

# New Exactly Solvable Isospectral Partners for $\mathcal{PT}$ Symmetric Potentials

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## Abstract

We examine in detail the possibility of applying Darboux transformation to non Hermitian hamiltonians. In particular we propose a simple method of constructing exactly solvable  $\mathcal{PT}$  symmetric potentials by applying Darboux transformation to higher states of an exactly solvable  $\mathcal{PT}$  symmetric potential. It is shown that the resulting hamiltonian and the original one are pseudo supersymmetric partners. We also discuss application of Darboux transformation to hamiltonians with spontaneously broken  $\mathcal{PT}$  symmetry.

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# 1 Introduction

Ever since it was conjectured by Bender et al that some non-Hermitian Hamiltonians exhibiting symmetry under the combined transformation of *parity* ( $\mathcal{P} : x \rightarrow -x$ ), and *time reversal* ( $\mathcal{T} : i \rightarrow -i$ ) admit real eigenvalues [1, 2], non-Hermitian Hamiltonians have been the basis of many recent works on  $\mathcal{PT}$  symmetry and pseudo-Hermiticity [3, 4], because of intrinsic interest and their possible applications in molecular physics, quantum chemistry, superconductivity, quantum field theory and others.

On the other hand there are not many examples of exactly solvable complex potentials (both  $\mathcal{PT}$  invariant as well as otherwise). However, as in the Hermitian case, there have been attempts to expand the class of exactly solvable non Hermitian potentials by using different methods [5, 6, 7]. In the Hermitian case a popular method to obtain new exactly solvable potentials is to apply Darboux transformation [8] to the ground state of an exactly solvable potential. However, when applied to the excited states, this transformation produces not just one isospectral potential, but a number (depending on the nodes of the wave function) of nearly isospectral potentials which are defined not over the whole domain, but in disjoint intervals [9]. Here our objective is to apply Darboux transformation to non Hermitian potentials and it will be shown that for such potentials, it is possible to have wave functions without nodes on the real line, by a reasonable choice of parameters. In the present article, we shall use this result to construct non trivial isospectral partners of exactly solvable complex potentials. In particular, we shall apply the Darboux transformation to the well known  $\mathcal{PT}$  symmetric Scarf II potential

$$V(x) = -V_1 \operatorname{sech}^2 x - iV_2 \operatorname{sech} x \tanh x, \quad V_1 > 0, V_2 \neq 0 \quad (1)$$

and generate a series of new exactly solvable non-Hermitian potentials with real spectrum.

We note that in the case of Hermitian quantum mechanics Darboux transformation is equivalent to supersymmetry [10]. However this is not so in the non Hermitian case. So it is natural to ask whether there exists any symmetry which relates the two Hamiltonians i.e., the original and the one obtained by Darboux transformation. The answer to this question is in the affirmative and it will be shown that the two Hamiltonians are related by pseudo supersymmetry [11]. In other words the hamiltonians obtained by intertwining are pseudo supersymmetric partners.

Finally we shall examine the problem of applying Darboux transformation to models with spontaneously broken  $\mathcal{PT}$  symmetry. It is known [12] that models with spontaneously broken  $\mathcal{PT}$  symmetry exhibit a complex spectrum and all the energy levels appear as complex conjugate pairs. It will be shown that if Darboux transformation is applied to such a system one gets a potential with complex energy eigenvalues but as singlets.

The organisation of the paper is as follows : in section 2 we briefly present the Darboux construction; in section 3 we construct new  $\mathcal{PT}$  symmetric potentials; in section 4 we show that the partner Hamiltonians are connected by pseudo supersymmetry; in section 5 we examine the nature of the spectrum obtained by applying Darboux transformation to a potential with spontaneously broken  $\mathcal{PT}$  symmetry and finally section 6 is devoted to a discussion.

## 2 Darboux Transformation

To make the article self contained we start with a brief review of the Darboux transformation [8, 9]. A particle moving in the potential  $v(x)$  (real or complex) is characterised by the Hamiltonian

$$H = -\frac{d^2}{dx^2} + v(x) \quad (2)$$

(The units used are  $\hbar = 2m = 1$  for convenience).

