Line discontinuities, local action with both the field and its dual, and spin from no spin in two-dimensional scalar theory

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Abstract

We consider a local action with both the real scalar field and its dual in two Euclidean dimensions. The role of singular line discontinuities is emphasized. Exotic properties of the correlation of the field with its dual, the generation of spin from scalar fields, and quantization of dual charges are pointed out. Wick's theorem and rotation properties of fermions are recovered for half-integer quantization.

1 Introduction

Duality transformations play important roles in many field theories and statistical physics models. They provide an equivalent description of the system. They often relate two models

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which appear to be different. They map weak coupling or high temperature regime of one model to the strong coupling or low temperature regime of the other. This provides an understanding of the phases of the model as well as non-perturbative calculational techniques. The degrees of freedom of the dual theory play the role of disorder variables of the original theory. They often provide valuable order parameters to characterize the phases of the model. In addition to all this, order and disorder variables close together have exotic properties. For instance, certain combinations of bosonic variables may behave as a fermion. Such combinations also play a crucial role in providing the description and properties of the model in some phases. Because of these important features, it is useful to have systematic techniques for handling both the field and its dual on an equal footing. In this work we do this for a free massless real scalar field in two (Euclidean) dimensions. This case, though simple, encodes rich physics. It is self-dual, so that the dual field is also free scalar field. We also want to handle correlation function of the field with its dual. We thus obtain a *local* action which has both the fields simultaneously present. We derive this action in the presence of source terms which bring out the role of line discontinuities in the field configurations. We emphasize how the line discontinuities result in novel properties of the mixed correlation functions, and explain how they lead to the generation of spin from scalar fields, the quantization of dual charges and the construction of the composite fermion operator. The local action involving both the field and its dual was studied in the context of string theory in Refs. [1, 2]. The action is also related to Hamiltonians used for describing condensed matter systems in one dimension [3, 4].

In Section 2.1, we express the partition function in terms of the dual field. In Section 2.2, we bring out the role of line discontinuities, in close analogy with the order-disorder formulation of the 2-d Ising model [5]. We emphasize that such discontinuities in a field configuration are meaningful without requiring a smoothening, and contribute a finite action. In Section 2.3, we arrive at the local action with both the field and its dual present. This local action is not manifestly invariant under rotations. The underlying reason is the choice of the line of discontinuity. It is also not real in spite of being equivalent to the free massless scalar theory. Even though it has double the number of fields, it represents the original theory.

The correlation of order and disorder variables has been discussed in various contexts in [6, 7]. In our case, the correlation function of the field with its dual has a number of exotic properties, which we explain in Section 2.4. First, it is discontinuous, reflecting that the field introduces a line of discontinuity in the configuration of dual field. The correlation function does not die off at large distances, representing the non-local effect the field has. Indeed the correlation function is proportional to the angle subtended by the vector joining the coordinates of the field and the dual field with the line of discontinuity. Both the field and its dual being scalars, the correlation functions involving only the field (or only the dual field) depend on the distance and not on the orientation. This is no longer true of the mutual correlations, again reflecting the role of line discontinuities. In addition to all this, the correlation function is pure imaginary even though the fields are real. This is traced to the action which is not real.

The free massless real scalar field in two Euclidean dimensions embodies all minimal models of conformal field theory through the use of boundary charges, the Coulomb gas formalism and the vertex operators [8, 9]. In Section 3, we show that the vertex operators of both the field and its dual can be handled together in the formalism we use. This exhibits the phenomenon of 'spin from no spin'. The correlation functions imply that the vertex operators of the dual field (or equivalently the original field) acquire spin though constructed from a scalar field. The mixed correlation functions appear to be multivalued. But we emphasize that they are simply discontinuous, a consequence of line discontinuities. For a specific 'quantization' of the 'charges' of the field and its dual, the single valuedness of the correlation function is restored. This is exactly parallel to Dirac quantization of electric and magnetic charges. Just as the Dirac string becomes invisible, the line discontinuities become invisible for the specific vertex operators which acquire spins. The correlation functions are now single valued and rotation covariant with spins.

The composite operators built out of order and disorder fields often have exotic properties and play important roles in the properties of the theory. In 2-d Ising model, the combination is a fermion as regards to rotational properties and spin-statistics connection [5, 10, 11]. Analogous to this is the 2-d bosonization, which builds a fermion from real massless boson [12, 13, 14, 15, 16, 17]. The formalism used in our work is well equipped to handle this. In Section 4, we consider half-integer quantization of the dual charges and show how Wick's theorem for fermions is recovered. Here again we point out the role played by the line discontinuities to yield the rotational properties of the fermions. The prescription for handling Mandelstam's [12] composite operator for the fermion is given. We also offer new insight into the Klein factors [18] required in bosonization.

We discuss our results in Section 5. Here we indicate how the line discontinuity is related to the soliton of the sine-Gordon theory. We also point out how to write the local action with both the field and its dual in the presence of self-interactions.

We choose straight line discontinuities through most of the paper. However, we have the freedom of choosing any path, as discussed in the Appendix.

2 A local action with both the field and its dual, and the role of line discontinuities

2.1 Partition function in terms of dual field

Consider a free massless scalar field $\phi(x)$ in two Euclidean dimensions. Its correlation functions can be obtained from the generating functional:

$$Z[\rho] = \mathcal{N}_1 \int \mathcal{D}\phi e^{\int d^2x \left[-\frac{1}{2}\sum_{i=1}^{2}(\partial_i \phi(x))^2 + i\rho(x)\phi(x)\right]} \tag{1}$$

In (1) $\rho(x)$ is the source coupling locally to $\phi(x)$ and the normalization $\mathcal{N}_1$ is defined so that $Z[\rho = 0] = 1$. We are inserting $\sqrt{-1}$ in the source term for two reasons: (i) the duality transformation carried out below is more transparent, (ii) the point sources[1]

$$\rho(x) = \sum_i e_i \delta^2(\vec{x} - \vec{x}^i) \tag{2}$$

give the vertex operators

$$V_i = e^{ie_i\phi(x^i)} \tag{3}$$

which play crucial role here as in CFTs. Formally:

$$Z[\rho] = e^{-\frac{1}{2}\int d^2x d^2y \rho(x)\triangle(x-y)\rho(y)}, \tag{4}$$

where $\triangle(x-y)$ is the 2-d Coulomb potential satisfying:

$$-\partial_x^2 \triangle (x-y) = \delta^2(\vec{x} - \vec{y}). \tag{5}$$

As the Coulomb potential in 2-d is infrared divergent, we use a finite area of linear dimension R. Also to handle ultraviolet divergences in certain correlation functions, we use the ultraviolet cutoff a to get [19]:

$$\triangle (x) = \frac{1}{4\pi} \ln\left(\frac{R^2}{\vec{x}^2 + a^2}\right). \tag{6}$$

Using an auxiliary field $\vec{b}_i(x)$ we linearize the $\phi(x)$ terms in (1):

$$Z[\rho] = \mathcal{N}_2 \int \mathcal{D}\phi \mathcal{D}b_i e^{\int d^2x\left(-\frac{1}{2}b_i^2(x) + ib_i(x)\partial_i\phi(x) + i\rho(x)\phi(x)\right)}. \tag{7}$$

A formal integration over ϕ gives a functional δ function:

$$Z[\rho] = \mathcal{N}_3 \int \mathcal{D}b_i \prod_x \delta\left(\partial_i b_i(x) - \rho(x)\right) e^{-\frac{1}{2}\int d^2x b_i^2(x)}. \tag{8}$$

