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On Generalized Quasi-Einstein Manifolds Admitting Certain Vector Fields

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Abstract. The object of the present paper is to study some geometric properties of a generalized quasi-Einstein manifold. The existence of such a manifold have been proved by several non-trivial examples.

1. Introduction

A Riemannian or semi-Riemannian manifold (M^n, g), $n = dimM \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n}g \tag{1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to Besse([3], p. 432), (1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition impossed on their Ricci tensor([3],p.432-433). For instance, every Einstein manifold belongs to the class of Riemannian or semi-Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{2}$$

where $a, b \in \mathbb{R}$ and A is a non-zero 1-form such that

$$g(X, U) = A(X), \tag{3}$$

for all vector fields X. Moreover, different structures on Einstein manifolds have been studied by several authors. In 1993, Tamassy and Binh[29] studied weakly symmetric structures on Einstein manifolds.
A non-flat semi-Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (2).
It is to be noted that Chaki and Maity[6] also introduced the notoin of quasi-Einstein manifolds in a different

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way. They have taken a, b as scalars and the vector field U metrically equivalent to the 1-form A as a unit vector field. Such an n-dimensional manifold is denoted by $(QE)_n$. Quasi-Einstein manifolds have been studied by several authors such as Bejan[2], De and Ghosh[11], De and De[12] and De, Ghosh and Binh[13] and many others.
Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. Also, quasi-Einstein manifolds can be taken as a model of the perfect fluid spacetime in general relativity[10]. So quasi-Einstein manifolds have some importance in the general theory of relativity.
Quasi-Einstein manifolds have been generalized by several authors in several ways such as generalized quasi-Einstein manifolds([7],[14],[15],[16],[21],[24]), generalized Einstein manifolds[1], super quasi-Einstein manifolds([8],[18],[23]), N(k)-quasi-Einstein manifolds([9],[22],[27],[28]) and many others.
In a paper De and Ghosh[14] introduced the notion of generalized quasi-Einstein manifolds in another way. A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y), \tag{4}$$

where $a, b, c \in \mathbb{R}$ and A, B are two non-zero 1-forms such that

$$g(A, B) = 0, \quad ||A|| = ||B|| = 1.$$

The unit vector fields U and V corresponding to the 1-forms A and B respectively, defined by

$$g(X, U) = A(X), \; g(X, V) = B(X),$$

for every vector field X are orthogonal, that is, $g(U, V) = 0$. Such a manifold is denoted by $G(QE)_n$. If $c = 0$, then the manifold reduces to a quasi-Einstein manifold[6].

Gray[19] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class A consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi type tensor, i.e.,

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z).$$

The class B consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized Ricci recurrent manifold[17] if its Ricci tensor S of type (0,2) satisfies the condition

$$(\nabla_X S)(Y, Z) = \gamma(X)S(Y, Z) + \delta(X)g(Y, Z),$$

where γ and δ are non-zero 1-forms. If $\delta = 0$, then the manifold reduces to a Ricci recurrent manifold[25].

The present paper is organized as follows:
After introduction in Section 2, it is shown that if the generators U and V are Killing vector fields, then the generalized quasi-Einstein manifold satisfies cyclic parallel Ricci tensor. Section 3 deals with $G(QE)_n$ satisfying Codazzi type of Ricci tensor. In the next two sections we consider $G(QE)_n$ with generators U and V both as concurrent and recurrent vector fields. Finally, we give some examples of generalized quasi-Einstein manifolds.

2. The generators U and V as Killing vector fields

In this section let us consider the generators U and V of the manifold are Killing vector fields. Then we have

$$(\pounds_U g)(X, Y) = 0 \tag{5}$$

and

$$(\pounds_V g)(X, Y) = 0, \tag{6}$$

where £ denotes the Lie derivative.
From (5) and (6) it follows that

$$g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0 \tag{7}$$

and

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0. \tag{8}$$

Since $g(\nabla_X U, Y) = (\nabla_X A)(Y)$ and $g(\nabla_X V, Y) = (\nabla_X B)(Y)$, we obtain from (7) and (8) that