If the particle is in the  $m^{\text{th}}$  state, (i.e.,  $m$  is the quantum number equal to the number of nodes of the

$m^{\text{th}}$  eigenfunction  $\psi_m(x)$  of the starting potential  $v(x)$ , and the energy scale is adjusted so that the  $m^{\text{th}}$  energy eigenvalue is exactly zero ( $E_m = 0$ ), then the Schrödinger equation reads

$$H\psi_m = \left(-\frac{d^2}{dx^2} + v(x)\right)\psi_m = 0 \quad (3)$$

Equation (3) has a potential

$$v(x) = \frac{\psi_m''}{\psi_m} \quad (4)$$

which is regular everywhere, so that the Hamiltonian in (3) may be represented as

$$H = \left(-\frac{d^2}{dx^2} + \frac{\psi_m''}{\psi_m}\right) \quad (5)$$

Thus if the general solution  $\psi = \psi(x)$  of the Schrödinger equation

$$\frac{d^2\psi}{dx^2} + [\epsilon - v(x)]\psi = 0 \quad (6)$$

is known for all values of  $\epsilon$ , and for a particular value of  $\epsilon = E_m$ , the particular solution is  $\psi_m$ , then the general solution of the equation

$$\frac{d^2\phi}{dx^2} + [E - u(x)]\phi = 0 \quad (7)$$

with

$$\begin{aligned} u(x) &= \psi_m(x) \frac{d^2}{dx^2} \left( \frac{1}{\psi_m(x)} \right) \\ &= 2 \left( \frac{\psi_m'}{\psi_m} \right)^2 - \left( \frac{\psi_m''}{\psi_m} \right) \end{aligned} \quad (8)$$

$$E = \epsilon - E_m \quad (9)$$

for  $E \neq 0$  is

$$\begin{aligned} \phi_n(x) &= \psi_m(x) \left\{ \frac{\psi_n(x)}{\psi_m(x)} \right\}' \\ &= \psi_n'(x) - \left( \frac{\psi_m'(x)}{\psi_m(x)} \right) \psi_n(x) \end{aligned} \quad (10)$$

Choosing different  $\psi_m$ , one obtains a series of non-trivial partners  $u(x)$  given by (8), which contain all the states of the original potential  $v(x)$  except the  $m^{\text{th}}$  state, i.e. the one corresponding to the eigenstate  $\psi_m$ . It is easy to observe that the partners  $u(x)$  and  $v(x)$  are related by

$$v(x) = W_m(x)^2 - W_m'(x) \quad (11)$$

$$u(x) = W_m(x)^2 + W_m'(x) \quad (12)$$

where

$$W_m(x) = \left( -\frac{\psi_m'}{\psi_m} \right) \quad (13)$$

Though  $\psi_m' \neq 0$  at the nodes  $x_j$ ,  $j = 1, 2, 3, \dots$  of  $\psi_m$ ,  $W_m(x)$  has singularities at these points. However, since the second derivative  $\psi_m''$  also vanishes at the nodes, from (4)  $v(x)$  is regular everywhere. In case of Hermitian SUSY QM,  $W_m$  is the superpotential, and  $W_m^2(x) \pm W_m'(x)$  are called SUSY- $m$  partner potentials [9]. However, one can construct the partners for  $m = 0$  only, as for non-zero  $m$ ,  $W_m$  becomes singular and as a consequence such potentials are not defined over  $R$  but over disjoint intervals, the

number of intervals depending on the value of  $m$ . Thus, for  $m = 1$ , there are two potentials, each of them defined on a semi-infinite domain, for  $m = 2$  there is one potential on a finite domain between nodes  $x_1$  and  $x_2$ , and two potentials on the two semi-infinite domains  $(-\infty, x_1]$  and  $[x_2, +\infty)$ , and so on [9]. In the case of non-Hermitian quantum mechanics,  $W_m(x)$  is no longer singular (and it is not the *superpotential* anymore). As a consequence the new potentials do not have singularities on the real axis and are defined on  $(-\infty, \infty)$ . Thus, if one of the partner potentials is exactly solvable, this formalism enables one to construct an infinite number of exactly solvable, non-trivial partners defined on the entire real line  $(-\infty, +\infty)$ , unlike in the case of Hermitian quantum mechanics.