We solve the 'Gauss law constraint' as follows:

$$b_i(x) = \epsilon_{ij}\partial_j\tilde{\phi}(x) + \delta_{i1}\partial_1^{-1}\rho(x) \tag{9}$$

where ∂_1^{-1} is defined as:

$$\partial_1^{-1}\rho(x) = \int_{-\infty}^{x_1} dx_1' \rho(x_1', x_2). \tag{10}$$

For a point source $\rho(x) = \delta^2(x - x^0)$,

$$\partial_1^{-1}\rho(x) = \theta(x_1 - x_1^0)\delta(x_2 - x_2^0), \tag{11}$$

[1]Throughout this paper, we use superscripts to label different points in 2-d space.

i.e. the flux is carried along a line from x^0 parallel to x_1 axis in the positive direction. The 'Gauss law' solution (9) implies,

$$Z[\rho] = \mathcal{N}_4 \int \mathcal{D}\tilde{\phi} e^{-\frac{1}{2}\int d^2x \left(\left(\partial_1 \tilde{\phi}(x)\right)^2 + \left(\partial_2 \tilde{\phi}(x) + \partial_1^{-1}\rho(x)\right)^2 \right)}. \tag{12}$$

This re-expresses the partition function (1) in terms of a new massless scalar field $\tilde{\phi}(x)$ which is dual of the field $\phi(x)$. The source ρ, which couples locally to ϕ, couples in a specific non-local manner to the dual field $\tilde{\phi}$. In spite of this, an integration over $\tilde{\phi}$ reproduces (4):

$$\begin{aligned} Z[\rho] &= \mathcal{N}_4 \int \mathcal{D}\tilde{\phi} e^{\int d^2x \left[-\frac{1}{2}\left(\partial_i \tilde{\phi}(x)\right)^2 + \tilde{\phi}(x)\partial_2\partial_1^{-1}\rho(x) - \frac{1}{2}\left(\partial_1^{-1}\rho(x)\right)^2 \right]} \\ &= e^{\frac{1}{2}\left[\int d^2x d^2y \partial_2\partial_1^{-1}\rho(x) \triangle(x-y) \partial_2\partial_1^{-1}\rho(y) - \int d^2x \left(\partial_1^{-1}\rho(x)\right)^2 \right]}. \end{aligned} \tag{13}$$

Now integrating by parts with respect to x_2 and y_2, and using $\left(\frac{\partial}{\partial x_2}\right)^2 \triangle(x-y) = -\left(\frac{\partial}{\partial x_1}\right)^2 \triangle (x-y) - \delta^2(x-y)$ and $\partial_1\partial_1^{-1}\rho(x) = \rho(x)$, we recover (4).

2.2 Interpretation in terms of lines of discontinuity

For the point sources (2) the partition function (12) takes the form:

$$Z(e_i) = \mathcal{N}_4 \int \mathcal{D}\tilde{\phi} e^{-\frac{1}{2}\int d^2x \left[\left(\partial_1 \tilde{\phi}(x)\right)^2 + \left(\partial_2 \tilde{\phi}(x) + \sum_i e_i \theta(x_1 - x_1^i)\delta(x_2 - x_2^i)\right)^2 \right]}. \tag{14}$$

This exhibits the non-local interaction of the dual field $\tilde{\phi}(x)$ with the local sources of $\phi(x)$. A point source at $\vec{x}^i$ of strength e_i corresponds to a singular flux line which starts at $\vec{x}^i$ and goes parallel to x_1 axis in the positive direction. Despite the singular line, the partition function (14) is meaningful and reproduces (4) as seen above. The singular line forces the configurations $\tilde{\phi}(x)$ which contribute to the partition function (14) to have discontinuities along the singular line, so as to cancel the singularities in the exponent and contribute a finite action. Thus *the configurations $\tilde{\phi}(x)$ which contribute to the partition function are no longer smooth and continuous configurations, but those with specific lines of discontinuity.*

2.3 A local action with both the field and its dual present

Using an auxiliary field $\tilde{\pi}(x)$, the partition function (12) can be written as:

$$Z[\rho] = \mathcal{N}_5 \int \mathcal{D}\tilde{\phi}\mathcal{D}\tilde{\pi} e^{\int d^2x \left[-\frac{1}{2}\left(\partial_1 \tilde{\phi}(x)\right)^2 - \frac{1}{2}\tilde{\pi}^2(x) + i\tilde{\pi}(x)\left(\partial_2\tilde{\phi}(x) + \partial_1^{-1}\rho(x)\right) \right]}. \tag{15}$$

Notice that ρ couples locally to $-\partial_1^{-1}\tilde{\pi}$. Therefore, we identify

$$-\partial_1^{-1}\tilde{\pi}(x) \equiv \phi(x). \tag{16}$$

We can now explore the correlation function of $\phi(x)$ with its dual field $\tilde{\phi}(x)$. For this we introduce a local source $\tilde{\rho}(x)$ and couple it to $\tilde{\phi}(x)$. The partition function (15) takes the form:

$$Z[\rho,\tilde{\rho}] = \mathcal{N}_6 \int \mathcal{D}\phi\mathcal{D}\tilde{\phi} e^{\int d^2x\left[-\frac{1}{2}(\partial_1\phi(x))^2 - \frac{1}{2}\left(\partial_1\tilde{\phi}(x)\right)^2 - i\partial_1\phi(x)\partial_2\tilde{\phi}(x) + i\left(\rho(x)\phi(x)+\tilde{\rho}(x)\tilde{\phi}(x)\right)\right]}. \tag{17}$$

The partition function (17) represents the original theory (1) with only $\phi(x)$ field as an equivalent local theory with both the field and its dual simultaneously present. Moreover it is self-dual as the action itself is invariant under: $\phi \leftrightarrow \tilde{\phi}$ and $\rho \leftrightarrow \tilde{\rho}$. The action in (17) also has discrete inversion symmetry: $\phi \to -\phi$, $\tilde{\phi} \to -\tilde{\phi}$, $\rho \to -\rho$, $\tilde{\rho} \to -\tilde{\rho}$. In the absence of sources, the action in (17) is also invariant under the following global transformations:

$$\phi(x_1,x_2) \to \phi(x_1,x_2) + \sigma, \quad \tilde{\phi}(x_1,x_2) \to \tilde{\phi}(x_1,x_2) + \tilde{\sigma}. \tag{18}$$

However, it is not manifestly invariant under rotations of the 2-d space. The traditional $(\partial_2\phi)^2$ and $(\partial_2\tilde{\phi})^2$ terms are not present. Also, the action is not real (even in the absence of the imaginary sources we are using). The term bilinear in ϕ and $\tilde{\phi}$ has $\sqrt{-1}$ factor. (This $\sqrt{-1}$ factor, however, would not be present in Minkowski space. This is the case considered in Refs. [1, 2].) In spite of all this, the equivalence of the action to the traditional partition function is seen by integrating over one or the other field (in the absence of sources). It should be noted that *the lack of manifest rotation invariance can be traced back to the specific choice of the x_1-direction in our solution (9) of the Gauss law constraint.*

We see from (15) that if we regard x_2 as the Euclidean time, the partition function (12) corresponds to the Hamiltonian density ($Z = \lim_{T\to\infty} \mathrm{Tr} e^{-TH}$):

$$\mathcal{H}(\tilde{\pi},\tilde{\phi}) = \frac{1}{2}\tilde{\pi}^2(x) + \frac{1}{2}\left(\partial_1\tilde{\phi}(x)\right)^2 \tag{19}$$

with canonically conjugate variables $(\tilde{\phi}(x),\tilde{\pi}(x))$. Thus (16) relates the field $\phi(x)$ non-locally to the momentum $\tilde{\pi}$ conjugate to the dual field $\tilde{\phi}(x)$ [20]. This connects the formalism of the present work to condensed matter system Hamiltonians [3, 4]. With this interpretation, (17) is not so exotic. Nevertheless, it is very useful as the field and its dual are on the same footing.

