

New classes of quasi-solvable potentials, their exactly-solvable limit and related orthogonal polynomials

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3 May 2002

Published in Nov. JMP issue

Abstract

We have generated, using an $\mathfrak{sl}(2, \mathbb{R})$ Lie-algebraic formalism several new classes of quasi-solvable elliptic potentials, which in the appropriate limit go over to the exactly solvable forms. We have obtained exact solutions of the corresponding spectral problem for some real values of the potential parameters. We have also given explicit expressions of the families of associated orthogonal polynomials in the energy variable.

1 Introduction

In recent times elliptic potentials have proved to be an important addition [1–3] to the class of solvable [4, 5] and quasi-solvable [6–8] potentials in quantum mechanics. In particular, within the $\mathfrak{sl}(2, \mathbb{R})$ algebra, exact solutions of Lamé and associated Lamé equation have been obtained [9–12] for various ranges of the potential parameters. Indeed a handful of theorems relating to the properties of elliptic potentials are known for a long time [13–16] including the study of the properties of the corresponding wavefunctions [14]. The solutions of associated Lamé equation have also been obtained [17] by using these theorems which, however, do not use the $\mathfrak{sl}(2, \mathbb{R})$ technique. Note that some new elliptic models based on Weierstrass \wp function have recently been proposed [18] wherein it is shown that the corresponding Hamiltonians possess the so-called energy-reflection symmetry [19].

By the term quasi-exactly solvable (QES) periodic potentials we mean potentials consisting of finite number of allowed bands and expressible as doubly-periodic elliptic functions which are either Jacobian elliptic functions $snx \equiv sn(x, k)$, $cnx \equiv cn(x, k)$, $dnx \equiv dn(x, k)$ of real elliptic modulus parameter k ($0 < k^2 < 1$) or Weierstrass \wp function. This is in sharp contrast with the ordinary ES periodic potentials with a single period.

There is an intriguing relation between ES and QES class. In fact an $\mathfrak{sl}(2)$ based construction with an n -dimensional finite space representation gives $(n+1)$ levels for a Hamiltonian designated as QES model. It was pointed out in Ref. 8 that if one can construct a Hamiltonian having no explicit dependence on n , then in the limit $n \rightarrow \infty$ the ES models are recovered. This provides a sufficient reason to believe that corresponding to every ES model there ought to exist a QES model which in the proper limit goes over to the former.

In this article we show that the elliptic parameter k can be used for passage from QES to ES in the periodic models. We derive three new QES periodic potentials involving Jacobian elliptic functions. The elliptic functions having real and imaginary period reduce to ordinary periodic functions namely hyperbolic(imaginary period) and trigonometric(real

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period) functions as the modulus parameter k goes to 1 and 0. We exploit this interesting property of elliptic functions to show that our QES models are connected with some well known ES periodic class. Note that we do not intend to take the limit $n \rightarrow \infty$ and as such we cannot expect to recover the whole spectrum for ES models. Rather we show that the limit $k \rightarrow 1$ and 0 correctly map the few lower states of the QES and ES periodic potentials.

The plan of this article is as follows. In Sec. 2 we briefly review the basics of the $\mathfrak{sl}(2, \mathbb{R})$ Lie-algebraic formalism and generate Type I, Type II and Type III models within this framework. The method of construction of the related orthogonal polynomials is also sketched here. Specific examples are constructed in Sec. 3 for each of them based on some real values of the potential parameters. In Sec. 4 we systematically analyze the ES-limit of our results to show how this limit can reproduce the ES results. Finally we present our conclusions in Sec. 5.

2 New QES potentials from $\mathfrak{sl}(2, \mathbb{R})$ and related orthogonal polynomials

To start with, let us adopt the following differential realization of the $\mathfrak{sl}(2, \mathbb{R})$ generators

$$T^+ = \xi^2 \partial_\xi - n\xi, \quad T^0 = \xi \partial_\xi - \frac{1}{2}n, \quad T^- = \partial_\xi, \quad (2.1)$$

obeying commutation relations

$$[T^+, T^-] = -2T^0, \quad [T^0, T^\pm] = \pm T^\pm, \quad (2.2)$$

where n is a non-negative integer. The gauged Hamiltonian is taken as the standard homogeneous quadratic combination of $\mathfrak{sl}(2, \mathbb{R})$ generators along with linear terms:

$$H_G = -C_{++}T^{+2} - C_{00}T^{02} - C_{--}T^{-2} - C_+T^+ - C_0T^0 - C_-T^- - d, \quad (2.3)$$

where $C_{ii}, C_j (i, j = 0, \pm)$ are numerical parameters and d is a suitably chosen constant taken as function of C_j . Note that d plays the role of an overall shift in the energy scale. This pseudo degree of freedom allows us to obtain QES models in the desired form.

Substitution of (2.1) into (2.3) yields

$$\begin{aligned} H_G(\xi) = & -(C_{++}\xi^4 + C_{00}\xi^2 + C_{--})\partial_\xi^2 - [2(1-n)C_{++}\xi^3 + C_+\xi^2 + \{(1-n)C_{00} \\ & + C_0\}\xi + C_-] \partial_\xi - [n(n-1)C_{++}\xi^2 - nC_+\xi + \frac{n^2}{4}C_{00} - \frac{n}{2}C_0 + d], \end{aligned} \quad (2.4)$$

which after a coordinate transformation

$$x(\xi) = \int^\xi d\tau / \sqrt{C_{++}\tau^4 + C_{00}\tau^2 + C_{--}}, \quad (2.5)$$

converts H_G into the form

$$\begin{aligned} H_G(x) = & -\partial_x^2 + \frac{2nC_{++}\xi^3(x) - C_+\xi^2(x) + (nC_{00} - C_0)\xi(x) - C_-}{\sqrt{C_{++}\xi^4(x) + C_{00}\xi^2(x) + C_{--}}} \partial_x \\ & - [n(n-1)C_{++}\xi^2(x) - nC_+\xi(x) + \frac{n^2}{4}C_{00} - \frac{n}{2}C_0 + d], \end{aligned} \quad (2.6)$$

where $\xi = \xi(x)$ is determined by (2.5).