### 3 New exactly solvable $\mathcal{PT}$ symmetric potentials

In this section we shall construct isospectral partners of the complexified Scarf II potential. This potential is given by

$$V(x) = -V_1 \operatorname{sech}^2 x - iV_2 \operatorname{sech} x \tanh x, \quad V_1 > 0, V_2 \neq 0 \quad (14)$$

and it has been studied by various authors as it is not only invariant under  $\mathcal{PT}$  symmetry, but also  $\mathcal{P}$ -pseudo Hermitian. This exactly solvable model has certain interesting properties. It has a discrete spectrum that admits both real as well as complex conjugate energies, depending on the relative strengths of its parameters  $V_1$  and  $V_2$ . The normalized wave functions for this potential are well known, being given by [12]

$$\psi_n(x) = \frac{\Gamma(n - 2p + \frac{1}{2})}{n! \Gamma(\frac{1}{2} - 2p)} z^{-p} (z^*)^{-q} P_n^{-2p - \frac{1}{2}, -2q - \frac{1}{2}}(i \sinh x) \quad (15)$$

where  $P_n^{\alpha, \beta}$  are the Jacobi polynomials given by [13]

$$P_n^{\alpha, \beta}(i \sinh x) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)} F(-n, n + \alpha + \beta + 1; \alpha + 1; z) \quad (16)$$

and

$$z = \frac{1 - i \sinh x}{2} \quad (17)$$

$$p = -\frac{1}{4} \pm \frac{1}{2} \sqrt{\frac{1}{4} + V_1 + V_2} = -\frac{1}{4} \pm \frac{t}{2} \quad (18)$$

$$q = -\frac{1}{4} \pm \frac{1}{2} \sqrt{\frac{1}{4} + V_1 - V_2} = -\frac{1}{4} \pm \frac{s}{2} \quad (19)$$

$t$  and  $s$  are defined with only the positive sign in the discriminant in  $p$  and  $q$ . The energy eigenvalues are obtained as

$$E_n = -(n - p - q)^2, \quad n = 0, 1, 2, \dots < \left(\frac{s + t - 1}{2}\right) \quad (20)$$

Since  $V_1 > 0$ , two cases arise for real  $V_2$ , depending on the relative strengths of the real and imaginary parts of the potential :

1. For  $|V_2| \leq V_1 + \frac{1}{4}$

In this case the potential and the wave functions are  $\mathcal{PT}$  invariant,  $p$  and  $q$  are real and one gets a real bound state spectrum. In addition to the potential (1), the wave functions  $\psi_n(x)$  given in (15) are also  $\mathcal{PT}$  invariant. Note that due to normalisation requirements only the values with the positive sign are allowed in (18) and (19).

2. For  $|V_2| > V_1 + \frac{1}{4}$

$\mathcal{PT}$  symmetry is spontaneously broken, as though the potential is  $\mathcal{PT}$  invariant, the wave functions are no longer so. Either  $p$  or  $q$  is complex, and all energies occur as complex conjugate pairs. Real energies are conspicuous by their absence.

For purely imaginary  $V_2$ , however, the potential (1) is real, possessing only real energies.

However, when  $V_2$  has both real and imaginary parts, the potential (1) loses its  $\mathcal{PT}$  invariance, and so will not be considered in this study.

We now consider the first case when  $\mathcal{PT}$  symmetry is unbroken. Now, using the explicit solution for the normalized wave function (15), we obtain

$$\begin{aligned} W_m(x) &= -\frac{\psi'_m(x)}{\psi_m(x)} \\ &= (p+q)\tanh x - i(p-q)\operatorname{sech} x + \frac{mb}{c} \frac{F(-m+1, b+1; c+1; z)}{F(-m, b; c; z)} \end{aligned} \quad (21)$$

where  $b$  and  $c$  stand for

$$b = -2p - 2q + 2 \quad (22)$$

$$c = -2p + \frac{1}{2} \quad (23)$$

The exactly solvable potential  $U^{(m)}(x)$ , which is isospectral to the Scarf II potential (except for the  $m^{\text{th}}$  state), is obtained from the formula

$$U^{(m)}(x) = W_m^2 + W'_m - \beta_m \quad (24)$$

In writing the last expression we have made use of the fact that if  $v(x)$  and  $u(x)$  are isospectral, so are  $V(x)$  and  $U^m(x)$ , given by

$$V(x) = \{v(x) - \beta_m\} \quad (25)$$

$$U^m(x) = \{u(x) - \beta_m\} \quad (26)$$

For the Scarf II potential,  $\beta_m$  is calculated to be

$$\beta_m = (p+q)^2 - 2m(p+q) + m^2 \quad (27)$$

Thus this approach yields new interesting potentials, with eigenfunctions for this particular case being given by (see eq.(10))

$$\phi_n(x) = \left( \frac{P_m P'_n - P'_m P_n}{P_m} \right) \psi_0(x) \quad (28)$$

In the above  $P_n$  stands for  $P_n^{\alpha, \beta}(i \sinh x)$  and  $P'_n$  denotes differentiation of  $P_n^{\alpha, \beta}(i \sinh x)$  with respect to  $x$ . Since  $P_n^{\alpha, \beta}$  is well defined on the entire real line, so also is  $\phi_n(x)$ .