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0 \tag{9}$$

and

$$(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0, \tag{10}$$

for all X, Y.
Similarly, we have

$$(\nabla_X A)(Z) + (\nabla_Z A)(X) = 0, \tag{11}$$

$$(\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0, \tag{12}$$

$$(\nabla_X B)(Z) + (\nabla_Z B)(X) = 0, \tag{13}$$

$$(\nabla_Z B)(Y) + (\nabla_Y B)(Z) = 0, \tag{14}$$

for all X, Y, Z.
Now from (4) we have

$$\begin{aligned}(\nabla_Z S)(X, Y) = b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]\\ +c[(\nabla_Z B)(X)B(Y) + B(X)(\nabla_Z B)(Y)].\end{aligned} \tag{15}$$

Using (15) we obtain

$$\begin{aligned}(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = b[\{(\nabla_X A)(Y)\\ +(\nabla_Y A)(X)\}A(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}A(Y)\\ +\{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}A(X)] + c[\{(\nabla_X B)(Y)\\ +(\nabla_Y B)(X)\}B(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}B(Y)\\ +\{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}B(X)].\end{aligned} \tag{16}$$

By virtue of (9)−(14) we obtain from (16) that

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

Thus we can state the following theorem:

Theorem 2.1. *If the generators of a $G(QE)_n$ are Killing vector fields, then the manifold satisfies cyclic parallel Ricci tensor.*

3. $G(QE)_n$ satisfying Codazzi type of Ricci tensor

A Riemannian or semi-Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{17}$$

for all X, Y, Z.
Using (15) and (17), we obtain

$$\begin{aligned} b[(\nabla_X A)(Y)A(Z) - (\nabla_Y A)(X)A(Z) + A(Y)(\nabla_X A)(Z) - \\ A(X)(\nabla_Y A)(Z)] + c[(\nabla_X B)(Y)B(Z) - (\nabla_Y B)(X)B(Z) \\ +B(Y)(\nabla_X B)(Z) - B(X)(\nabla_Y B)(Z)] = 0. \end{aligned} \tag{18}$$

Putting $Z = U$ in (18) and using $(\nabla_X A)(U) = 0$ we get

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0, \quad i.e., \ dA(X,Y) = 0.$$

Similarly, putting $Z = V$ in (18) and using $(\nabla_X B)(V) = 0$ yields $dB(X,Y) = 0$.
Thus we can state the following:

Theorem 3.1. *If a $G(QE)_n$ satisfies the Codazzi type of Ricci tensor, then the associated 1-forms A and B are closed.*

Again putting $X = Z = U$ in (18) we have

$$(\nabla_U A)(Y) = 0, \tag{19}$$

which means that $g(X, \nabla_U U) = 0$ for all Y, that is, $\nabla_U U = 0$.
Similarly, putting $X = Z = V$ in (18) we have

$$(\nabla_V B)(Y) = 0, \tag{20}$$

which yields $\nabla_V V = 0$. This leads to the following theorem:

Theorem 3.2. *If a generalized quasi-Einstein manifold satisfies Codazzi type of Ricci tensor, then the integral curves of the vector fields U and V are geodesic.*

4. The generators U and V as concurrent vector fields

A vector field ξ is said to be concurrent if[26]

$$\nabla_X \xi = \rho X, \tag{21}$$

where ρ is a non-zero constant. If $\rho = 0$, the vector field reduces to a parallel vector field.

In this section we consider the vector fields U and V corresponding to the associated 1-forms A and B respectively are concurrent. Then

$$(\nabla_X A)(Y) = \alpha g(X,Y) \tag{22}$$

and

$$(\nabla_X B)(Y) = \beta g(X,Y), \tag{23}$$

where α and β are non-zero constants.
Using (22) and (23) in (15) we get

$$\begin{aligned} (\nabla_Z S)(X,Y) = b[\alpha g(X,Z)A(Y) + \alpha g(Y,Z)A(X)] \\ +c[\beta g(X,Z)B(Y) + \beta g(Y,Z)B(X)]. \end{aligned} \tag{24}$$

Contracting (24) over X and Y we obtain

$$dr(Z) = 2[b\alpha A(Z) + c\beta B(Z)], \tag{25}$$

where r is the scalar curvature of the manifold.
Again from (4) we have

$$r = an + b + c. \tag{26}$$

Since, $a, b, c \in \mathbb{R}$, it follows that $dr(X) = 0$, for all X. Thus equation (25) yields

$$b\alpha A(Z) + c\beta B(Z) = 0. \tag{27}$$

Since α and β are not zero, using (27) in (4), we finally get

$$S(X, Y) = ag(X, Y) + (b + \frac{b^2\alpha^2}{c\beta^2})A(X)A(Y).$$

Thus the manifold reduces to a quasi-Einstein manifold. Hence we can state the following theorem:

Theorem 4.1. *If the associated vector fields of a $G(QE)_n$ are concurrent vector fields, then the manifold reduces to a quasi-Einstein manifold.*

5. The generators U and V as recurrent vector fields

A vector fiels ξ corresponding to the associated 1-form η is said to be recurrent if[26]

$$(\nabla_X \eta)(Y) = \psi(X)\eta(Y), \tag{28}$$

where ψ is a non-zero 1-form.

