

Ideals in Rings and Intermediate Rings of Measurable Functions

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ABSTRACT. The set of all maximal ideals of the ring $\mathcal{M}(X, \mathcal{A})$ of real valued measurable functions on a measurable space (X, \mathcal{A}) equipped with the hull-kernel topology is shown to be homeomorphic to the set \hat{X} of all ultrafilters of measurable sets on X with the Stone-topology. This yields a complete description of the maximal ideals of $\mathcal{M}(X, \mathcal{A})$ in terms of the points of \hat{X} . It is further shown that the structure spaces of all the intermediate subrings of $\mathcal{M}(X, \mathcal{A})$ containing the bounded measurable functions are one and the same and are compact Hausdorff zero-dimensional spaces. It is observed that when X is a P -space, then $C(X) = \mathcal{M}(X, \mathcal{A})$ where \mathcal{A} is the σ -algebra consisting of the zero-sets of X .

1. introduction

In what follows (X, \mathcal{A}) stands for a nonempty set X equipped with a family \mathcal{A} of subsets of X , which is closed under countable union and complementation. Such a family \mathcal{A} is known as a σ -algebra over X and the pair (X, \mathcal{A}) is called a measurable space, and members of \mathcal{A} are called \mathcal{A} -measurable sets. A function $f : X \rightarrow \mathbb{R}$ is called \mathcal{A} -measurable if for any real number α , $f^{-1}(\alpha, \infty)$ is a member of \mathcal{A} . It is a standard result in measure theory that the aggregate $\mathcal{M}(X, \mathcal{A})$ of all real valued \mathcal{A} -measurable functions on X , constitutes a commutative lattice ordered ring with unity if the relevant operations are defined point wise on X [4]. The chief object of study in this article is this ring $\mathcal{M}(X, \mathcal{A})$ together with some of its chosen subrings *viz* those rings which contain all the bounded \mathcal{A} -measurable functions on X . The first paper concerning this ring dates back to 1966 [9]. It was followed by a series of articles in 1974, 1977, 1978, 1981 in [18], [19], [20], [21]. After a long gap of more than twenty five years, the articles [1], [4], and [6] appeared, which deal with various problems related to these rings.

In Section 2, we initiate a kind of duality between ideals (maximal ideals) of the ring $\mathcal{M}(X, \mathcal{A})$ and appropriately defined filters *viz* \mathcal{A} -filters (\mathcal{A} -ultrafilters) on X . An \mathcal{A} -filter on X is simply a filter whose members are \mathcal{A} -measurable sets. By exploiting this duality, we show that the set of all maximal ideals of $\mathcal{M}(X, \mathcal{A})$ endowed with the familiar hull-kernel topology, also called the structure space of $\mathcal{M}(X, \mathcal{A})$, is homeomorphic to the set \hat{X} of all \mathcal{A} -ultrafilters on X , equipped with the Stone-topology (Theorem 2.10). This is the first important technical result in this article. This further yields a complete description of the maximal ideals of

2010 *Mathematics Subject Classification.* Primary 54C40; Secondary 46E30.

Key words and phrases. Rings of Measurable functions, intermediate rings of measurable functions, \mathcal{A} -filter on X , \mathcal{A} -ultrafilter on X , \mathcal{A} -ideal, absolutely convex ideals; hull-kernel topology; Stone-topology; conditionally complete lattice; P -space.

The second author thanks the NBHM, Mumbai-400 001, India, for financial support.

$\mathcal{M}(X, \mathcal{A})$ in terms of \hat{X} (Theorem 2.11). Incidentally, if $\mathcal{M}(X, \mathcal{A})$ is equipped with the m -topology, then all ideals in $\mathcal{M}(X, \mathcal{A})$ are closed (Theorem 2.15). We further note that the σ -algebra \mathcal{A} on X is finite when and only when each ideal (maximal ideal) of $\mathcal{M}(X, \mathcal{A})$ is fixed (Theorem 2.13).