2.4 Correlations of the field ϕ with its dual $\tilde{\phi}$

Evaluating (17) and taking functional derivatives with respect to the sources $\rho(x)$ and $\tilde{\rho}(x)$ we get correlation functions of field $\phi(x)$ with its dual $\tilde{\phi}(x)$:

$$\begin{aligned} Z[\rho,\tilde{\rho}] &= \mathcal{N}_6 \int \mathcal{D}\phi\mathcal{D}\tilde{\phi} e^{\int d^2x\left[-\frac{1}{2}\left(\phi(x)\ \tilde{\phi}(x)\right)\begin{pmatrix} -\partial_1^2 & -i\partial_1\partial_2 \\ -i\partial_1\partial_2 & -\partial_1^2\end{pmatrix}\begin{pmatrix}\phi(x)\\ \tilde{\phi}(x)\end{pmatrix} + i\left(\phi(x)\ \tilde{\phi}(x)\right)\begin{pmatrix}\rho(x)\\ \tilde{\rho}(x)\end{pmatrix}\right]}, \\ &= e^{-\frac{1}{2}\int d^2x d^2y\left[\left(\rho(x)\ \tilde{\rho}(x)\right)\begin{pmatrix} -\partial_1^2 & -i\partial_1\partial_2 \\ -i\partial_1\partial_2 & -\partial_1^2\end{pmatrix}^{-1}(x,y)\begin{pmatrix}\rho(y)\\ \tilde{\rho}(y)\end{pmatrix}\right]}. \end{aligned} \tag{20}$$

Now using

$$\begin{pmatrix} -\partial_1^2 & -i\partial_1\partial_2 \\ -i\partial_1\partial_2 & -\partial_1^2\end{pmatrix}^{-1}(x,y) = \begin{pmatrix} 1 & -i\partial_1^{-1}\partial_2 \\ -i\partial_1^{-1}\partial_2 & 1\end{pmatrix} \triangle(x-y),$$

we get:

$$\langle \phi(x)\phi(y)\rangle = \triangle(x-y) \tag{21}$$

which is consistent with (4), and

$$\langle \tilde{\phi}(x)\tilde{\phi}(y)\rangle = \triangle(x-y) \tag{22}$$

as expected for a massless real scalar field. The correlation function of the field $\phi(x)$ with its dual $\tilde{\phi}(y)$ is:

$$\langle \phi(x)\tilde{\phi}(y)\rangle = -\frac{i}{2}[\partial_1^{-1}\partial_2 \triangle (\vec{x}-\vec{y}) + \ x \leftrightarrow y \ ]. \tag{23}$$

In (23) the operator $\partial_1^{-1}\partial_2$ acting on $\triangle(\vec{x}-\vec{y})$ is with respect to the first coordinate $\vec{x}$. We define:

$$\frac{\Theta(\vec{x})}{2\pi} \equiv \partial_1^{-1}\partial_2 \triangle (x) = -\frac{1}{2\pi}\partial_1^{-1}\frac{x_2}{r^2} = -\frac{x_2}{2\pi}\int_{-\infty}^{x_1} dx_1' \frac{1}{(x_1')^2+(x_2)^2} = -\frac{1}{2\pi}\left(\tan^{-1}\frac{x_1}{x_2}+\frac{\pi}{2}\right). \tag{24}$$

We now have to directly address the integral to choose the branch for the definition of inverse of tan. The integral

$$\int_{-\infty}^{x_1} dx_1' \frac{1}{(x_1')^2+(x_2)^2} \tag{25}$$

is positive definite and well defined as long as $x_2 \neq 0$. Even for $x_2 = 0$, it is well defined if $x_1 < 0$ because the integrand never blows up. Making a change of variables from x_1' to $t \equiv \frac{x_1'}{|x_2|}$, we get

$$\Theta(\vec{x}) = -\mathrm{sgn}(x_2)\int_{-\infty}^{\frac{x_1}{|x_2|}} dt \frac{1}{1+t^2} \tag{26}$$

where $\mathrm{sgn}(x_2)$ equals $+1(-1)$ for $x_2 > 0$ ($x_2 < 0$). Therefore, as shown in Figure 1, $\Theta(\vec{x})$ is the angle subtended by the vector $\vec{x}$ measured in the anti-clockwise direction with starting value $-\pi$ from the positive x_1-axis. Thus

$$\langle \phi(x)\tilde{\phi}(y)\rangle = -\frac{i}{4\pi}\left(\Theta(\vec{x}-\vec{y})+\Theta(\vec{y}-\vec{x})\right). \tag{27}$$

Note that $\Theta(\vec{x})$ is discontinuous along the positive x_1 axis because it changes from $-\pi$ to $+\pi$ as x_2 changes from a positive value to negative value as shown in Figure 1-a. Thus the correlation function of the field with its dual (27) is discontinuous whenever $\vec{x}-\vec{y}$ is parallel to the x_1-axis. As shown in Figure 1-b, it changes by $\frac{i}{2}\mathrm{sgn}(x_1-y_1)$ as (x_2-y_2) changes sign from negative to positive. Note that the x_1-axis is singled out in (9), leading to ∂_1^2 in the diagonal elements in (20), and then to ∂_1^{-1} in (23). Thus *this discontinuity is a reflection of the boundary condition forced by the line discontinuity generated by the field $\phi(x)$ in the configuration space of its dual field $\tilde{\phi}(y)$.*

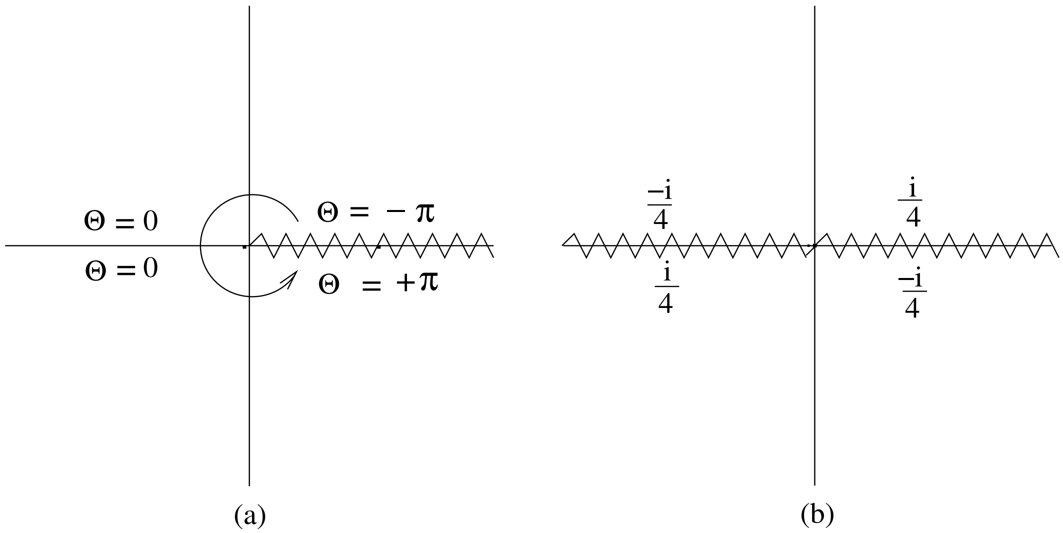

Figure 1: The discontinuities in (a) the angle $\Theta(\vec{x})$, (b) the correlation function $\langle\phi(x)\tilde{\phi}(0)\rangle = -\frac{i}{4\pi}\left(\Theta(\vec{x}) + \Theta(-\vec{x})\right)$ across the (horizontal) x_1-axis.