Let us now consider the Schrödinger equation with the potential $V(x)$

$$H(x)\psi(x) \equiv [-\partial_x^2 + V(x)]\psi(x) = E\psi(x) \quad (2. 7)$$

Writing $\psi(x)$ in the form

$$\psi(x) = \mu(x)\chi(x), \quad (2. 8)$$

we obtain

$$H_G(x)\chi(x) \equiv [-\partial_x^2 - 2(\frac{\mu'}{\mu})\partial_x - (\frac{\mu'}{\mu})^2 - (\frac{\mu'}{\mu})' + V]\chi(x) = E\chi(x). \quad (2. 9)$$

Comparing (2.6) and (2.9) we find for the potential $V(x)$ and the gauge factor $\mu(x)$ the following relationships

$$V(x) = (\frac{\mu'}{\mu})^2 + (\frac{\mu'}{\mu})' - [n(n-1)C_{++}\xi^2 - nC_+\xi + \frac{n^2}{4}C_{00} - \frac{n}{2}C_0 + d], \quad (2. 10)$$

$$\mu(x) = [C_{++}\xi^4 + C_{00}\xi^2 + C_{--}]^{-\frac{n}{4}} \exp\left[\int^\xi \frac{C_+\tau^2 + C_0\tau + C_-}{2(C_{++}\tau^4 + C_{00}\tau^2 + C_{--})} d\tau\right]. \quad (2. 11)$$

Note that the choice of numerical parameters C_{ii} must be such that equation (2.5) may be invertible in terms of $\xi = \xi(x)$. For our purpose $\xi(x)$ needs to be expressed in terms of Jacobian elliptic functions. In Ref. 12 we gave an almost exhaustive list of the choice of C_{ii} leading to various new classes of elliptic potentials. Here we consider the following three types of combinations of parameters namely

$$\text{Type I : } \quad C_{++} = -k'^2 \quad C_{00} = 2k^2 - 1 \quad C_{--} = k'^2, \quad (2. 12)$$

$$\text{Type II : } \quad C_{++} = k^2 \quad C_{00} = -(1 + k^2) \quad C_{--} = 1, \quad (2. 13)$$

$$\text{Type III : } \quad C_{++} = k'^2 \quad C_{00} = 1 + k'^2 \quad C_{--} = 1, \quad (2. 14)$$

where $k^2 \in (0, 1)$ and $k'^2 = 1 - k^2$. Each of the above types defines different coordinate transformations through (2.5). These give respectively $\xi = -cnx$, $-cnx/dnx$ and snx/cnx for the three types mentioned above.

In this way we are then led to the following new classes of elliptic potentials:

$$\text{Type I : } \quad V(x) = [B^2 + A(A+1)]\frac{dn^2x}{sn^2x} - 2B(A + \frac{1}{2})\frac{cnx}{sn^2x}, \quad x \in (0, 2K) \quad (2. 15)$$

$$\text{Type II : } \quad V(x) = B(B+1)\frac{dn^2x}{sn^2x} - A(A+1)dn^2x, \quad x \in (0, 2K) \quad (2. 16)$$

$$\text{Type III : } \quad V(x) = [B^2 - A(A+1)]k^2cn^2x + 2Bk^2(A + \frac{1}{2})snxcnx, \quad x \in (-\infty, \infty) \quad (2. 17)$$

where $K = \int_0^{\pi/2} d\alpha/\sqrt{1 - k^2 \sin^2 \alpha}$ is the complete elliptic integral of 1st kind. Note that in (2.15)–(2.17) the potential parameters $A, B \in \mathbb{R}$ and the choices of C_j and the spin parameter n in (2.10) in terms of A, B are given in Table 1.

The constant d is chosen as

$$\text{Type I : } \quad d = \frac{C_0}{4}\{C_0 - 4k'^2(n+1)\} + \frac{C_+}{2k^2}(k^2C_- - k'^2C_+) + \frac{n(n+2)}{2}k'^2 \quad (2. 18)$$

$$\text{Type II : } \quad d = (\frac{C_+}{2k})^2 - \frac{C_0}{2}(n+1) + \frac{n(n+2)}{4}(1+k^2) \quad (2. 19)$$

$$\text{Type III : } \quad d = \frac{1}{4}[(\frac{C_+}{k'})^2 - n(n+2)(2-k^2) - 2C_0(n+1)] \quad (2. 20)$$

Table 1: Different algebraizations for Type I–III potentials are given. Last column gives restrictions on potential parameters to keep n to a non-negative integer

Type name	Solution no.	n	C_+	C_-	C_0	Restrictions on A, B
I	1.1	A	$2k^2B$	$2k'^2B$	A	$A \in \mathbb{N} - 1, B \in \mathbb{R}$
	1.2	$A - 1$	$2k^2B$	$2k'^2B$	$A + 1$	$A \in \mathbb{N}, B \in \mathbb{R}$
	1.3	$B - 1$	$2k^2(A + \frac{1}{2}) - ikk'$	$2k'^2(A + \frac{1}{2}) + ikk'$	B	$A \in \mathbb{R}, B \in \mathbb{N}$
	1.4	$A - \frac{1}{2}$	$2k^2B - ikk'$	$2k'^2B + ikk'$	$A + \frac{1}{2}$	$A \in \mathbb{N} - \frac{1}{2}, B \in \mathbb{R}$
	1.5	$B - \frac{1}{2}$	$2k^2(A + \frac{1}{2})$	$2k'^2(A + \frac{1}{2})$	$B - \frac{1}{2}$	$A \in \mathbb{R}, B \in \mathbb{N} - \frac{1}{2}$
	1.6	$B - \frac{3}{2}$	$2k^2(A + \frac{1}{2})$	$2k'^2(A + \frac{1}{2})$	$B + \frac{1}{2}$	$A \in \mathbb{R}, B \in \mathbb{N} + \frac{1}{2}$
II	2.1	$A - \frac{1}{2}$	$-2k^2(B + \frac{1}{2})$	$2(B + \frac{1}{2})$	$-k'^2(A + \frac{1}{2})$	$A \in \mathbb{N} - \frac{1}{2}, B \in \mathbb{R}$
	2.2	$A - \frac{1}{2}$	$2k^2(B + \frac{1}{2})$	$-2(B + \frac{1}{2})$	$-k'^2(A + \frac{1}{2})$	$A \in \mathbb{N} - \frac{1}{2}, B \in \mathbb{R}$
III	3.1	A	$-2k'^2B$	$-2B$	$-Ak^2$	$A \in \mathbb{N} - 1, B \in \mathbb{R}$
	3.2	$A - 1$	$-2k'^2B$	$-2B$	$-(A + 1)k^2$	$A \in \mathbb{N}, B \in \mathbb{R}$
	3.3	$A - \frac{1}{2}$	$-2k'^2B + ik'$	$-2B + ik'$	$-(A + \frac{1}{2})k^2$	$A \in \mathbb{N} - \frac{1}{2}, B \in \mathbb{R}$

Note that while the Type III potential is defined over the entire x -axis, Type I and Type II potentials are singular at $x = 0, 2K$ and so defined over an open domain $(0, 2K)$. It follows from the oscillation theorem that we need to find the periodic solutions [of period $4K$ (or $8K$)] for $4K$ -periodic potential of Type I and periodic solutions [of period $2K$ (or $4K$)] for $2K$ -periodic potentials of Type II and Type III at $E = E_j$. The monotonic increasing sequence $\{E_j\}$, where $E_0 < E_1 \leq E_2 < E_3 \leq E_4 \dots$, gives the characteristic values of the energy parameter.