In Section 3, we consider the order on the quotient ring of $\mathcal{M}(X, \mathcal{A})/I$ for an ideal I . It turns out that $\mathcal{M}(X, \mathcal{A})/I$ is a lattice ordered ring with respect to the natural order induced by the order of the original ring $\mathcal{M}(X, \mathcal{A})$. We have the following characterization of the maximal ideals of $\mathcal{M}(X, \mathcal{A})$: the ideal I is maximal if and only if $\mathcal{M}(X, \mathcal{A})/I$ is totally ordered (Theorem 3.5). This is the main result in Section 3. We define real and hyperreal maximal ideals of $\mathcal{M}(X, \mathcal{A})$ in an analogous manner to their counterparts in rings of continuous functions and provide a characterization of those ideals in terms of the associated \mathcal{A} -ultrafilters on X (Theorem 3.9).

In Section 4, we initiate the study of intermediate rings of measurable functions. By an intermediate ring of measurable functions we mean a subring $\mathcal{N}(X, \mathcal{A})$ of $\mathcal{M}(X, \mathcal{A})$ which contains $\mathcal{M}^*(X, \mathcal{A})$ of all the bounded measurable functions on X . The main technical tool in this section, which we borrow from the articles [13], [14], [15], is that of local invertibility of measurable functions on measurable sets in the given intermediate ring. With each maximal ideal M in $\mathcal{N}(X, \mathcal{A})$, we associate an \mathcal{A} -ultrafilter $\mathcal{Z}_{\mathcal{N}}[M]$ on X which leads to a bijection between the set of all maximal ideals of $\mathcal{N}(X, \mathcal{A})$ and the family of all \mathcal{A} -ultrafilters on X (Theorems 4.6 and 4.7). It is interesting to note that this bijective map becomes a homeomorphism provided the former set is equipped with the hull-kernel topology and the later with the Stone-topology (Theorem 4.8). This in essence says that the structure space of each intermediate ring of measurable functions is one and the same as that of the original ring $\mathcal{M}(X, \mathcal{A})$. In the concluding portion of Section 4, we highlight a number of special properties which characterize $\mathcal{M}(X, \mathcal{A})$ among all the intermediate rings $\mathcal{N}(X, \mathcal{A})$ (Theorems 4.9, 4.10, and 4.11).

In Section 5, we highlight several properties enjoyed by the ring $\mathcal{M}(X, \mathcal{A})$ and the ring $C(Y)$ of all real-valued continuous functions defined over a P -space Y . We conclude by raising a few questions about the relationship between rings of continuous functions on P -spaces and rings of measurable functions.

2. Ideals in $\mathcal{M}(X, \mathcal{A})$ versus \mathcal{A} -filters on X

Throughout the paper, when we speak of ideal unmodified, we will always mean a proper ideal. In this section, we introduce filters on the lattice of measurable sets, which we call \mathcal{A} -filters, and we show that each ideal (maximal ideal) of $\mathcal{M}(X, \mathcal{A})$ corresponds to an \mathcal{A} -filter (\mathcal{A} -ultrafilter) on X . We also describe the structure space of $\mathcal{M}(X, \mathcal{A})$.

DEFINITION 2.1. A subfamily \mathfrak{F} of \mathcal{A} is called an \mathcal{A} -filter on X if it excludes the empty set and is closed under finite intersection and formation of supersets from the family \mathcal{A} . An \mathcal{A} -filter on X is said to be an \mathcal{A} -ultrafilter if it is not properly contained in any \mathcal{A} -filter on X .

By using Zorn's Lemma, it is easy to see that each \mathcal{A} -filter on X extends to an \mathcal{A} -ultrafilter on X . Indeed \mathcal{A} -ultrafilters on X are precisely those subfamilies of \mathcal{A} , which possess the finite intersection property and are maximal with respect to this property. Before formally initiating the duality between ideals in $\mathcal{M}(X, \mathcal{A})$ and the \mathcal{A} -filters on X , we write down the following well known technique of construction of measurable functions from smaller domains to larger ones.