Let us analyse three low lying cases  $m = 0, 1, 2$ .

For  $m = 0$

$$U^{(0)}(x) = -\{2(p^2 + q^2) - (p+q)\} \operatorname{sech}^2 x - i(p-q)[2(p+q) - 1] \operatorname{sech} x \tanh x \quad (29)$$

with eigenenergies

$$E_n = -(n+1-p-q)^2, \quad n = 0, 1, 2, 3, \dots \quad (30)$$

and the ground state

$$\phi_0 = N_0 \left( \frac{1 - i \sinh x}{2} \right)^{-(p-\frac{1}{2})} \left( \frac{1 + i \sinh x}{2} \right)^{-(q-\frac{1}{2})} \quad (31)$$

Thus for  $m = 0$ , the partners belong to the family of the so-called *satellite* potentials. (29) is also a Scarf II potential, with a different set of parameters, and shares all the energies of (1) except for the ground state of  $V(x)$ , which is missing in (29). So, this is analogous to the Hermitian case.

$m = 1$  gives the first of the non-trivial potentials.

$$U^{(1)}(x) = -\{2(p^2 + q^2) - (p + q)\} \operatorname{sech}^2 x - i(p - q)[2(p + q) - 1] \operatorname{sech} x \tanh x \\ + 2 \left( \frac{f'_1}{f_1} \right)^2 - \frac{2(p - q)}{f_1} - 2 \quad (32)$$

with

$$f_1(x) = F \left( -1, -p - q - \lambda; 2p + \frac{3}{2}; z \right) \\ = \left\{ -(p - q) + \frac{i}{2} (-2p - 2q + 1) \sinh x \right\} \quad (33)$$

The ground state of the partner (32) is obtained from (10) as

$$\phi_0 = \frac{2i \left( \frac{1}{2} - p - q \right)}{(-p - q) + i \left( \frac{1}{2} - p - q \right) \sinh x} z^{-(p-\frac{1}{2})} (z^*)^{-(q-\frac{1}{2})} \quad (34)$$

with energy

$$E_0 = -(-p - q)^2 \quad (35)$$

It can be shown that the state corresponding to  $\psi_1$  is excluded from the spectrum as it turns out to be non-normalizable. All other states share identical energies with the original potential (1). The excited states are obtained from (10) as

$$\phi_n = \left( \frac{P_1 P'_{n+2} - P'_1 P_{n+2}}{P_1} \right) \psi_0 \quad (36)$$

with energies

$$E_n = -(n + 2 - p - q)^2, n = 0, 1, 2, 3, \dots \quad (37)$$

As  $P_n^{\alpha, \beta}(i \sinh x)$  has no zeroes on the real line, so the eigenfunctions are well defined. It is easily observed from (36) that  $\phi_n$  are normalizable. Moreover, the potential  $U^{(1)}(x)$  so constructed has no singularity on the real line, and hence is defined on the entire domain  $(-\infty, +\infty)$ . Also, for real values of the parameters  $p$  and  $q$  (corresponding to real energies) the new potential, too, is invariant under  $\mathcal{PT}$  transformation.

In an analogous way, the isospectral partner for  $m = 2$  is found to be

$$U^{(2)}(x) = -\{2(p^2 + q^2) - (p + q)\} \operatorname{sech}^2 x - i(p - q)[2(p + q) - 1] \operatorname{sech} x \tanh x \\ + 2 \left( \frac{f'_2}{f_2} \right)^2 + \frac{\sigma - 6(p - q)i \sinh x}{f_2} - 8 \quad (38)$$

with

$$f_2(x) = F \left( -2, -p - q - \lambda; 2p + \frac{3}{2}; z \right) \quad (39)$$

$$\sigma = \frac{-2p - 2q + 2}{(-2p + \frac{1}{2})(-2p + \frac{3}{2})} \left\{ 2(p - q)^2 - \frac{(-2q + \frac{3}{2})(-2p + \frac{3}{2})}{(-2p - 2q + 2)} + \frac{1}{2}(3 + 2p + 2q)(3 - 2p - 2q) \right\} \quad (40)$$

For a visual representation as well as for comparison, we have plotted the real and imaginary parts of the potentials  $V$ ,  $U^{(i)}$  ( $i = 0, 1, 2$ ). In fig 1, we have plotted the real parts of  $V$ ,  $U^{(i)}$  ( $i = 0, 1, 2$ ) for the parameter values  $V_1 = 24, V_2 = 18$  while in fig 2, we have plotted the imaginary parts of the same potentials for the same values of  $V_1$  and  $V_2$ .