Following the analysis made in [12] we can find the effective combinations of the potential parameters A, B . We see that the Type I and Type III potentials are invariant under the translation $A, B \rightarrow A', B'$ where $A' = -A - 1, B' = -B$ and Type II potential is invariant under $A, B \rightarrow A', B'$ where $A' = -A - 1, B' = -B - 1$. Further due to the periodic relations $sn(x + 2K) = -snx, cn(x + 2K) = -cnx, dn(x + 2K) = dnx$, it results that the effective regions in the A - B plane for Type I–III models are bounded by the constraints $A \geq -1/2, B \geq 0; A, B \geq -1/2$ and $A \geq -1/2, B \in \mathbb{R}$ respectively. It may be pointed out that the eigenstates and spectra of Type I potential for $B < 0$ can be obtained from those for $B > 0$ under the coordinate translation $x \rightarrow x + 2K$.

Before concluding this section let us briefly describe the method of construction of families of orthogonal polynomials [12, 20, 21] generated by the eigenstates of a QES Hamiltonian. Let us consider a gauged eigenvalue equation

$$H_G(\xi)\chi(\xi) = E\chi(\xi) \quad (2. 21)$$

where we identify $\chi(\xi(x)) \equiv \chi(x)$ as given in (2.8). We now expand the gauged eigenfunction $\chi(\xi)$ in the form

$$\chi(\xi) = \left(\frac{\xi_2 - \xi}{\xi_1 - \xi_2}\right)^n \sum_{j=0}^{\infty} \frac{P_j(E)}{j!} \left(\frac{\xi_1 - \xi}{\xi_2 - \xi}\right)^j \quad (2. 22)$$

where ξ_1, ξ_2 are two distinct roots of the coefficient of ∂_ξ^2 in (2.4). Now a suitable choice of ξ_1, ξ_2 (note that ξ_1, ξ_2 can be chosen in six ways) gives us a three-term recursion relation satisfied by $\{P_j(E)\}$

$$\begin{aligned} -[(2j - n + 1)\hat{C}_{0-} + \hat{C}_-]P_{j+1} &= [E + d_1 + \hat{C}_0(j - \frac{n}{2}) + \hat{C}_{00}(j - \frac{n}{2})^2]P_j \\ &+ j(j - 1 - n)[(2j - n - 1)\hat{C}_{+0} + \hat{C}_+]P_{j-1}, \\ &(j \geq 0), (2. 23) \end{aligned}$$

where \hat{C}_{ij} are determined from the relations

$$\begin{aligned}\hat{C}_{+0} &= -\frac{1}{(\xi_1 - \xi_2)^2} [2\xi_1\xi_2^3 C_{++} + \xi_2(\xi_1 + \xi_2)C_{00} + 2C_{--}], \\ \hat{C}_{00} &= \frac{1}{(\xi_1 - \xi_2)^2} [6\xi_1^2\xi_2^2 C_{++} + (\xi_1^2 + \xi_2^2 + 4\xi_1\xi_2)C_{00} + 6C_{--}], \\ \hat{C}_{0-} &= -\frac{1}{(\xi_1 - \xi_2)^2} [2\xi_1^3\xi_2 C_{++} + \xi_1(\xi_1 + \xi_2)C_{00} + 2C_{--}],\end{aligned}\tag{2. 24}$$

and \hat{C}_j, d_1 are given by

$$\begin{aligned}\hat{C}_+ &= \frac{1}{\xi_1 - \xi_2} [\xi_2^2 C_+ + \xi_2 C_0 + C_-], \\ \hat{C}_0 &= -\frac{1}{\xi_1 - \xi_2} [2\xi_1\xi_2 C_+ + (\xi_1 + \xi_2)C_0 + 2C_-], \\ \hat{C}_- &= \frac{1}{\xi_1 - \xi_2} [\xi_1^2 C_+ + \xi_1 C_0 + C_-], \\ d_1 &= d + \frac{n(n+2)}{12} (C_{00} - \hat{C}_{00}).\end{aligned}\tag{2. 25}$$

From equation (2.23)–(2.25) it transpires that the eigenstates of Type I–III Hamiltonians generate, in general, different orthogonal family of polynomials in the energy variable corresponding to each algebraization of Table 1 provided

$$(2j - n + 1)\hat{C}_{0-} + \hat{C}_- \neq 0, \quad \forall j \geq 0\tag{2. 26}$$

The family $\{P_j(E)\}$ can be expressed in terms of monic polynomials $\{\tilde{P}_j(E)\}$ satisfying the recurrence relation

$$\tilde{P}_{j+1} = (E - \lambda_j)\tilde{P}_j - \rho_j\tilde{P}_{j-1},\tag{2. 27}$$

$$\tilde{P}_j = \omega_j P_j, \quad j \geq 0,\tag{2. 28}$$

where $\tilde{P}_{-1} = P_1 \equiv 0$ and $\tilde{P}_0 = P_0 \equiv 1$. It is now straightforward to write down the expression of eigenfunctions from (2.8) and (2.22). It follows from equation (2.23) that $\rho_0 = 0$ and $\rho_{n+1} = 0$; so the infinite power series expansion in (2.22) truncates after the $(n+1)$ -th term since the coefficients $P_j(E_i)$ vanishes for $j > n$, where E_i ($i = 0, 1, \dots, n$) are the zeros of the critical polynomial $\tilde{P}_{n+1}(E)$. This points to the fact that Type I–Type III potentials belong to QES class having $(n+1)$ levels for each algebraization. The final expression of the band-edge eigenfunctions may be written in the form

$$\psi_{E_i}(x) = \mu(x)(\xi(x) - \xi_2)^n \sum_{j=0}^n \frac{P_j(E_i)}{j!} \left(\frac{\xi(x) - \xi_1}{\xi(x) - \xi_2} \right)^j, \quad (i = 0, 1, \dots, n),\tag{2. 29}$$

where $\mu(x)$ is determined from (2.11) for each of the three types given by (2.12)–(2.14) and the band-edge eigenvalues are E_0, E_1, \dots, E_n . Note that n is to be computed for each of the algebraization in Table 1.