In this section we suppose that the generators U and V corresponding to the associated 1-forms A and B are recurrent. Then we have

$$(\nabla_X A)(Y) = \lambda(X)A(Y) \tag{29}$$

and

$$(\nabla_X B)(Y) = \mu(X)B(Y), \tag{30}$$

where λ and μ are non-zero 1-forms.
Now, using (29) and (30) in (15) we get

$$(\nabla_Z S)(X, Y) = 2b\lambda(Z)A(X)A(Y) + 2c\mu(Z)B(X)B(Y). \tag{31}$$

We assume that the 1-forms λ and μ are equal, i.e.,

$$\lambda(Z) = \mu(Z), \tag{32}$$

for all Z. Then we obtain from (31) and (32) that

$$(\nabla_Z S)(X, Y) = 2\lambda(Z)[bA(X)A(Y) + cB(X)B(Y)]. \tag{33}$$

Using (4) and (33) we have

$$(\nabla_Z S)(X, Y) = \alpha_1(Z)S(X, Y) + \alpha_2(Z)g(X, Y),$$

where $\alpha_1(Z) = 2\lambda(Z)$ and $\alpha_2(Z) = -2a\lambda(Z)$.
Thus we can state the following:

Theorem 5.1. *If the generators of a $G(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence, then the manifold is a generalized Ricci recurrent manifold.*

6. Examples of $G(QE)_n$

In this section we prove the existence of generalized quasi-Einstein manifolds by constructing some non-trivial concrete examples.

Example 6.1. Let us consider a semi-Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = x^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \tag{34}$$

where $i, j = 1, 2, 3, 4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensors and the derivatives of the components of curvature tensors are

$$\Gamma^2_{11} = \Gamma^2_{33} = -\frac{1}{2x^2}, \quad \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^3_{23} = \frac{1}{2x^2},$$

$$R_{1221} = R_{2332} = -\frac{1}{2x^2}, \quad R_{1331} = \frac{1}{4x^2}, \quad R_{1232} = 0,$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensor R_{ij} are

$$R_{11} = R_{33} = -\frac{1}{4(x^2)^2}, \quad R_{22} = -\frac{1}{(x^2)^2}.$$

It can be easily shown that the scalar curvature of the resulting manifold $(\mathbb{R}^4, g)$ is $-\frac{3}{2(x^2)^3} \neq 0$. We shall now show that $(\mathbb{R}^4, g)$ is a generalized quasi-Einstein manifold.
Let us now consider the associated scalars as follows:

$$a = \frac{1}{(x^2)^3}, \quad b = -\frac{5}{2(x^2)^3}, \quad c = -\frac{2}{(x^2)^3}. \tag{35}$$

Again let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} \frac{1}{\sqrt{2}}\sqrt{x^2}, & \text{for i=1, 3} \\ 0, & \text{otherwise,} \end{cases} \tag{36}$$

$$B_i(x) = \begin{cases} \sqrt{x^2}, & \text{for i=2} \\ 0, & \text{otherwise,} \end{cases} \tag{37}$$

at any point $x \in \mathbb{R}^4$. To verify the relation (4), it is sufficient to check the following equations:

$$R_{11} = ag_{11} + bA_1A_1 + cB_1B_1, \tag{38}$$

$$R_{22} = ag_{22} + bA_2A_2 + cB_2B_2, \tag{39}$$

$$R_{33} = ag_{33} + bA_3A_3 + cB_3B_3, \tag{40}$$

since for the other cases (4) holds trivially. By virtue of (35), (36), (37) and (38) we get

$$\begin{aligned}\text{R.H.S. of (38)} &= ag_{11} + bA_1A_1 + cB_1B_1 \\ &= \frac{1}{(x^2)^3}x^2 + (-\frac{5}{2(x^2)^3})\frac{1}{2}(x^2) + 0 \\ &= -\frac{1}{4(x^2)^2} = R_{11} \\ &= \text{L.H.S. of (38).}\end{aligned}$$

By similar argument it can be shown that (39) and (40) are also true. We shall now show that the associated vectors A_i and B_i are unit.
Here

$$g^{ij}A_iA_j = 1, \;\; g^{ij}B_iB_j = 1, \;\; g^{ij}A_iB_j = 0.$$

Therefore the vectors A_i and B_i are unit and also they are orthogonal.
So, $(\mathbb{R}^4, g)$ is a generalized quasi-Einstein manifold.