THEOREM 2.2 (Pasting Lemma). (*See [3, Lemma 6]*) *If $\{A_i\}_{i=1}^{\infty}$ is a countable family of members of \mathcal{A} and $f : \cup_{i=1}^{\infty} A_i \mapsto \mathbb{R}$ is a function such that $f|_{A_i}$ is a measurable function for each i , then f is also a measurable function.*

For any $f \in \mathcal{M}(X, \mathcal{A})$, we let $Z(f)$ denote the zero-set of f and $cZ(f) = X \setminus Z(f)$ the co-zero set of f ; here $Z(f) = \{x \in X : f(x) = 0\}$. It is clear that zero-sets and co-zero sets of functions lying in $\mathcal{M}(X, \mathcal{A})$ are all members of \mathcal{A} . Conversely, each set $E \in \mathcal{A}$ is the zero set of some function in $\mathcal{M}(X, \mathcal{A})$, indeed $E = Z(\chi_{E^c})$, where χ_{E^c} is the characteristic function of $E^c = X \setminus E$ on X .

The following proposition is an immediate consequence of Pasting Lemma.

THEOREM 2.3. *For $f, g \in \mathcal{M}(X, \mathcal{A})$, $Z(f) \supseteq Z(g)$ if and only if f is a multiple of g .*

PROOF. If f is a multiple of g in $\mathcal{M}(X, \mathcal{A})$, then it is trivial that $Z(f) \supseteq Z(g)$. Conversely let $Z(f) \supseteq Z(g)$. Define a function $h : (X, \mathcal{A}) \mapsto \mathbb{R}$ by the following rule: $h(x) = \frac{f(x)}{g(x)}$ if $x \notin Z(g)$ and $h(x) = 0$ if $x \in Z(g)$. Then by the Pasting Lemma h is a member of $\mathcal{M}(X, \mathcal{A})$ and clearly $f = gh$. \square

It follows from Theorem 2.3 that each $f \in \mathcal{M}(X, \mathcal{A})$ is a multiple of f^2 , and hence $\mathcal{M}(X, \mathcal{A})$ is a Von-Neumann regular ring. It is well-known that any commutative reduced ring is Von-Neumann regular if and only if each of its prime ideals is maximal (see [8, Theorem 1.16]). Thus we have the following corollary.

COROLLARY 2.4. Every prime ideal of $\mathcal{M}(X, \mathcal{A})$ is maximal.

An ideal I in a commutative ring R with unity is called z° -ideal if for each $a \in I$, $P_a \subseteq I$, where P_a is the intersection of all minimal prime ideals containing a . Since each ideal in a Von-Neumann regular ring is a z° -ideal [3, Remark 1.6(a)], it follows that all ideals of $\mathcal{M}(X, \mathcal{A})$ are z° -ideals. This fact is also independently observed by [6, Proposition 9].

For any ideal I in $\mathcal{M}(X, \mathcal{A})$, let $Z[I] = \{Z(f) : f \in I\}$, and for any \mathcal{A} -filter on X , let $Z^{-1}[\mathfrak{F}] = \{f \in \mathcal{M}(X, \mathcal{A}) : Z(f) \in \mathfrak{F}\}$. The following theorem entailing a duality between ideals in $\mathcal{M}(X, \mathcal{A})$ and \mathcal{A} -filters on X is a measure-theoretic analog to [7, Theorem 2.3], and can be established by using some routine arguments. See also [6, Proposition 3].