It is convenient to define the standard azimuthal angle $\theta(\vec{x})$ $(0 < \theta(\vec{x}) < 2\pi)$ by

$$\theta(\vec{x}) \equiv \Theta(\vec{x}) + \pi, \tag{28}$$

obeying the standard relation:

$$\theta(-\vec{x}) = \theta(\vec{x}) + \pi \mathrm{sgn}(x_2). \tag{29}$$

The correlation function (27) can now be written as:

$$\langle\phi(x)\tilde{\phi}(y)\rangle = -i\left(\frac{\theta(\vec{x}-\vec{y})}{2\pi} + \frac{\mathrm{sgn}(x_2-y_2)}{4} - \frac{1}{2}\right). \tag{30}$$

Thus the correlation function of the field $\phi(x)$ with its dual $\tilde{\phi}(y)$ has the following unusual properties:

1. It does not fall off with the distance, reflecting the non-local effect $\phi(x)$ has in the configuration space of $\tilde{\phi}(x)$.

2. *The correlation depends on the orientation of the relative vector $\vec{x}-\vec{y}$ in space.* Note that the fields $\phi(x)$ and $\tilde{\phi}(x)$ are by themselves scalar fields and therefore their correlations would be expected to depend only on the distance of separations and not on the orientation of the relative vector. This scalar nature is true if we consider the correlations of the field ϕ only (or $\tilde{\phi}$ only) at various points. However, as discussed in detail in Section 3, the correlations of $\tilde{\phi}$ and ϕ acquire a direction dependence. *The reason again is the non-local effect produced by ϕ in the configurations of $\tilde{\phi}$ or vice-versa.*

3. Even though ϕ and $\tilde{\phi}$ are real fields, their mutual correlations are pure imaginary. The technical reason is that the action in the partition function (17) is no longer real.

4. The $\langle\phi(x)\tilde{\phi}(y)\rangle$ correlations given by (27) are symmetric under the interchange of $\vec{x}$ and $\vec{y}$. The reason can be traced to (20) where we inverted a symmetric matrix.

We finally have the master formula (see equations (20)-(23)):

$$Z[\rho,\tilde{\rho}] = e^{\int d^2x d^2y \left[-\frac{1}{2}\rho(x)\triangle(x-y)\rho(y)-\frac{1}{2}\tilde{\rho}(x)\triangle(x-y)\tilde{\rho}(y)+i\rho(x)\left(\frac{\theta(\vec{x}-\vec{y})}{2\pi}+\frac{\mathrm{sgn}(x_2-y_2)}{4}-\frac{1}{2}\right)\tilde{\rho}(y)\right]}. \tag{31}$$

One can also use the linear combinations of the fields $\phi(x)$ and its dual $\tilde{\phi}(x)$:

$$\xi_\pm(x) = \frac{1}{2}\left(\phi(x) \pm \tilde{\phi}(x)\right). \tag{32}$$

The action in (17) decouples in ξ_+ and ξ_- [1, 2]:

$$S[\phi,\tilde{\phi}] = S\left[\xi_+,\xi_-\right] = \int d^2x \left((\partial_1\xi_+)^2 + i\partial_1\xi_+\partial_2\xi_+ + (\partial_1\xi_-)^2 - i\partial_1\xi_-\partial_2\xi_-\right). \tag{33}$$

The correlation functions of these combinations are:

$$\begin{aligned}\langle\xi_+(x)\xi_+(0)\rangle &= -\frac{1}{4\pi}\left(\ln\frac{z}{R} + i\pi\left(\frac{\mathrm{sgn}\ x_2}{2} - 1\right)\right),\\ \langle\xi_-(x)\xi_-(0)\rangle &= -\frac{1}{4\pi}\left(\ln\frac{\bar{z}}{R} - i\pi\left(\frac{\mathrm{sgn}\ x_2}{2} - 1\right)\right),\end{aligned} \tag{34}$$

where $z = x_1 + ix_2$. The fields $\xi_\pm$ are related to holomorphic and anti-holomorphic fields $\varphi(z)$ and $\bar{\varphi}(\bar{z})$ used in CFT, and are involved in bosonization also. (See, for example, Refs. [19] and [20]; see also our equation (52).)

3 Vertex operators of the field ϕ and its dual $\tilde{\phi}$

Point sources (2) correspond to vertex operators $e^{ie_i\phi(x^i)}$ of well known Coulomb gas formalism for minimal CFTs [8, 9]. Their well known correlation functions are read off from the master formula (4) and the regularized propagator (6),

$$\begin{aligned}\langle\prod_i e^{ie_i\phi(x^i)}\rangle &= e^{-\frac{1}{2}\int d^2x\, d^2y (\sum_i e_i\delta^2(x-x^i))\Delta(x-y)(\sum_j e_j\delta^2(y-x^j))}\\ &= \left(\frac{R}{a}\right)^{-\frac{1}{4\pi}(\sum_i e_i)^2}\prod_{i<j}\left(\frac{|x^i-x^j|}{a}\right)^{\frac{e_ie_j}{2\pi}}.\end{aligned} \tag{35}$$

When $R \to \infty$, only the net charge zero ($\sum_i e_i = 0$) correlation functions survive. Then, using $\sum_{i<j} e_i e_j = -\frac{1}{2}\sum_i e_i^2$, we can write the a-dependence in Eq. (35) as $\prod_i a^{e_i^2/4\pi}$. So, to get finite correlations as the ultraviolet regulator $a \to 0$, we 'renormalize' the vertex operators [19]:

$$: e^{ie_i\phi(x)} := a^{-\frac{e_i^2}{4\pi}} e^{ie_i\phi(x)}. \tag{36}$$

This amounts to removing self correlations $\langle\phi(x)\phi(x)\rangle$ in $e^{ie\phi(x)}$. The expression in Eq. (35) then reduces to just $\prod_{i<j}|x^i - x^j|^{\frac{e_ie_j}{2\pi}}$. For the neutral combination,

$$\langle : e^{ie\phi(x)} :: e^{-ie\phi(y)} :\rangle = \frac{1}{|x-y|^{\frac{e^2}{2\pi}}}. \tag{37}$$

We may define vertex operators for the dual field $\tilde{\phi}$ in a similar manner:

$$\langle : e^{ig\tilde{\phi}(x)} :: e^{-ig\tilde{\phi}(y)} :\rangle = \frac{1}{|x-y|^{\frac{g^2}{2\pi}}}. \tag{38}$$

The mixed correlation functions can be computed using the master formula (31). We need charge neutrality separately for field ϕ and dual field $\tilde{\phi}$, for finite correlation function in $R \to \infty$ limit. It is as if we have two Coulomb gas partition functions, except that cross-correlations are also present:

$$\langle\prod_i : e^{ie_i\phi(x^i)} : \prod_k : e^{ig_k\tilde{\phi}(y^k)} :\rangle = \prod_{i<j}|\vec{x}^i - \vec{x}^j|^{\frac{e_ie_j}{2\pi}} \prod_{k<l}|\vec{y}^k - \vec{y}^l|^{\frac{g_kg_l}{2\pi}} \prod_{i,k} e^{ie_ig_k\left[\frac{\theta(\vec{x}^i - \vec{y}^k)}{2\pi} + \frac{\mathrm{sgn}(x_2^i - y_2^k)}{4} - \frac{1}{2}\right]} \tag{39}$$

with $\sum_i e_i = 0, \sum_k g_k = 0$.