Example 6.2. We consider a Riemannian manifold (M^4, g) endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^idx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 + (dx^4)^2, \tag{41}$$

where $i, j = 1, 2, 3, 4$. The only non-vanishing components of Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = -x^1, \;\; \Gamma^2_{33} = -\frac{x^2}{(x^1)^2}, \;\; \Gamma^2_{12} = \frac{1}{x^1}, \;\; \Gamma^3_{23} = \frac{1}{x^2},$$

$$R_{1332} = -\frac{x^2}{x^1}, \quad S_{12} = -\frac{1}{x^1x^2}.$$

It can be easily shown that the scalar curvature of the manifold is zero. We shall now show that $(\mathbb{R}^4, g)$ is a generalized quasi-Einstein manifold.
We take the associated scalars as follows:

$$a = \frac{1}{x^1(x^2)^2}, \;\; b = -\frac{8}{3(x^1)^2x^2}, \;\; c = -\frac{2}{3(x^1)^2x^2}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} \frac{1}{\sqrt{3}}, & \text{for i=1} \\ \frac{x^1}{\sqrt{3}}, & \text{for i=2} \\ \frac{x^2}{\sqrt{3}}, & \text{for i=3} \\ 0, & \text{for i=4} \end{cases}$$

and

$$B_i(x) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{for i=1} \\ -\frac{x^1}{\sqrt{2}}, & \text{for i=2} \\ 0, & \text{otherwise} \end{cases}$$

at any point $x \in M$. In our (M^4, g), (4) reduces with these associated scalars and 1-forms to the following equation:

$$S_{12} = ag_{12} + bA_1A_2 + cB_1B_2 \tag{42}$$

It can be easily prove that the equation (42) is true.
We shall now show that the associated vectors A_i and B_i are unit and also they are orthogonal.
Here,

$$g^{ij}A_iA_j = 1, \;\; g^{ij}B_iB_j = 1, \;\; g^{ij}A_iB_j = 0.$$

So, the manifold under consideration is a generalized quasi-Einstein manifold.

Example 6.3. [16] A 2-quasi-umbilical hypersurface of a space of constant curvature is a $G(QE)_n$, which is not a quasi-Einstein manifold.

Example 6.4. [16] A quasi-umbilical hypersurface of a Sasakian space form is a $G(QE)_n$, which is not a quasi-Einstein manifold.

Example 6.5. De and Mallick [16] considered a Riemannian metric g on R^4 by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2. \tag{43}$$

Then they showed that (M^4, g) is a generalized quasi-Einstein manifold, which is not a quasi-Einstein manifold.

Example 6.6. *Özgür* and Sular [24] assumed an isometrically immersed surface $\bar{M}$ in E^3 with non-zero distinct principal curvatures λ and μ. Then they considered the hypersurface $M = \bar{M} \times E^{n-2}$ in $E^{n+1}, n \geq 4$. The principal curvatures of M are $\tilde{\lambda}$, $\tilde{\mu}$, 0,..., 0, where 0 occures (n-2)-times. Hence the manifold is a 2-quasi umbilical hypersurface and so it is generalized quasi-Einstein.

Example 6.7. *Özgür* and Sular [24] assumed a sphere S^2 in E^{k+2} given by the immersion $f : S^2 \to E^{k+2}$ and BS^2 be the bundle of unit normal to S^2. The hypersurface M defined by the map $\varphi_t : BS^2 \to E^{k+2}$, $\varphi_t(x, \xi) = F(x, t\xi) = f(x) + t\xi$ is called the tube of radius t over S^2. It was proved in [5] that if (λ, λ) are the principal curvature of S^2 then the principal curvatures of M are $(\frac{\lambda}{1-t\lambda}, \frac{\lambda}{1-t\lambda}, -\frac{1}{t}, ..., -\frac{1}{t})$, where $-\frac{1}{t}$ occures (k-1)-times. So M is 2-quasi umbilical and hence it is generalized quasi-Einstein.