3 Eigenstates and spectra of Type I–III models for some real values of the potential parameters

In this section we construct some examples based upon the general results obtained in the previous section. It may be useful to collect the following identities and differential relations

Table 2: The coefficients ρ_j and λ_j of the recurrence relation (2.27) are given for each of the six algebraizations of Type I model.

Solution			
no.	n	ρ_j	λ_j
1.1	A	$(\frac{1}{2})^2 j(j-1-A)(2j+2B-1) \times (2B-2A+2j-1)$	$\frac{1-2k^2}{2}[A(A+1) + (A-2j)(2B-A+2j)]$
1.2	$A-1$	$(\frac{1}{2})^2 j(j-A)(2j+2B+1) \times (2B-2A+2j-1)$	$\frac{1-2k^2}{2}[A(A+1) + (A-1-2j) \times (2B+2j-A+1)]$
1.3	$B-1$	$(\frac{1}{2})^2 j(j-B)(2j+1) \times (2j-2B+1)$	$\frac{1-2k^2}{4}[2B^2-1 + 2(B-1-2j)(2j-B+2)] + \frac{1}{2}ikk'(2A+1)(2B-2A-4j-3)$
1.4	$A-\frac{1}{2}$	$(\frac{1}{2})^2 j(2j-2A-1)(j-A) \times (2j+1)$	$\frac{1-2k^2}{8}[4A(A+1)-1 + (2A-4j-1) \times (4j-2A+3)] + 2Bikk'(A-2j-1)$
1.5	$B-\frac{1}{2}$	$(\frac{1}{2})^2 j(j+A)(2j-2B-1) \times (2A-2B+2j+1)$	$\frac{1-2k^2}{8}[4B^2-1 + (2B-4j-1) \times (4j+4A-2B+3)]$
1.6	$B-\frac{3}{2}$	$(\frac{1}{2})^2 j(2j-2B+1)(j+A+1) \times (2A-2B+2j+1)$	$\frac{1-2k^2}{8}[4B^2-1 + (2B-4j-3) \times (4j+4A-2B+5)]$

among the Jacobian elliptic functions which will be frequently used:

$$sn^2x + cn^2x = 1, \quad dn^2x + k^2sn^2x = 1$$

$$sn'x = cnx dnx, \quad cn'x = -snx dnx, \quad dn'x = -k^2snx cnx$$

In the following examples we denote the eigenstates[spectra] by $\phi_r(x)[e_r]$ whenever ordering is possible. Otherwise we denote them by $\psi_{E_i}[E_i]$ and $\psi_{E'_i}[E'_i]$ indicating different algebraizations.

3.1 Type I model [defined on the domain (0,2K)]

We have got six algebraizations (see the solution 1.1–1.6 of Table 1) for Type I Hamiltonian (2.15). For each of them the corresponding eigenstates generate an orthogonal family of polynomials satisfying the recurrence relation (2.27). The explicit expressions of ρ_j and λ_j corresponding to each solutions of n are given in Table 2. The corresponding expressions of ω_j together with the choice of ξ_1, ξ_2 and the overall restrictions on potential parameters are given in Table 3.

It is clear that the algebraic solutions are obtained for the following two cases:

Case 1. $A \in (\mathbb{N}-1) \cup (\mathbb{N}-\frac{1}{2}), B \in \mathbb{R}$

Here B is any real parameter and for each real values of B , A is allowed to take $0, 1/2, 1, 3/2, \dots, \Lambda \leq B$ (Λ is an integer or half-integer). For integer values of A , from the algebraizations 1.1–1.2 we get $(2A+1)$ band edge eigenstates and eigenvalues. Also for half-integer values of A the algebraization 1.4 gives $(A+1/2)$ solutions of the Schrödinger equation.

Case 2. $B \in \mathbb{N} \cup \mathbb{N}-\frac{1}{2}, A \in \mathbb{R}$

Here A is any real number and B is allowed to take values $1/2, 1, 3/2, \dots, \Lambda \leq A+1$ (Λ is integer or half-integer). It is to be noted that algebraization 1.3 is considered for integer values of B and 1.5–1.6 for half-integer B .

Table 3: The expressions for ω_j of the relation (2.28) along with the choice of roots ξ_1, ξ_2 and the overall restrictions on potential parameters are provided for each algebraization for Type I model.

Solution		Overall restrictions		
no.	n	ω_j	(ξ_1, ξ_2)	on A, B
1.1	A	$(\frac{1}{2})^j \frac{\prod_{r=0}^{[B]-A+j} (2B-2A+2j-1-2r)}{\prod_{r=0}^{[B]-A} (2B-2A-1-2r)}$	$(-1, 1)$	$A \in \mathbb{N} - 1, B \in \mathbb{R}, B - A \geq 0$
1.2	$A - 1$	$(\frac{1}{2})^j \frac{\prod_{r=0}^{[B]-A+j} (2B-2A+2j-1-2r)}{\prod_{r=0}^{[B]-A} (2B-2A-1-2r)}$	$(-1, 1)$	$A \in \mathbb{N}, B \in \mathbb{R}, B - A \geq 0$
1.3	$B - 1$	$(\frac{-1}{2})^j \prod_{r=0}^j (2j + 1 - 2r)$	$(\frac{ik'}{k}, \frac{-ik'}{k})$	$A \in \mathbb{R}, B \in \mathbb{N}, A - B + 1 \geq 0$
1.4	$A - \frac{1}{2}$	$(\frac{-1}{2})^j \prod_{r=0}^j (2j + 1 - 2r)$	$(\frac{ik'}{k}, \frac{-ik'}{k})$	$A \in \mathbb{N} - \frac{1}{2}, B \in \mathbb{R}, B - A \geq 0$
1.5	$B - \frac{1}{2}$	$(\frac{1}{2})^j \frac{\prod_{r=0}^{[A-B]+j} (2A-2B+2j-2r+1)}{\prod_{r=0}^{[A-B]} (2A-2B-2r+1)}$	$(-1, 1)$	$A \in \mathbb{R}, B \in \mathbb{N} - \frac{1}{2}, A - B + 1 \geq 0$
1.6	$B - \frac{3}{2}$	$(\frac{1}{2})^j \frac{\prod_{r=0}^{[A-B]+j} (2A-2B+2j-2r+1)}{\prod_{r=0}^{[A-B]} (2A-2B-2r+1)}$	$(-1, 1)$	$A \in \mathbb{R}, B \in \mathbb{N} + \frac{1}{2}, A - B + 1 \geq 0$