Generally these correlation function have jumps in phase whenever a relative vector $\vec{x}^i - \vec{y}^k$ tends to be parallel to the x_1-axis (see Figure 1-b). We may alternatively interpret that the correlation functions are multivalued. The discontinuity, as explained after (27), reflects the line discontinuity produced by ϕ in the configuration space of $\tilde{\phi}$. The specific role of x_1-axis is due to our choice of the solution of the Gauss' law constraint. *This parallels the role of Dirac string in Dirac formulation of the magnetic monopole.* Let $e_i = m_i e, g_k = n_k g$ (where m_i and n_k are integers) with the Schwinger quantization condition

$$eg = 4\pi\,. \tag{40}$$

Consider the lowest values $m_i = 1$ and $n_k = 1$. The corresponding phase in (39) is then $\exp[-4\pi\langle\phi(x)\tilde{\phi}(y)\rangle]$, and equals $e^{\pm i\pi}$ across the x_1-axis (refer to Figure 1-b). Thus the phase is continuous, the 'string' is no longer visible and the correlation functions are single valued.

We now briefly explain when we get the Schwinger quantization condition, and when the Dirac quantization condition $eg = 2\pi$. By introducing field ϕ in (17) and inverting the symmetric matrix in (20), we got $\frac{1}{2}(\Theta(\vec{x}-\vec{y}) + \Theta(\vec{y}-\vec{x}))$ in (27). This is the analogue of the Schwinger potential, since the line discontinuity is distributed to both positive and negative x_1 axis (as in Figure 1-b). Thus this is due to insisting on a symmetric quadratic form in ϕ and $\tilde{\phi}$. On the other hand, having only $\Theta(\vec{x}-\vec{y})$ in (27) is the analogue of the Dirac potential, since it has the line discontinuity only across the positive x_1 axis. This is obtained from the partition function in terms of the dual field $\tilde{\phi}$ alone, after adding a source for $\tilde{\phi}$ and integrating over $\tilde{\phi}$. This has been done in the Appendix. In this case the 'string' becomes invisible for the Dirac quantization condition $eg = 2\pi$.

Rotation covariance now manifests in an unusual manner. Though ϕ and $\tilde{\phi}$ were separately scalar fields, the mutual correlation functions (39) undergo transformation under the rotation $\theta(\vec{x}-\vec{y}) \to \theta(\vec{x}-\vec{y}) + \omega$. In other words, (39) are not invariant, but only covariant, under rotation. Thus the vertex operators have acquired a 'spin'. This is in analogy with the Saha-Wilson [21, 22, 23] contribution (of the electromagnetic field) to the total angular momentum of a charge-monopole system. With $eg = 2\pi$, (39) transforms under the above rotation by a factor of $e^{\pm i\omega}$. So the vertex operators may be assigned the following transformation property under rotation:

$$\begin{aligned} e^{\pm ie\phi(x)} &\to e^{\pm ie\phi(x)}, \\ e^{\pm ig\tilde{\phi}(y)} &\to e^{\pm i\omega} e^{\pm ig\tilde{\phi}(y)}, \end{aligned} \tag{41}$$

i.e., the dual field vertex operators $e^{\pm ig\tilde{\phi}(y)}$ have acquired spin $\pm$ 1. Equivalently we may assign spin to $e^{\pm ie\phi(x)}$. We emphasize that *the underlying reason for the generation of spin from scalar fields is that the correlation function between $\phi(x)$ and $\tilde{\phi}(y)$ depends on the orientation of the relative vector $\vec{x}-\vec{y}$.*

4 Fermions from Bosons

In the previous section, we obtained vertex operators with spin from scalar fields. With $\frac{eg}{2\pi} = 1$, the spins were integer valued (in particular ± 1 in (41)). In this section, we consider the case of half-integer quantization, viz $\frac{eg}{2\pi} = \frac{1}{2}$. In this case, the discontinuity in phase is π for the correlation functions (39), and so the line of discontinuity is now visible. Despite this, the case is of significance. We generate operators that have spin $\frac{1}{2}$. They change sign under rotation by 2π. Indeed in the case of $e^2 = g^2 = \pi$ we recover the correlation functions of the free fermions [19, 20].

The action for the free massless (Dirac) fermions in two Euclidean dimensions is:

$$S = \frac{1}{2\pi} \int d^2x \left(\bar{\psi}_+(x)\partial_+\psi_+(x) + \bar{\psi}_-\partial_-\psi_-(x) \right) \tag{42}$$

where $\partial_\pm = \partial_1 \pm i\partial_2$. Under rotation by an angle ω,

$$\partial_\pm \to e^{\mp i\omega}\partial_\pm. \tag{43}$$

The corresponding transformations on $\psi_\pm(x)$ that leave the action (42) invariant are:

$$\psi_\pm(x) \to e^{\pm i\frac{\omega}{2}}\psi_\pm(x), \quad \bar{\psi}_\pm(x) \to e^{\pm i\frac{\omega}{2}}\bar{\psi}_\pm(x). \tag{44}$$

Thus $\psi_+(x)$ and $\bar{\psi}_+(x)$ transform the same way by half the angle. (ψ and $\bar{\psi}$ are independent Grassmann fields in the Euclidean functional integral). Thus $\psi_\pm, \bar{\psi}_\pm$ change by a sign under full rotation by 2π and carry spin $\pm\frac{1}{2}$. The propagators:

$$\langle \psi_+(x)\bar{\psi}_+(y)\rangle = 2\pi\langle x|\frac{1}{\partial_+}|y\rangle = -2\pi\partial_- \triangle (x-y) = \frac{1}{(x-y)_+} = \frac{e^{-i\theta(\vec{x}-\vec{y})}}{|\vec{x}-\vec{y}|}. \tag{45}$$

Here $x_\pm = x_1 \pm ix_2$ and $\theta(\vec{x})$ is the angle subtended by $\vec{x}$ with the x_1-axis. We also have:

$$\langle \psi_-(x)\bar{\psi}_-(y)\rangle = \frac{1}{(x-y)_-} = \frac{e^{i\theta(\vec{x}-\vec{y})}}{|\vec{x}-\vec{y}|}. \tag{46}$$

All other correlation functions vanish:

$$\langle \psi_-(x)\bar{\psi}_+(y)\rangle = 0, \langle \psi_+(x)\bar{\psi}_-(y)\rangle = 0, \langle \psi_-(x)\psi_-(y)\rangle = 0, \langle \psi_+(x)\psi_+(y)\rangle = 0. \tag{47}$$

The $2n$-point functions are obtained by Wick's theorem for fermions:

$$\begin{aligned}\langle \psi_+(x^1)\bar{\psi}_+(y^1)\psi_+(x^2)\bar{\psi}_+(y^2)\cdots\psi_+(x^n)\bar{\psi}_+(y^n)\rangle &= \frac{1}{(x^1-y^1)_+}\frac{1}{(x^2-y^2)_+}\frac{1}{(x^3-y^3)_+}\cdots\frac{1}{(x^n-y^n)_+} \\ -\frac{1}{(x^1-y^2)_+}\frac{1}{(x^2-y^1)_+}\frac{1}{(x^3-y^3)_+}\cdots\frac{1}{(x^n-y^n)_+} &+ \cdots \text{all possible permutations of } y^1, y^2, \cdots y^n.\end{aligned} \tag{48}$$

The right hand side of (48) contains $n!$ terms and each term appears with $\pm$ sign depending upon the corresponding sign of permutation of $(y^1, y^2, \cdots y^n)$. Therefore, the above correlation functions are simply:

$$\langle \psi_+(x^1)\bar{\psi}_+(y^1)\psi_+(x^2)\bar{\psi}_+(y^2)\cdots\psi_+(x^n)\bar{\psi}_+(y^n)\rangle = det\left[\frac{1}{(x^i-y^j)_+}\right]_{1\le i,j\le n} \tag{49}$$

This can be further written as the Cauchy determinant formula (see equation (12.195) of [9]):

$$\langle \psi_+(x^1)\bar{\psi}_+(y^1)\psi_+(x^2)\bar{\psi}_+(y^2)\cdots\psi_+(x^n)\bar{\psi}_+(y^n)\rangle = (-1)^{\frac{n(n-1)}{2}}\frac{\prod_{i<j}(x^i-x^j)_+(y^i-y^j)_+}{\prod_{i,j}(x^i-y^j)_+} \tag{50}$$

We have similar correlation functions for $\psi_-, \bar{\psi}_-$ with x_+, y_+ replaced by x_-, y_-.