Since the Scarf II potential is always  $\mathcal{PT}$  symmetric, so are its partners  $U^{(m)}(x)$ , for real values of the parameters  $p$  and  $q$  ( $V_1 + 1/4 \geq |V_2|$ ), as the functions  $f_m(x)$  remain invariant under  $\mathcal{PT}$ . Moreover, all the new potentials so constructed are defined over the entire domain  $(-\infty, +\infty)$ , admit real bound state spectrum, and possess all the energies of the original potential except for the  $m^{th}$  state, if one starts with the  $m^{th}$  order eigenfunction. Though  $m = 0$  gives the usual shape-invariant form, highly non-trivial, non-shape-invariant potentials are obtained for non-zero  $m$ .

## 4 Pseudo supersymmetry and intertwining

It is well known that in the case of Hermitian quantum mechanics, Darboux transformation and supersymmetric quantum mechanics are equivalent. Although this is not so in the case of non Hermitian quantum mechanics, Darboux transformation can still be implemented in terms of intertwining operators. To see this we consider the intertwining operators  $A$  and  $B$  :

$$A = \frac{d}{dx} + W_m \quad (41)$$

$$B = -\frac{d}{dx} + W_m \quad (42)$$

where  $W_m$  is defined by (13), then the partner Hamiltonians

$$H_{\pm} = -\frac{d^2}{dx^2} + W_m^2 \pm W'_m \quad (43)$$

can be written as  $H_- = BA$  and  $H_+ = AB$ , where

$$H_- = \left( -\frac{d^2}{dx^2} + v(x) \right) \quad (44)$$

$$H_+ = \left( -\frac{d^2}{dx^2} + u(x) \right) \quad (45)$$

Evidently, if  $\psi_n$  is an eigenfunction of  $H_-$  with energy eigenvalue  $E_n^-$ , then  $\phi_n = A\psi_n$  is also an eigenfunction of  $H_+$  with the same eigenvalue  $E_n^-$ , except for  $n = m$ , since in this case  $A\psi_m = 0$ .

$$H_+ A\psi_n = (AB)A\psi_n = A(H_- \psi_n) = E_n^- (A\psi_n) \quad (46)$$

For Hermitian Hamiltonians,  $A$  and  $B$  are mutually adjoint operators ( $B = A^\dagger$ ), giving the well known results of supersymmetry, viz.,  $AH_- = H_+A$  or  $H_-A^\dagger = A^\dagger H_+$ .

To extend the idea of supersymmetry to include non Hermitian hamiltonians, we assume the existence of a linear, invertible, Hermitian operator  $\eta$ , such that [11]

$$B = A^\# = \eta^{-1}A^\dagger\eta \quad (47)$$

This allows one to rewrite the partner Hamiltonians as

$$H_+ = B^\# B \quad H_- = B B^\# \quad (48)$$

so that

$$B H_+ = H_- B \quad H_+ B^\# = B^\# H_- \quad (49)$$

From (49) it is clear that  $B$  maps eigenfunctions of  $H_+$  to those of  $H_-$ , and  $A(= B^\#)$  does the converse. Thus the mutually adjoint operators  $A$  and  $A^\dagger$  of conventional supersymmetric quantum mechanics are replaced by their pseudo supersymmetric counterparts  $A$  and  $B$  when the potential is non Hermitian. However, we would like to point out that the choice of  $\eta$  is not unique.

To determine a form of  $\eta$ , let us note that a simple representation is given by [11]:

$$\eta = \mathcal{P} \quad , \quad \mathcal{P}f(x) = f(-x) \quad (50)$$

It follows that for real potentials, (50) leads to  $B = A^\dagger$ , thus reproducing the conventional result of supersymmetry. We note that the above results are quite general since they do not depend on a specific  $W_m$ . We thus conclude that in the complex case the hamiltonians  $H_\pm$  are pseudo supersymmetric partners of each other. Finally to cast the above results in a formal pseudo supersymmetric form let us define the pseudo supercharges  $Q$  and  $Q^\#$  in the following way :

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad , \quad Q^\# = \eta^{-1} Q^\dagger \eta = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \quad (51)$$

Thus the pseudo supercharges  $Q$  and  $Q^\#$  are nilpotent and they satisfy the following closed algebra :

$$H = \{Q, Q^\#\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \quad , \quad [Q, H] = [Q^\#, H] = 0 \quad (52)$$

We thus conclude that in the non Hermitian case the hamiltonians obtained by intertwining are pseudo supersymmetric partners.