We now consider some specific examples when both of A and B are integer or half-integer.

$$1. \mathbf{A = 0, B=1/2} : \quad V(x) = \frac{1}{4} \frac{dn^2 x}{sn^2 x} - \frac{1}{2} \frac{cnx}{sn^2 x}$$

$$\phi_0(x) = \sqrt{\frac{snx}{1+cnx}}, \quad e_0 = 0 \quad (3.1)$$

$$2. \mathbf{A=0, B=3/2} : \quad V(x) = \frac{9}{4} \frac{dn^2 x}{sn^2 x} - \frac{3}{2} \frac{cnx}{sn^2 x}$$

$$\phi_0(x) = \left(\frac{snx}{1+cnx}\right)^{3/2}, \quad e_0 = 0 \quad (3.2)$$

$$3. \mathbf{A = 1/2, B=1} : \quad V(x) = \frac{7}{4} \frac{dn^2 x}{sn^2 x} - 2 \frac{cnx}{sn^2 x}$$

$$\phi_0(x) = \frac{\sqrt{snx} dn x}{1+cnx} \exp\left(-\frac{i}{2} \tan^{-1} \frac{kcnx}{k'}\right), \quad e_0 = \frac{1-2k^2}{4} - ikk' \quad (3.3)$$

$$4. \mathbf{A=1/2, B=2} : \quad V(x) = \frac{19}{4} \frac{dn^2 x}{sn^2 x} - 4 \frac{cnx}{sn^2 x}$$

$$\phi_0(x) = \frac{\sqrt{dnx} sn^{3/2} x}{(1+cnx)^2} \exp\left(-\frac{i}{2} \tan^{-1} \frac{kcnx}{k'}\right), \quad e_0 = \frac{1-2k^2}{4} - 2ikk' \quad (3.4)$$

$$5. \mathbf{A = 1, B=3/2} : \quad V(x) = \frac{17}{4} \frac{dn^2 x}{sn^2 x} - \frac{9}{2} \frac{cnx}{sn^2 x}$$

For $0 < k^2 < 1/2$,

$$\phi_{0,1}(x) = \frac{\sqrt{snx}}{(1+cnx)^{5/2}} [g_{\mp}(k)cn^2 x + 4cnx + 4 - g_{\mp}(k)], \quad e_{0,1} = 1 - 4k^2 + \frac{1}{2}g_{\mp}(k), \quad (3.5)$$

$$\phi_2(x) = \frac{dnx\sqrt{snx}}{(1+cnx)^{3/2}}, \quad e_2 = 1 - 2k^2, \quad (3.6)$$

Table 4: The coefficients ρ_j and λ_j of the recurrence relation (2.27) are provided for Type II(1st two rows) and Type III models.

Solution			
no.	n	ρ_j	λ_j
2.1	$A - \frac{1}{2}$	$(\frac{k'^2}{2})^2 j(2j - 2A - 1)(j - A + B) \times (2j + 2B + 1)$	$-\frac{k^2}{4}(2B + 1)^2 - \frac{k'^2}{8}(2A + 1)^2 + \frac{1+k^2}{8} \times (2A - 4j - 1)(4B - 2A + 4j + 3)$
2.2	$A - \frac{1}{2}$	$(\frac{k'^2}{2})^2 j(2j - 2A - 1)(j - A - B - 1) \times (2j - 2B - 1)$	$-\frac{k^2}{4}(2B + 1)^2 - \frac{k'^2}{8}(2A + 1)^2 + \frac{1+k^2}{8} \times (2A - 4j - 1)(4j - 2A - 4B - 1)$
3.1	A	$(\frac{k^2}{2})^2 j(j - A - 1)(2j - 1)(2j - 2A - 1)$	$\frac{2j-A}{2} [(2j - A)(2 - k^2) + 4Bik'] - B^2 k'^2 - A(A + 1)k^2/2$
3.2	$A - 1$	$(\frac{k^2}{2})^2 j(j - A)(2j - 2A - 1)(2j + 1)$	$\frac{2j-A+1}{2} [(2j - A + 1)(2 - k^2) + 4Bik'] - B^2 k'^2 - A(A + 1)k^2/2$
3.3	$A - \frac{1}{2}$	$(\frac{k^2}{2})^2 j(j - A)(2j - 2A - 1)(2j + 1)$	$(\frac{1+2Bik'}{2})^2 - \frac{k^2}{8}(2A + 1)^2 + \frac{1}{8}(2A - 4j - 1) \times [(k^2 - 2)(4j - 2A + 3) - 8Bik']$

where $g_{\pm}(k) = 6k^2 - 1 \pm \sqrt{1 - 36k^2k'^2}$

For $1/2 < k^2 < 1$, the suffixes 0,1,2 have to be replaced by 1,2,0 respectively.

Proceeding in the same fashion, we can find the eigenstates and spectra for higher values of A and B.

3.2 Type II model [defined on the domain (0,2K)]

Here two algebraizations are obtained (see the solution 2.1–2.2 of Table 1). The related orthogonal polynomials are determined by the entries 2.1–2.2 of Table 4 and Table 5.

Note that both algebraizations are valid provided A is restricted to positive half-integer values only while the other parameter B is arbitrary.