All these correlation functions are reproduced by combinations of vertex operators of ϕ and the dual field $\tilde{\phi}$ 'close together' in the special case of

$$e^2 = g^2 = \pi\,. \tag{51}$$

To show this we consider the composite operators:

$$\begin{aligned}\Psi^{\vec{\epsilon}}_\pm(\vec{x}) &\equiv\ :e^{\,ie\phi(\vec{x}+\vec{\epsilon})}:\ \ :e^{\,\pm ig\tilde{\phi}(\vec{x})}:e^{\mp ieg\left(\frac{\theta(\vec{\epsilon})}{2\pi}+\frac{\mathrm{sgn}(\epsilon_2)}{4}-\frac{1}{4}\right)}, \\ \bar{\Psi}^{\vec{\epsilon}}_\pm(\vec{x}) &\equiv\ :e^{\,-ie\phi(\vec{x}+\vec{\epsilon})}:\ \ :e^{\,\mp ig\tilde{\phi}(\vec{x})}:e^{\mp ieg\left(\frac{\theta(\vec{\epsilon})}{2\pi}+\frac{\mathrm{sgn}(\epsilon_2)}{4}-\frac{1}{4}\right)}.\end{aligned} \tag{52}$$

$\Psi^{\vec{\epsilon}}_{\pm}(x)$ in (52), for the case (51), correspond to a point source charge $e(=+\sqrt{\pi})$ at $\vec{x}+\vec{\epsilon}$ for the field ϕ and $g(=\pm\sqrt{\pi})$ for the dual field $\tilde{\phi}$ at a nearby point $\vec{x}$. These are the analogues of the order-disorder composites $\sigma(n)\mu(n^*)$ in the 2-d Ising model case (n and n^* being points on the lattice and the dual lattice respectively) which are known to behave as fermions [5]. Note that both $\Psi^{\vec{\epsilon}}_{+}(\Psi^{\vec{\epsilon}}_{-})$ and $\bar{\Psi}^{\vec{\epsilon}}_{+}(\bar{\Psi}^{\vec{\epsilon}}_{-})$ have the same, and not opposite, phase factor. This fact leads to the rotation property by half angle under $\theta(\vec{\epsilon})\to\theta(\vec{\epsilon})+\omega$ just like $\psi_+(\psi_-)$ in (44). Under rotation by 2π, $\Psi^{\vec{\epsilon}}_{\pm}$ and $\bar{\Psi}^{\vec{\epsilon}}_{\pm}$ all change by a sign as $\theta(\vec{\epsilon})$ changes by 2π and $\mathrm{sgn}(\epsilon_2)$ does not change.

As explained in Section 2.3, the field ϕ is non-locally related via (16) to the the momentum $\tilde{\pi}=\partial_2\tilde{\phi}$ conjugate to $\tilde{\phi}$. So $\phi(x_1,x_2)=-\int_{-\infty}^{x_1}dx_1'\,(\partial/\partial x_2)\tilde{\phi}(x_1',x_2)$. Thus with $e^2=g^2=\pi$, the operators $\Psi^{\vec{\epsilon}}_{\pm}$ in (52) correspond to Mandelstam's construction of the fermion operator in the case of the free Fermi field (as given by equations (2.8) of [12] for the case $\beta^2=4\pi$.) In (52), we give the prescription for handling this composite fermion operator in our formalism.

From (39) and (52) we get:

$$\langle\Psi^{\vec{\epsilon}}_{+}(x)\bar{\Psi}^{\vec{\epsilon}'}_{+}(y)\rangle=\frac{1}{|\vec{x}-\vec{y}|^{\frac{e^2+g^2}{2\pi}}}\,e^{-ieg\left(\frac{\theta(\vec{x}-\vec{y})}{\pi}+\frac{\mathrm{sgn}(x_2-y_2)}{2}-\frac{1}{2}\right)}\,. \tag{53}$$

Note that the phase factors in (52) have been chosen to cancel out the contribution of the self-correlation $\langle\phi(\vec{x}+\vec{\epsilon})\tilde{\phi}(\vec{x})\rangle$ within each composite operator (compare (39)). Thus the correlations of $\Psi^{\vec{\epsilon}}$ defined as in (52) are insensitive to $\vec{\epsilon}$, and we drop this superscript henceforth. For the special case (51) we get:

$$\langle\Psi_+(x)\bar{\Psi}_+(y)\rangle=\ \mathrm{sgn}(x_2-y_2)\frac{e^{-i\theta(\vec{x}-\vec{y})}}{|\vec{x}-\vec{y}|} \tag{54}$$

as $e^{i\frac{\pi}{2}\mathrm{sgn}(x_2-y_2)}=i\mathrm{sgn}(x_2-y_2)$. We can now reproduce the fermion propagator (45) with the identification:

$$\langle\psi_+(x)\bar{\psi}_+(y)\rangle\equiv\ \mathrm{sgn}(x_2-y_2)\langle\Psi_+(x)\bar{\Psi}_+(y)\rangle. \tag{55}$$

The correlator $\langle\bar{\psi}_+(y)\psi_+(x)\rangle$ is identified as the negative of the right-hand side of (55), as ψ's are Grassmann variables.

The factor $\mathrm{sgn}(x_2-y_2)$ *provides the crucial anti-symmetric term to relate the bosonic correlation functions to fermionic ones. The Klein factor required in bosonization [18] comes from this factor.* Consider the multi-fermion correlation function

$$\langle\Psi_+(x^1)\bar{\Psi}_+(y^1)\Psi_+(x^2)\bar{\Psi}_+(y^2)\cdots\Psi_+(x^n)\bar{\Psi}_+(y^n)\rangle\,. \tag{56}$$

As Ψ_+, $\bar{\Psi}_+$ are bosonic operators, (56) is invariant under permutations of $\{x^i\}$ and also of $\{y^i\}$. However, when we express a correlator like $\langle\phi(x^i)\tilde{\phi}(x^j)\rangle$, which involves the symmetric combination $\theta(\vec{x}^i-\vec{x}^j)+\theta(\vec{x}^j-\vec{x}^i)$, in terms of $\theta(\vec{x}^i-\vec{x}^j)$ alone, the resulting expression involves the asymmetric term $\mathrm{sgn}(x_2^i-x_2^j)$. In this way, (56) gets split into parts which are individually non-invariant under permutations of $\{x^i\}$ (or of $\{y^i\}$). Thus, using (39), the correlator (56) is equal to (with all self-correlations within each composite operator cancelling out)

$$\frac{\prod_{i<j}\left(x^i-x^j\right)_+ \mathrm{sgn}\left(x_2^i-x_2^j\right)\prod_{k<l}\left(y^k-y^l\right)_+ \mathrm{sgn}\left(y_2^k-y_2^l\right)}{\prod_{i,k}\left(x^i-y^k\right)_+ \mathrm{sgn}\left(x_2^i-y_2^k\right)}\,. \tag{57}$$

