Exponential operators such as $\exp(\lambda D^2)$ appear extensively in the literature on the Laplace transform. In particular, with the parameter $\lambda = -\frac{1}{4}$, one can easily see from the explicit representation (1) that[3]

$$H_n(x) = \exp(-\tfrac{1}{4}D^2)\ (2x)^n, \tag{36}$$

which provides yet another operational representation for the Hermite polynomials. Now, recalling the property:

$$\exp(\lambda D^2)\{f(x)\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-t^2)\ f(x+2t\sqrt{\lambda})\,dt, \tag{37}$$

we find from (36) that (*cf.* [15, p. 254])

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-t^2)\ (x+it)^n\ dt, \tag{38}$$

which expresses the Hermite polynomial as an integral involving elementary functions.

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[3]See also a relevant paper by Gould and Hopper [12, p. 58 *et seq.*] in which an interesting generalization of the Hermite polynomials is accomplished by use of the general operator $\exp(\lambda D^r)$ $(r \in \mathbb{N})$.

The operational representation (36) was applied recently by Chatterjea [10] in his study of certain *trilinear* generating functions for the Hermite polynomials. Chatterjea [9], on the other hand, made use of the integral representation (38) to derive a number of integral formulas from various generating functions for the Hermite polynomials. For example, the integral formula:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-t^2+2iyt) \ (x+it)^n \ dt$$

$$= 2^{-n} \exp(-y^2) \ H_n(x-y) \tag{39}$$

was derived in this manner by Sultan and Chatterjea [25, p. 46, Equation (2.3)], and also by Chatterjea [9, p. 10, Equation (9)], by appealing to (38) and a well-known generating function for the Hermite polynomials (*cf.* [16, p. 197, Equation (1)]; see also [22, p. 419, Equation 8.4(13)]).

Other noteworthy examples of integral formulas resulting from (38) are

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(2i(y-zx)zt - (1-z^2)t^2) \ (x+it)^n \ dt$$

$$= 2^{-n} \ (1-z^2)^{-(n+1)/2} \exp\left[-\frac{(y-zx)z}{1-z^2}\right] H_n\left[\frac{x-yz}{\sqrt{1-z^2}}\right] \tag{40}$$

and

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-t^2+2iyte^{ia}) \ L_n^{(-n)}(2y^2-2y(x+it)e^{ia}) \ dt$$

$$= \frac{(y\ e^{ia})^n}{n!} \exp(-y^2 \ e^{2ia}) \ H_n(x-2y \cos a), \tag{41}$$

which were also derived by Chatterjea [9] using the aforementioned technique. Notice, however, that the *special* Laguerre polynomial $L_n^{(-n)}(z)$ involved in the last formula (41) is reducible *significantly* by virtue of the fact that

$$z^n = (-1)^n\, n!\; L_n^{(-n)}(z) \qquad (n \in \mathbb{N}_0), \tag{42}$$

which follows readily from the relatively more familiar relationship [26, p. 102, Equation (5.2.1)]:

$$L_m^{(-n)}(z) = (-z)^n \frac{(m-n)!}{m!}\; L_{m-n}^{(n)}(z) \qquad (n = 0,1,\cdots,m). \tag{43}$$

Making use of (42), the integral formula (41) assumes the *simpler* form:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-t^2+2iyt\; e^{ia})\; \{(x+it)e^{ia}-y\}^n\, dt$$

$$= (\tfrac{1}{2}e^{ia})^n \exp(-y^2\; e^{2ia})\; H_n(x-2y\, \cos\, a). \tag{44}$$

Now recall that

$$\cos\, a = \frac{1}{2}(e^{ia} + e^{-ia}),$$

so that (44) becomes

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-t^2+2iyte^{ia})\; (x-y\; e^{-ia}+it)^n\, dt$$

$$= 2^{-n} \exp(-y^2\; e^{2ia})\; H_n(x-y\; e^{-ia} - y\; e^{ia}). \tag{45}$$

Replacing $x$ by $x + y\, e^{-ia}$, and then $y$ by $y\, e^{-ia}$, (45) leads us immediately to the integral formula (39). Conversely, if we first replace $y$ in (39) by $y\, e^{ia}$, and then $x$ by $x - y\, e^{-ia}$, (39) would readily yield the integral formula (41) in its simplified form (44) or (45).

Numerous other applications of the operational representations for the Hermite polynomials can be found in the works cited in this paper.

**ACKNOWLEDGEMENTS**

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant A–7353.

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