Some examples are furnished below.

$$1. \mathbf{A} = \frac{1}{2} : \quad V(x) = B(B + 1) \frac{dn^2 x}{sn^2 x} - \frac{3}{4} dn^2 x \tag{3. 7}$$

$$\phi_0(x) = \frac{sn^{B+1} x}{(cnx + dnx)^{B+1/2}}, \quad e_0 = -\frac{k'^2}{2} - \frac{k^2}{4}(2B + 1)^2$$

For $B \leq 0$, the following degenerated state is found:

$$\psi_0(x) = (cnx + dnx)^{B+1/2} / sn^B x$$

$$2. \mathbf{A} = \frac{3}{2} : \quad V(x) = B(B + 1) \frac{dn^2 x}{sn^2 x} - \frac{15}{4} dn^2 x \tag{3. 8}$$

$$\begin{aligned} \phi_{0,1}(x) = & \frac{sn^{B+1} x}{(cnx + dnx)^{B+3/2}} \left[\left\{ (1 + k^2) \left(B + \frac{3}{2} \right) + \eta_{\mp}(k) \right\} cn^2 x \right. \\ & \left. + 2 \left(B + \frac{3}{2} \right) cnx dnx + k'^2 \left(B + \frac{3}{2} \right) - \eta_{\mp}(k) \right], \end{aligned}$$

$$e_{0,1} = \frac{6k^2 - 10 - k^2(2B + 1)^2}{4} \mp \sqrt{2(1 + k^4) \left(B + \frac{1}{2} \right)^2 - k'^4},$$

Table 5: The expressions for ω_j of the relation (2.28) together with the choice of ξ_1, ξ_2 and the overall restrictions on potential parameters are provided for Type II and Type III models.

Solution		ω_j	(ξ_1, ξ_2)	Overall Restrictions
no.	n			A, B
2.1	$A - \frac{1}{2}$	$(\frac{k'^2}{2})^j \frac{\prod_{r=0}^{j+ B } (2j+2B-2r+1)}{\prod_{r=0}^{ B } (2B-2r+1)}$	$(-1, 1)$	$A \in \mathbb{N} - \frac{1}{2}, B \in \mathbb{R}$
2.2	$A - \frac{1}{2}$	$(\frac{k'^2}{2})^j \frac{\prod_{r=0}^j (2j-2B-2r-1)}{-2B-1}$	$(-1, 1)$	$A \in \mathbb{N} - \frac{1}{2}, B \in \mathbb{R}, B \leq 0$
3.1	A	$(\frac{k^2}{2})^j \frac{\prod_{r=0}^j (2j-1-2r)}{(-1)}$	$(\frac{i}{k'}, \frac{-i}{k'})$	$A \in \mathbb{N} - 1, B \in \mathbb{R}$
3.2	$A - 1$	$(\frac{k^2}{2})^j \prod_{r=0}^j (2j+1-2r)$	$(\frac{i}{k'}, \frac{-i}{k'})$	$A \in \mathbb{N}, B \in \mathbb{R}$
3.3	$A - \frac{1}{2}$	$(\frac{k^2}{2})^j \prod_{r=0}^j (2j+1-2r)$	$(\frac{i}{k'}, \frac{-i}{k'})$	$A \in \mathbb{N} - \frac{1}{2}, B \in \mathbb{R}$

$$\text{where } \eta_{\pm}(k) = -(1+k^2)(B + \frac{1}{2}) \pm \sqrt{2(1+k^4)(B + \frac{1}{2})^2 - k'^4}$$

For $B \leq 0$ two other degenerated states are obtained:

$$\begin{aligned} \psi_{0,1}(x) &= \frac{(cnx + dnx)^{B-1/2}}{sn^B x} [\{(1+k^2)(B + \frac{3}{2}) + \eta_{\mp}(k)\} cn^2 x \\ &\quad + (1-2B)cnx dnx - k'^2(B - \frac{1}{2}) - (1+k^2)(2B+1) - \eta_{\mp}(k)] \end{aligned}$$

3.3 Type III model [defined on the entire real line]

This corresponds to three algebraizations (see the solution 3.1–3.3 of Table 1), the first two of which are for an integer A while the third one is for a half-integer A . As before, the eigenstates generate an orthogonal family of polynomials in the energy variable for each algebraization.

The recurrence relation (2.27) is determined by the entries 3.1–3.3 of Table 4 and Table 5. The other parameter B takes arbitrary values.

We now consider some examples.

$$\mathbf{1. A = 0:} \quad V(x) = B^2 k^2 cn^2 x + B k^2 snx cnx \quad (3.9)$$

$$\phi_0(x) = \exp(-B \tan^{-1} \frac{snx}{cnx}), \quad e_0 = -B^2 k'^2$$

$$\mathbf{2. A = \frac{1}{2}:} \quad V(x) = (B^2 - \frac{3}{4}) k^2 cn^2 x + 2B k^2 snx cnx \quad (3.10)$$

$$\phi_0(x) = \sqrt{dnx} \exp[-B \tan^{-1}(\frac{snx}{cnx}) + \frac{i}{2} \tan^{-1}(\frac{k' snx}{cnx})], \quad e_0 = -\frac{k^2}{2} + (\frac{1+2Bik'}{2})^2$$

$$\mathbf{3. A = 1:} \quad V(x) = (B^2 - 2) k^2 cn^2 x + 3B k^2 snx cnx \quad (3.11)$$

$$\phi_0(x) = dnx \exp(-B \tan^{-1} \frac{snx}{cnx}), \quad e_0 = -B^2 k'^2 - k^2$$

$$\phi_{1,2}^{(i)}(x) = [(k^2 \mp \sqrt{k^4 - 16k'^2 B^2}) snx + 4B cnx] \exp(-B \tan^{-1} \frac{snx}{cnx}),$$

$$\phi_{1,2}^{(ii)}(x) = [4B k'^2 snx + (k^2 \pm \sqrt{k^4 - 16k'^2 B^2}) cnx] \exp(-B \tan^{-1} \frac{snx}{cnx}),$$

$$e_{1,2} = 1 - \frac{3}{2}k^2 - B^2k'^2 \mp \frac{1}{2}\sqrt{k^4 - 16k'^2B^2},$$

where the superscripts in the eigenstates indicate their double-degeneracy.

4 ES-limit of QES models

We have so far constructed three new classes of QES potentials and explicitly obtained their eigenstates and spectra. Our purpose in this section is to show that corresponding to each type there is associated an ES class potential. It is useful to write down the following results of the ES-limit

$$sn(x, k) \xrightarrow[k \rightarrow 0]{k \rightarrow 1} \begin{cases} \tanh x \\ \sin x \end{cases}, \quad cn(x, k) \xrightarrow[k \rightarrow 0]{k \rightarrow 1} \begin{cases} \operatorname{sech} x \\ \cos x \end{cases}, \quad dn(x, k) \xrightarrow[k \rightarrow 0]{k \rightarrow 1} \begin{cases} \operatorname{sech} x \\ 1 \end{cases} \quad (4. 1)$$

Each of the three types of potentials are doubly periodic, one is real and the other imaginary. As $k \rightarrow 1$ (or 0) we get an ES potential having an imaginary(or real) period.