We can now reproduce (50), by identifying

$$\begin{aligned} &\langle\psi_+(x^1)\bar{\psi}_+(y^1)\psi_+(x^2)\bar{\psi}_+(y^2)\cdots\psi_+(x^n)\bar{\psi}_+(y^n)\rangle \\ \equiv\;& (-1)^{n(n-1)/2}\prod_{i<j}\mathrm{sgn}\left(x_2^i-x_2^j\right)\prod_{k<l}\mathrm{sgn}\left(y_2^k-y_2^l\right)\prod_{i,k}\mathrm{sgn}\left(x_2^i-y_2^k\right) \\ &\times\langle\Psi_+(x^1)\bar{\Psi}_+(y^1)\Psi_+(x^2)\bar{\Psi}_+(y^2)\cdots\Psi_+(x^n)\bar{\Psi}_+(y^n)\rangle\,. \end{aligned} \tag{58}$$

Let us first choose $\{x^i\}$ and $\{y^i\}$ such that

$$x_2^1>x_2^2>x_2^3>\cdots>x_2^n \quad\text{and}\quad y_2^1>y_2^2>y_2^3>\cdots>y_2^n\,. \tag{59}$$

In this case, all $\mathrm{sgn}\left(x_2^i-x_2^j\right)=+1$ for $i<j$ and all $\mathrm{sgn}\left(y_2^k-y_2^l\right)=+1$ for $k<l$. Then any other ordering of $\{x^i\}$ and $\{y^i\}$ on the left-hand side of (58) will change it by $\mathrm{sgn}\mathcal{P}_x\mathrm{sgn}\mathcal{P}_y$, where $\mathrm{sgn}\mathcal{P}$ is the sign of the permutation $\mathcal{P}$ and $\mathrm{sgn}\mathcal{P}_x$ ($\mathrm{sgn}\mathcal{P}_y$) is the permutation which takes $(x^1,x^2,\cdots x^n)$ (respectively $(y^1,y^2,\cdots y^n)$) to the choice (59).

We can also check that a correlator like $\langle\Psi_+(x^1)\bar{\Psi}_+(y^1)\Psi_-(x^2)\bar{\Psi}_-(y^2)\rangle$ factorizes into $\langle\Psi_+(x^1)\bar{\Psi}_+(y^1)\rangle$ and $\langle\Psi_-(x^2)\bar{\Psi}_-(y^2)\rangle$. This happens because terms such as $|\vec{x}^1-\vec{y}^2|$ involving arguments of Ψ_+ and Ψ_- appear in both numerator and denominator and hence cancel. Also the phase factors such as $\theta(\vec{x}^1-\vec{y}^2)$ cancel because the combinations e_ig_k have opposite signs.

Finally, we note that the two fermion number conservations, i.e. conservation of ψ_+ and of ψ_-, can be traced to the two global symmetries (18) of the bosonic fields $\phi(x)$ and $\tilde{\phi}(x)$ [20]. As a consequence of these symmetries, the correlations $\langle\prod_i e^{ie_i\phi(x^i)}\prod_k e^{ig_k\tilde{\phi}(x^k)}\rangle$ are zero unless $\sum_i e_i=0$ and $\sum_k g_k=0$. Thus these charge neutrality conditions are valid even for a finite area, though in Section 3 we obtained them by requiring infrared finiteness.

5 Discussion

Our aim in this paper is to develop techniques for handling a field and its dual on an equal footing. We have both fields present in a local, self-dual action. The dual field is interpreted in terms of line discontinuities in the configurations of the original field (or vice versa). We have emphasized that the dual field explores new configurations of the original field. In many contexts the dual field and also composite operators made of both fields play an important role. Here we illustrate this in the simple case of a free massless real scalar field in two Euclidean dimensions. The action we consider has been used in string theory and is also related to Hamiltonians useful in one-dimensional condensed matter physics.

This action is not manifestly invariant under rotations. The underlying reason is the specific choice of the line discontinuity to be made. This also gives unusual features of the correlations of the field with its dual. Even though the correlation of either field with itself is as expected for scalar fields, the mutual correlations have a dependence on the orientation. This leads to the generation of spin from scalar fields. This is in analogy with the Saha-Wilson contribution of the electromagnetic field to the angular momentum of a charge-monopole system. For correlations of vertex operators, rotation covariance is recovered at the cost of assigning a spin to these operators. Demanding single valuedness of the correlaton functions, or equivalently, invisibility of the line discontinuity, gives quantization condition for the charges of the field and the dual field. This parallels Dirac's argument for a charge-monopole system. For half-integer quantization, a system of point sources for the field and the dual field close together behaves as a fermion, in close analogy with the situation in 2-d Ising model.

The formalism used by us reproduces Mandelstam's construction of the fermion operator and clarifies sticky points in the bosonization programme. We have precise techniques for handling composite operators. We throw light on how antisymmetric correlation functions of fermions can be related to the bosonic correlation functions. Kinematical factors of $\mathrm{sgn}(x_2-y_2)$ which relate the two are naturally obtained, giving the Klein factors required in bosonization.

We now explain how our line discontinuity is related to soliton of zero width in the sine-Gordon model. For this we regard the x_1-axis as the time direction and the x_2 axis as the space direction. A point source of unit charge for the field ϕ at the point $\vec{y}$ produces a singular flux line, starting at $\vec{y}$ and along the positive x_1-direction, in the partition function in terms of the dual field $\tilde{\phi}$. So there is a discontinuity $\Delta\tilde{\phi} = 1$ in the configurations of $\tilde{\phi}$ at all times $x_1 > y_1$ at the spatial point $x_2 = y_2$. Thus a soliton configuration of zero width is generated at time y_1 by the vertex operator $e^{i\phi(y)}$.

The dual field is non-locally related to the conjugate momentum. This is the reason that even with double the number of fields, we are not introducing new degrees of freedom but only reinterpreting the original model. Even though we considered a free theory here our framework can accomodate interacting theories as well. The crucial point is that the dual field is related to the conjugate momentum and therefore involves only the kinetic energy part of the action and not the potential energy part. Thus for the theory $\mathcal{L} = \frac{1}{2}(\partial_i\phi)^2 + V(\phi)$ our local theory with both the field and its dual is simply:

$$\mathcal{L} = \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}(\partial_1\tilde{\phi})^2 + i\partial_1\phi(x)\partial_2\tilde{\phi}(x) + V(\phi).$$

The presence of $V(\phi)$ makes important differences. In order that the action be finite in the presence of the line discontinuity, ϕ can only take the values corresponding to the minima of $V(\phi)$. For $V(\phi) = \lambda(\phi^2 - v^2)^2$, these are only $\pm v$ and not arbitrary as in the absence of $V(\phi)$. Also there is only one global invariance $\tilde{\phi} \to \tilde{\phi} + \tilde{\sigma}$ and only one conserved fermion number.

We also want to emphasize that our considerations are not restricted to two dimensions or scalar theories. In Ref. [24], we have developed local field theoretical formulation with both the field and its dual simultaneously present for the case of Abelian gauge theory in three dimensions and also in four dimensions. We can handle even non-abelian gauge theories in three and four dimensions. This will be explored elsewhere.