Example 6.8. The study of warped product manifold was initiated by Kru*č*kovi*č* [20] in 1957. Again in 1969 Bishop and O'Neill [4] also obtained the same notion of the warped product manifolds while they were constructing a large class of manifolds of negative curvature. Warped product are generalizations of the Cartesian product of Riemannian manifolds. Let $(\bar{M}, \bar{g})$ and (M^*, g^*) be two Riemannian or semi-Riemannian manifolds. Let $\bar{M}$ and M^* be covered with coordinate charts $(U; x^1, x^2, ..., x^p)$ and $(V; y^{p+1}, y^{p+2}, ..., y^n)$ respectively. Then the warped product $M = \bar{M} \times_f M^*$ is the product manifold of dimension n furnished with the metric

$$g = \pi^*(\bar{g}) + (f \circ \pi)\sigma^*(g^*), \tag{44}$$

where $\pi : M \to \bar{M}$ and $\sigma : M \to M^*$ are natural projections such that the warped product manifold $\bar{M} \times_f M^*$ is covered with the coordinate chart

$$(U \times V; x^1, x^2, ..., x^p, x^{p+1} = y^{p+1}, x^{p+2} = y^{p+2}, ..., x^n = y^n).$$

Then the local components of the metric g with respect to this coordinate chart are given by

$$g_{ij} = \begin{cases} \bar{g}_{ij} & \text{for i=a and j=b,} \\ f g^*_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{45}$$

Here $a, b, c, ... \in \{1, 2, ..., p\}$ and $\alpha, \beta, \gamma, ... \in \{p+1, p+2, ..., n\}$ and $i, j, k, ... \in \{1, 2, ..., n\}$. Here $\bar{M}$ is called the base, M^* is called the fiber and f is called warping function of the warped product $M = \bar{M} \times_f M^*$. We denote by Γ^i_{jk}, R_{ijkl}, R_{ij} and r as the components of Levi-Civita connection ∇, the Riemann-Christoffel curvature tensor R, Ricci tensor S and the scalar curvature of (M, g) respectively. Moreover we consider that, when Ω is a quantity formed with respect to g, we denote by $\bar{\Omega}$ and Ω^*, the similar quantities formed with respect to $\bar{g}$ and g^* respectively. Then the non-zero local components of Levi-Civita connection ∇ of (M, g) are given by

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc}, \ \ \Gamma^\alpha_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma}, \ \ \Gamma^a_{\beta\gamma} = -\frac{1}{2}\bar{g}^{ab} f_b g^*_{\beta\gamma}, \ \ \Gamma^\alpha_{a\beta} = \frac{1}{2f} f_a \delta^\alpha_\beta, \tag{46}$$

where $f_a = \partial_a f = \frac{\partial f}{\partial x^a}$. The local components $R_{hijk} = g_{hl}R^l_{ijk} = g_{hl}(\partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ij}\Gamma^l_{mk} - \Gamma^m_{ik}\Gamma^l_{mj}), \partial_k = \frac{\partial}{\partial x^k}$, of the Riemann-Christoffel curvature tensor R of (M, g) which may not vanish identically are the following:

$$R_{abcd} = \bar{R}_{abcd}, R_{a\alpha b\beta} = -f T_{ab} g^*_{\alpha\beta}, R_{\alpha\beta\gamma\delta} = f R^*_{\alpha\beta\gamma\delta} - f^2 G^*_{\alpha\beta\gamma\delta}, \tag{47}$$

where $G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ and

$$T_{ab} = -\frac{1}{2f}(\nabla_b f_a - \frac{1}{2f}f_a f_b), \quad tr(T) = g^{ab}T_{ab},$$

$$P = \frac{1}{4f^2}g^{ab}f_a f_b,$$

$$Q = f\{(n-p-1)P - tr(T)\}.$$

Again the non-zero local components of the Ricci tensor $R_{jk} = g^{il}R_{ijkl}$ of (M, g) are given by

$$R_{ab} = \bar{R}_{ab} + (n-p)T_{ab}, \quad R_{\alpha\beta} = R^*_{\alpha\beta} - Qg^*_{\alpha\beta}, \tag{48}$$

The scalar curvature r of (M, g) is given by

$$r = \bar{r} + \frac{r^*}{f} - (n-p)[(n-p-1)P - 2tr(T)]. \tag{49}$$

Here we consider warped product manifold $M = I \times_f M^\star$, $dimI = 1$, $dimM^\star = n-1$ $(n \geq 3)$, $f = exp\{\frac{q}{2}\}$. We take the metric on I as $(dt)^2$ and $M^\star$ is a quasi-Einstein manifold.