4.1 ES classes associated to Type I model

We have already shown that the Type I model [cf. equation (2.15)] belongs to the QES periodic class. The potential has a real period $4K$ and imaginary period $4K'$, where $K' \equiv K(k')$. Using the relation (4.1) and the relation

$$K [K'] \xrightarrow[k \rightarrow 0]{k \rightarrow 1} \begin{cases} \infty [\pi/2] \\ \pi/2 [\infty] \end{cases}, \quad (4. 2)$$

we see that the QES model is also exactly solvable when the modulus parameter $k \rightarrow 1$ and 0. The associated ES classes are given by

$$k \rightarrow 1 : V_1(x) = [B^2 + A(A + 1)]\operatorname{cosech}^2 x - 2B(A + \frac{1}{2})\operatorname{cosech} x \coth x, \quad x \in (0, \infty) \quad (4. 3)$$

$$k \rightarrow 0 : V_2(x) = [B^2 + A(A + 1)]\operatorname{cosec}^2 x - 2B(A + \frac{1}{2})\operatorname{cosec} x \cot x, \quad x \in (0, \pi), \quad (4. 4)$$

whose eigenstates and spectra are [4]

$$\begin{aligned} \psi_r^{(1)}(x) &= (\cosh x - 1)^{\frac{B-A}{2}} (\cosh x + 1)^{-\frac{B+A}{2}} P_r^{(B-A-\frac{1}{2}, -B-A-\frac{1}{2})}(\cosh x), \\ \psi_r^{(2)}(x) &= (1 - \cos x)^{\frac{B-A}{2}} (1 + \cos x)^{-\frac{B+A}{2}} P_r^{(B-A-\frac{1}{2}, -B-A-\frac{1}{2})}(\cos x), \end{aligned} \quad (4. 5)$$

$$E_r^{(1)} = -(A - r)^2, \quad E_r^{(2)} = (A - r)^2, \quad (r = 0, 1, 2, \dots) \quad (4. 6)$$

where $P_r^{(\alpha, \beta)}(x)$ is the Jacobi polynomial and the superscripts 1 and 2 indicates the potentials $V_1(x)$ and $V_2(x)$ respectively. Note that we have solved the Type I spectral problem (see Subsec. 3.1) when at least one of the parameters A, B is an integer or an half-integer. Thus the ES results (4.5) and (4.6) can be reproduced from the associated QES results for this restricted domain of potential parameters only.

We now consider the ES-limits of the examples given in Subsec. 3.1 [see equations (3.1)–(3.5)]. In the following the superscripts in the eigenstates indicate whether they correspond to the potentials $V_1(x)$ and $V_2(x)$.

1. $\mathbf{A} = \mathbf{0}, \mathbf{B} = \mathbf{1/2}$:

$$V_1(x) = \frac{1}{4} \operatorname{cosech}^2 x - \frac{1}{2} \operatorname{cosech} x \coth x \quad (4.7)$$

$$V_2(x) = \frac{1}{4} \operatorname{cosec}^2 x - \frac{1}{2} \operatorname{cosec} x \cot x \quad (4.8)$$

$$\phi_0^{(1)}(x) = \sqrt{\tanh \frac{x}{2}}, \quad \phi_0^{(2)}(x) = \sqrt{\tan \frac{x}{2}}, \quad e_0^{(1)} = e_0^{(2)} = 0$$

From (4.5) and (4.6) it follows that we can reproduce the ground states for the two ES classes.

2. $\mathbf{A} = \mathbf{0}, \mathbf{B} = \mathbf{3/2}$:

$$V_1(x) = \frac{9}{4} \operatorname{cosech}^2 x - \frac{3}{2} \operatorname{cosech} x \coth x \quad (4.9)$$

$$V_2(x) = \frac{9}{4} \operatorname{cosec}^2 x - \frac{3}{2} \operatorname{cosec} x \cot x \quad (4.10)$$

$$\phi_0^{(1)}(x) = \tanh^{\frac{3}{2}} \frac{x}{2}, \quad \phi_0^{(2)}(x) = \tan^{\frac{3}{2}} \frac{x}{2}, \quad e_0^{(1)} = e_0^{(2)} = 0$$

Again ground levels for two ES potentials (4.9),(4.10) are reproduced.

3. $\mathbf{A} = \mathbf{1/2}, \mathbf{B} = \mathbf{1}$:

$$V_1(x) = \frac{7}{4} \operatorname{cosech}^2 x - 2 \operatorname{cosech} x \coth x \quad (4.11)$$

$$V_2(x) = \frac{7}{4} \operatorname{cosec}^2 x - 2 \operatorname{cosec} x \cot x \quad (4.12)$$

$$\phi_0^{(1)}(x) = \frac{\sqrt{\operatorname{sech} x \tanh x}}{1 + \operatorname{sech} x}, \quad \phi_0^{(2)}(x) = \frac{\sqrt{\sin x}}{1 + \cos x}, \quad e_0^{(1)} = -\frac{1}{4} = -e_0^{(2)}$$

These are the ground states for the ES classes.

4. $\mathbf{A} = \mathbf{1/2}, \mathbf{B} = \mathbf{2}$:

$$V_1(x) = \frac{19}{4} \operatorname{cosech}^2 x - 4 \operatorname{cosech} x \coth x \quad (4.13)$$

$$V_2(x) = \frac{19}{4} \operatorname{cosec}^2 x - 4 \operatorname{cosec} x \cot x \quad (4.14)$$

$$\phi_0^{(1)}(x) = \frac{\sinh^{3/2} x}{(1 + \cosh x)^2}, \quad \phi_0^{(2)}(x) = \frac{\sin^{3/2} x}{(1 + \cos x)^2}, \quad e_0^{(1)} = -\frac{1}{4} = -e_0^{(2)}$$

These are the ground levels for ES potentials.

5. $\mathbf{A} = \mathbf{1}, \mathbf{B} = \mathbf{3/2}$:

$$V_1(x) = \frac{17}{4} \operatorname{cosech}^2 x - \frac{9}{2} \operatorname{cosech} x \coth x \quad (4.15)$$

$$V_2(x) = \frac{17}{4} \operatorname{cosec}^2 x - \frac{9}{2} \operatorname{cosec} x \cot x \quad (4.16)$$

$$\phi_0^{(1)}(x) = \frac{\sqrt{\sinh x}}{(1 + \cosh x)^{3/2}}, \quad \phi_1^{(2)}(x) = \frac{\sqrt{\sin x}}{(1 + \cos x)^{3/2}}, \quad e_0^{(1)} = -1 = -e_1^{(2)}$$

$$\phi_1^{(1)}(x) = \frac{\sqrt{\sinh x}(\cosh x - 3)}{(1 + \cosh x)^{3/2}}, \quad \phi_0^{(2)}(x) = \frac{\sqrt{\sin x}(\cos x - 3)}{(1 + \cos x)^{3/2}}, \quad e_1^{(1)} = 0 = e_0^{(2)}$$

Here two ES levels are reproduced.