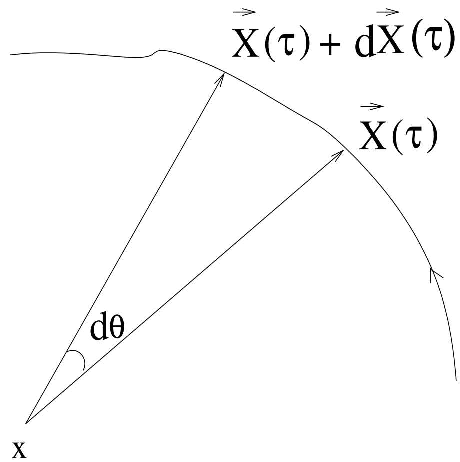

Figure 2: An arbitrary line of discontinuity.

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Appendix

On the choice of the line of discontinuity

In Section 2, when solving the Gauss law (9), we chose the line of discontinuity to be along the x_1 axis. We could have chosen an arbitray path to carry the conserved flux. The general solution of the Gauss law constraint is:

$$b_i(x) = \epsilon_{ij}\partial_j\tilde{\phi}(x) + \beta_i(x) \tag{60}$$

where

$$\beta_i(x) = -\int_{-\infty}^{0} d\tau \frac{dX_i(\tau)}{d\tau}\delta^2\left(x - X(\tau)\right) \tag{61}$$

brings the flux from $\vec{X}(-\infty)$ to $\vec{y}$ along an arbitrary path $\vec{X}(\tau)$ parametrized by a parameter τ. As $\vec{y}$ is the end point, $\vec{X}(0) = \vec{y}$. Indeed,

$$\begin{aligned}\partial_i\beta_i(x) &= -\int_{-\infty}^{0} d\tau \frac{dX_i}{d\tau}\frac{\partial}{\partial x_i}\delta^2\left(x - X(\tau)\right) = \int_{-\infty}^{0} d\tau \frac{dX_i}{d\tau}\frac{\partial}{\partial X_i}\delta^2\left(x - X(\tau)\right) \\ &= \int_{-\infty}^{0} d\tau \frac{d}{d\tau}\delta^2\left(x - X(\tau)\right) = \delta^2(\vec{x} - \vec{y}) - \delta^2\left(\vec{x} - \vec{X}(-\infty)\right) = \delta^2\left(\vec{x} - \vec{y}\right).\end{aligned} \tag{62}$$

In the last step above we have taken the other end point $\vec{X}(-\infty)$ of the flux line to be located outside the region of interest. For different point sources located at $\vec{y}^r$ $(r = 1, 2, \cdots N)$ we may choose different discontinuity lines $\vec{X}^r(\tau)$ ending at $\vec{y}^r$. For a distributed source $\rho(x)$,

$$\beta_i(x) = -\sum_r \int_{-\infty}^0 d\tau \frac{dX_i^r(\tau)}{d\tau} \delta^2(\vec{x} - \vec{X}^r(\tau)) \tag{63}$$

with the boundary conditions $\vec{X}^r(\tau = 0) = \vec{y}^r$. Using (60) in (8) and introducing source term for $\tilde{\phi}$ we get:

$$Z[\rho, \tilde{\rho}] = \int D\tilde{\phi}\, e^{\int d^2x \left(-\frac{1}{2}\left(\epsilon_{ij}\partial_j\tilde{\phi}(x) + \beta_i(x)\right)^2 + i\ \tilde{\rho}(x)\tilde{\phi}(x)\right)}. \tag{64}$$

Integration over $\tilde{\phi}$ gives,

$$Z[\rho, \tilde{\rho}] = e^{-\frac{1}{2}\int d^2x \vec{\beta}^2(x) + \frac{1}{2}\int d^2x d^2y [\partial_1\beta_2(x) - \partial_2\beta_1(x) + i\ \tilde{\rho}(x)] \triangle(x-y) [\partial_1\beta_2(y) - \partial_2\beta_1(y) + i\ \tilde{\rho}(y)]}. \tag{65}$$

Using $\partial_i\beta_i(x) = \rho(x)$, we get

$$Z[\rho, \tilde{\rho}] = e^{\int d^2x d^2y \left(-\frac{1}{2}\rho(x)\triangle(x-y)\rho(y) - \frac{1}{2}\tilde{\rho}(x)\triangle(x-y)\tilde{\rho}(y) + i\tilde{\rho}(x)\triangle(\vec{x}-\vec{y})(\partial_1\beta_2(y) - \partial_2\beta_1(y))\right)}. \tag{66}$$

Now

$$\int d^2x d^2y \tilde{\rho}(x) \triangle (\vec{x} - \vec{y}) \left(\partial_1\beta_2(y) - \partial_2\beta_1(y)\right) \tag{67}$$
$$= -\frac{1}{2\pi} \int d^2x d^2y \tilde{\rho}(x) \int_{-\infty}^0 d\tau \frac{dX_i}{d\tau} \epsilon_{ij} \frac{(x_j - X_j(\tau))}{\left(\vec{x} - \vec{X}(\tau)\right)^2} \rho(y).$$

We also have

$$\epsilon_{ij} \frac{(X_i(\tau) - x_i)}{\left(\vec{X}(\tau) - \vec{x}\right)^2} d\vec{X}_j(\tau) = d\theta \tag{68}$$

where $d\theta$ is the angle subtended by $d\vec{X}$ at $\vec{x}$, as in Figure 2, with the following sign convention: $d\theta$ is positive (negative) according as the angle between $(\vec{X} - \vec{x})$ and $d\vec{X}$ (measured in the anti-clockwise sense from the former to latter) is $> \pi$ $(< \pi)$. This convention is shown in Figure 3. Thus (67) becomes:

$$\frac{1}{2\pi} \int d^2x d^2y \tilde{\rho}(x) \theta(x - y) \rho(y) \tag{69}$$

where

$$\theta(x - y) = \int_{-\infty}^0 d\theta(\mathrm{P}) \tag{70}$$

is the net change in the angle θ along the infinite path P. This effectively measures the angle from the asymptote of the path. If we choose the negative x_1-axis along this asympote, then (70) is the same as $\Theta(\vec{x} - \vec{y})$ of Section 2.4. We want to point out that here we got only $\Theta(\vec{x} - \vec{y})$, and not the symmetric combination $\frac{1}{2}(\Theta(\vec{x} - \vec{y}) + \Theta(\vec{y} - \vec{x}))$ as in (27), because we have not explicitly introduced the field ϕ and integrated over a symmetric quadratic form involving ϕ and $\tilde{\phi}$.

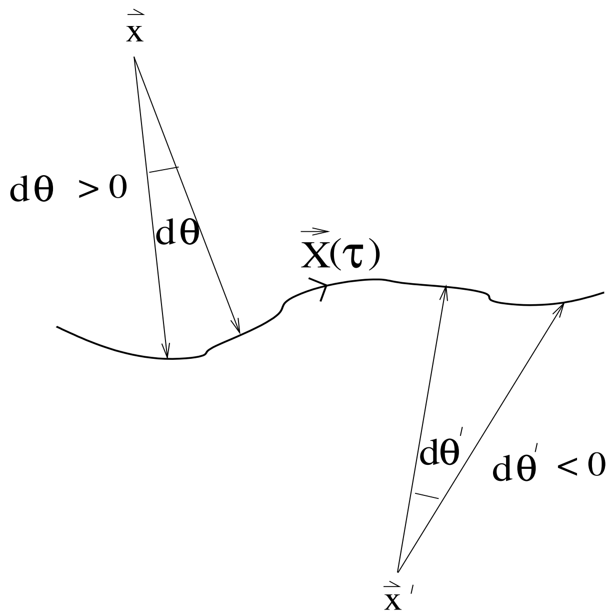

Figure 3: Sign convention for $d\theta$ defined in (68)

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