## 5 A model with spontaneously broken $\mathcal{PT}$ symmetry

As mentioned earlier, there are two cases for the  $\mathcal{PT}$  symmetric Scarf II potential, depending on the relative strengths of  $V_1$  and  $V_2$ , viz.,

(i)  $|V_2| \leq V_1 + \frac{1}{4}$  :

In this case, where the spectrum is real and discrete, each state is a singlet. This has already been discussed in Section 3.

(ii)  $|V_2| > V_1 + \frac{1}{4}$  :

In this case  $\mathcal{PT}$  symmetry is spontaneously broken. We can choose  $p$  to be real, taking a single value with only the positive sign in (18) while  $q$  can take either of the following values

$$q^\pm = -\frac{1}{4} \pm i\frac{s}{2} \quad (53)$$

where

$$s = \sqrt{V_2 - V_1 - \frac{1}{4}} \quad (54)$$

giving rise to complex conjugate pairs of energies

$$\begin{aligned} E_n^\pm &= -\mu_n^2 \pm i\mu_n s \quad , \quad n = 0, 1, 2, \dots \\ \mu_n &= n - p + \frac{1}{4} \end{aligned} \quad (55)$$

This case makes quite an interesting study and we shall investigate it further. Though the original potential (14) is still  $\mathcal{PT}$  invariant, the partners are no longer so. We note that while the original potential  $V(x)$  does not depend explicitly on the parameters  $p$  and  $q$ , the partner potential is explicitly dependent on these parameters. As a consequence there are two partner potentials corresponding to  $V(x)$  (this is due to the fact that in this case both the values of  $q$  are allowed). If the Darboux transformation is carried out by the  $m^{\text{th}}$  eigenstate  $\psi_m^-$  ( $\psi_m^+$ ) of the original potential, then straightforward calculations show that the corresponding state will be missing in the partner. Furthermore, the entire positive (negative) sector  $E_n^+ = -\left(\mu_n^2 - \frac{s^2}{4}\right) + i\mu_n s$  ( $E_n^- = -\left(\mu_n^2 - \frac{s^2}{4}\right) - i\mu_n s$ ) will be absent in the partner. Thus the spectrum of the partner potential is quite different from that of the Scarf II potential, as the former has only singlet complex energies. So, if one starts with the ground state eigenfunction  $\psi_0^-$ , then the partner potential (29) is of the form

$$U_-^0(x) = -\left\{(2p^2 - p - \frac{s^2}{2} + \frac{3}{8}) + is\right\} \text{sech}^2 x - \left\{i(2p^2 - p + \frac{s^2}{2} - \frac{3}{8}) + s\right\} \text{sech} x \tanh x \quad (56)$$

while the wave functions and the corresponding energy values are given by

$$\phi_n^-(x) = \frac{ab}{c} z^{-p} (z^*)^{-\frac{1}{4} + \frac{is}{2}} F(a+1, b+1; c+1; z) , \quad E_n^- = -\mu_{n+1}^2 - i\mu_{n+1} s , \quad n = 0, 1, 2, \dots \quad (57)$$

where

$$a = -(n+1) , \quad b = \left(n + \frac{3}{2} - 2p\right) + is , \quad c = -2p + \frac{1}{2} \quad (58)$$

## 6 Discussion

In this article we have explored the idea of applying Darboux transformation to non Hermitian quantum mechanical systems. In particular we have obtained a series of new exactly solvable  $\mathcal{PT}$  symmetric potentials with real bound state spectra. The symmetry aspect of the potentials has also been investigated and it has been shown that they are pseudo supersymmetric. On the other hand when Darboux transformation is applied to a system with spontaneously broken  $\mathcal{PT}$  symmetry the resulting potential admits a single tower of energy. We would like to point out that this is not a characteristic feature of only models with spontaneously broken  $\mathcal{PT}$  symmetry but in general the same result would be obtained whenever the original model possesses doublet of states [14].

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## Figure Captions

Fig. 1 : Graph of real parts of  $V$  (solid),  $U^{(0)}$  (dotted),  $U^{(1)}$  (small dash),  $U^{(2)}$  (large dash).

Fig. 2 : Graph of imaginary parts of  $V$  (solid),  $U^{(0)}$  (dotted),  $U^{(1)}$  (small dash),  $U^{(2)}$  (large dash).