4.2 ES class associated to Type II model

Using as before the relations (4.1) and (4.2), we see that in the limit $k \rightarrow 1$ Type II potential (2.16) goes over to the following ES class known as the generalized Pöschl-Teller potential:

$$V_3(x) = B(B+1)\operatorname{cosech}^2 x - A(A+1)\operatorname{sech}^2 x, \quad (4.17)$$

its eigenstates and spectra being given by

$$\psi_r^{(3)}(x) = (\cosh 2x - 1)^{-B/2}(\cosh 2x + 1)^{-A/2} P_r^{(-B-\frac{1}{2}, -A-\frac{1}{2})}(\cosh 2x),$$

$$E_r^{(3)} = -(A+B-2r)^2, \quad (r = 0, 1, 2, \dots), \quad (4.18)$$

where the superscripts indicate correspondance with $V_3(x)$.

We recall that the QES levels for Type II model is obtained for half-integer values of A . Let us now take the ES-limit of the examples given in Subsec. 3.2 [see equations (3.6)–(3.7)]. The other parameter B is taken as negative real number.

$$1. \mathbf{A} = \frac{1}{2}: \quad V_3(x) = B(B+1)\operatorname{cosech}^2 x - \frac{3}{4}\operatorname{sech}^2 x \quad (4.19)$$

$$\psi_0^{(3)}(x) = \operatorname{cosech}^B x \sqrt{\operatorname{sech} x}, \quad e_0^{(3)} = -\frac{(2B+1)^2}{4}.$$

Here $\psi_0^{(3)}, e_0^{(3)}$ are ground level for the ES potential $V_3(x)$.

$$2. \mathbf{A} = \frac{3}{2}: \quad V_3(x) = B(B+1)\operatorname{cosech}^2 x - \frac{15}{4}\operatorname{sech}^2 x \quad (4.20)$$

$$\psi_0^{(3)}(x) = \sinh^{-B} x \cosh^{-3/2} x, \quad e_0^{(3)} = -(B + \frac{3}{2})^2,$$

$$\psi_1^{(3)}(x) = \frac{\sinh^{-B} x}{\cosh^{3/2} x} [(2B+1) \cosh^2 x - 2] \quad e_1^{(3)} = -(B - \frac{1}{2})^2$$

Here $\psi_{0,1}^{(3)}, e_{0,1}^{(3)}$ are two levels for the ES potential (4.20).

4.3 ES class associated to Type III model

The Type III potential (2.17) is defined on the entire real line. This is a QES periodic potential having a real period $2K$ and an imaginary period $4K'$. The algebraic sector is determined for an integer or a half-integer A , while B is arbitrary real parameter. In the limit $k \rightarrow 1$, this potential coincides to the following ES class:

$$V_4(x) = [B^2 - A(A+1)]\operatorname{sech}^2 x + 2B(A + \frac{1}{2})\operatorname{sech} x \tanh x. \quad (4.21)$$

The eigenstates and spectra of the potential (4.21) are

$$\begin{aligned}\psi_r^{(4)}(x) &= \operatorname{sech}^A x \exp[-B \tan^{-1}(\sinh x)] P_r^{(-iB-A-\frac{1}{2}, iB-A-\frac{1}{2})}(i \sinh x), \\ E_r^{(4)} &= -(A-r)^2, \quad (r = 0, 1, 2, \dots)\end{aligned}\tag{4.22}$$

We now take the ES-limit (here $k \rightarrow 1$ only) of the examples given in Subsec 3.3. Note that the limit $k \rightarrow 0$ gives a free particle motion.

$$\mathbf{1. A = 0 :} \quad V_4(x) = B^2 \operatorname{sech}^2 x + B \operatorname{sech} x \tanh x \tag{4.23}$$

$$\phi_0^{(4)}(x) = \exp(-B \tan^{-1} \sinh x), \quad e_0^{(4)} = 0$$

Clearly, the ground state for (4.23) agrees with the general results(4.22) for $A = 0$.

$$\mathbf{2. A = \frac{1}{2} :} \quad V_4(x) = (B^2 - \frac{3}{4}) \operatorname{sech}^2 x + 2B \operatorname{sech} x \tanh x \tag{4.24}$$

$$\phi_0^{(4)}(x) = \sqrt{\operatorname{sech} x} \exp[-B \tan^{-1}(\sinh x)], \quad e_0^{(4)} = -1/4$$

This is also in agreement with (4.22) for $A = 1/2$.

$$\mathbf{3. A = 1 :} \quad V_4(x) = (B^2 - 2) \operatorname{sech}^2 x + 3B \operatorname{sech} x \tanh x \tag{4.25}$$

$$\phi_0^{(4)}(x) = \operatorname{sech} x \exp(-B \tan^{-1} \sinh x) = \phi_1^{(4)}(x) \quad , \quad e_0^{(4)} = -1 = e_1^{(4)}$$

$$\phi_2^{(4)}(x) = \operatorname{sech} x (\sinh x + 2B) \exp(-B \tan^{-1} \sinh x), \quad e_2^{(4)} = 0$$

Thus two ES levels are reproduced from the corresponding QES levels.

Hence we have shown that corresponding to three QES models Type I, Type II and Type III, there is associated some definite ES classes namely V_1, V_2 (for Type I), V_3 (for Type II) and V_4 (for Type III) respectively. In the ES limit we can reproduce some ES levels from the corresponding QES levels as well.

5 Conclusion

To conclude, we have constructed three new QES elliptic potentials Type I–III using $\mathfrak{sl}(2, \mathbb{R})$ Lie-algebraic scheme and obtained their algebraic levels analytically. Further we have shown that the eigenstates of QES Hamiltonians generate an orthogonal family of polynomials in the energy variable. The interesting point is that the elliptic parameter k ($0 < k^2 < 1$) in the models turns out to be responsible for QES class and when it touches the end-points 0 and 1 of its domain the ES classes are revealed. We have explicitly shown that some ES levels can be reproduced on a restricted domain of potential parameters.

Acknowledgement

I would like to thank Professor B Bagchi for guidance.

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