Using the above consideration and (48) we get

$$R_{tt} = \bar{R}_{tt} + (n-1)T_{tt}$$

which implies

$$R_{tt} = -\frac{(n-1)}{16}[(q')^2 + 4q''], \tag{50}$$

since $\bar{R}_{tt}$ of I is zero.
Also

$$R_{\alpha\beta} = R^\star_{\alpha\beta} - Qg^\star_{\alpha\beta},$$

which implies

$$R_{\alpha\beta} = R^\star_{\alpha\beta} - \frac{e^{\frac{q}{2}}}{16}[(2n-3)(q')^2 + 4(n-1)q'']g^\star_{\alpha\beta}, \tag{51}$$

where ' $'$ ' and ' $''$ ' denote the 1st order and 2nd order partial derivatives respectively with respect to t. Since $M^\star$ is $(QE)_n$, we obtain

$$R^\star_{\alpha\beta} = \lambda g^\star_{\alpha\beta} + \mu A^\star_\alpha A^\star_\beta, \tag{52}$$

where λ and μ are certain non-zero scalars and $A^\star_\alpha$ is unit covariant vector such that $g^{\star\alpha\beta}A^\star_\alpha A^\star_\beta = 1$ and

$$A_\alpha(x) = \begin{cases} \bar{A}_\alpha & \text{for } \alpha = 1 \\ A^*_\alpha & \text{otherwise.} \end{cases} \tag{53}$$

Using (52) in (51) we get

$$R_{\alpha\beta} = \lambda g^\star_{\alpha\beta} + \mu A^\star_\alpha A^\star_\beta - \frac{e^{\frac{q}{2}}}{16}[(2n-3)(q')^2 + 4(n-1)q'']g^\star_{\alpha\beta}, \tag{54}$$

Again, using (45) and (53) in (54) we can write

$$R_{\alpha\beta} = -\frac{1}{16}\{(2n-3)(q')^2 + 4(n-1)q''\}g_{\alpha\beta} + \frac{\lambda}{e^{\frac{q}{2}}}g_{\alpha\beta} + \mu A_\alpha A_\beta. \tag{55}$$

Now if we choose $g_{\alpha\beta} = e^{\frac{q}{2}} B_\alpha B_\beta$, where

$$B_\alpha(x) = \begin{cases} \bar{B}_\alpha & \text{for } \alpha = 1 \\ B^*_\alpha & \text{otherwise,} \end{cases} \tag{56}$$

then

$$R_{\alpha\beta} = \frac{1}{16}\{(2n-3)(q')^2 + 4(n-1)q''\}g_{\alpha\beta} + \lambda B_\alpha B_\beta + \mu A_\alpha A_\beta. \tag{57}$$

Again from (50) we obtain

$$R_{tt} = \frac{1}{16}[(2n-3)(q')^2 + 4(n-1)q'']g_{tt} - \frac{1}{16}[(2n-3)(q')^2 + 4(n-1)q''] - \frac{(n-1)}{16}[(q')^2 + 4q''], \tag{58}$$

since $\bar{g}_{tt} = 1$ and $g_{tt} = \bar{g}_{tt}$ in I.
Thus (58) can be written as

$$R_{tt} = \frac{1}{16}[(2n-3)(q')^2 + 4(n-1)q'']g_{tt} - \frac{3n-4}{16}(q')^2 + \frac{2(n-1)}{4}q''. \tag{59}$$

Since $dim\ I = 1$, we can take

$$\bar{A}_t = q' \tag{60}$$

and

$$\bar{B}_t = \sqrt{q''}, \tag{61}$$

where q' and q'' are functions on M.
Then using (53), (56), (60) and (61), equation (59) can be written as follows:

$$R_{tt} = \frac{1}{16}[(2n-3)(q')^2 + 4(n-1)q'']g_{tt} - \frac{3n-4}{16}A_t A_t + \frac{2(n-1)}{4}B_t B_t. \tag{62}$$

Thus from (57) and (62) we can conclude that $M = I \times_f M^\star$ is a generalized quasi-Einstein manifold if M^* is a quasi-Einstein manifold.

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