THEOREM 2.5. *Let I be an ideal in $\mathcal{M}(X, \mathcal{A})$, and \mathfrak{F} be an \mathcal{A} -filter on X . Then $Z[I]$ is an \mathcal{A} -filter on X , and $Z^{-1}[\mathfrak{F}]$ is an ideal in $\mathcal{M}(X, \mathcal{A})$*

PROPOSITION 2.6. *If I is an ideal in $\mathcal{M}(X, \mathcal{A})$ containing a function f , then any g in $\mathcal{M}(X, \mathcal{A})$ with $Z(g) = Z(f)$ is also a member of I .*

The first of the following is a consequence of Proposition 2.6 and the second directly from the definitions: if I is an ideal of $\mathcal{M}(X, \mathcal{A})$ and \mathfrak{F} is an \mathcal{A} -filter on X , then

$$(2.1) \quad Z^{-1}Z[I] = I \quad \text{and} \quad ZZ^{-1}[\mathfrak{F}] = \mathfrak{F}.$$

As a result, we have the following correspondence.

THEOREM 2.7. *The map $Z : I \mapsto Z[I]$ is a bijective correspondence between ideals in $\mathcal{M}(X, \mathcal{A})$ and the \mathcal{A} -filters on X . Moreover, if M is a maximal ideal, then $Z[M]$ is an \mathcal{A} -ultrafilter, and if \mathcal{U} is an \mathcal{A} -ultrafilter, then $Z^{-1}[\mathcal{U}]$ is a maximal ideal.*

An ideal I in $\mathcal{M}(X, \mathcal{A})$ is called fixed if $\cap Z[I] \neq \emptyset$, otherwise I is called a free ideal. It was observed in [6, Proposition 6] by adapting the arguments in [7, Theorem 4.6(a)] that the complete list of fixed maximal ideals in $\mathcal{M}(X, \mathcal{A})$ is

given by $\{M_p : p \in X\}$, where $M_p = \{f \in \mathcal{M}(X, \mathcal{A}) : f(p) = 0\}$. If in addition, \mathcal{A} separates points of X in the sense that given any two distinct points a, b in X , there is a member E of \mathcal{A} , which contains exactly one of them, then $M_p \neq M_q$, whenever $p \neq q$ in X . It is established in [11, Theorem 1.2] that if a commutative ring R with unity is also a Gelfand ring meaning that each prime ideal in R extends to a unique maximal ideal, then the structure space of R is Hausdorff. It follows therefore from Corollary 2.4 that, the structure space of the ring $\mathcal{M}(X, \mathcal{A})$ is Hausdorff. It also follows from a more general result Theorem 4.7, that we prove later in this paper. Nevertheless, we shall produce an alternative proof of this assertion, by exploiting the duality between maximal ideals and \mathcal{A} -ultrafilters in Theorem 2.7.

THEOREM 2.8. *The structure space of $\mathcal{M}(X, \mathcal{A})$ is a (compact) Hausdorff space.*

PROOF. Let M_1 and M_2 be two distinct maximal ideals of $\mathcal{M}(X, \mathcal{A})$. Then by Theorem 2.7, the \mathcal{A} -ultrafilters $Z[M_1]$ and $Z[M_2]$ are also different, this implies in view of the maximality of an \mathcal{A} -ultrafilter on X with respect to having the finite intersection property that, there exists $f_1 \in M_1$ and $f_2 \in M_2$ such that $Z(f_1) \cap Z(f_2) = \phi$. Let $g = \frac{(f_1)^2}{(f_1)^2 + (f_2)^2}$. Then $g \in \mathcal{M}(X, \mathcal{A})$. Let $Z_1 = \{x \in X : g(x) \leq \frac{1}{2}\}$ and $Z_2 = \{x \in X : g(x) \geq \frac{1}{2}\}$. then Z_1 and Z_2 are \mathcal{A} -measurable sets in X and $Z_1 \cup Z_2 = X$. We can write $Z_1 = Z(h_1)$ and $Z_2 = Z(h_2)$, where $h_1, h_2 \in \mathcal{M}(X, \mathcal{A})$. We see that $Z(h_2) \cap Z(f_1) = Z(h_1) \cap Z(f_2) = \phi$. Hence $h_2 \notin M_1$ and $h_1 \notin M_2$. Also $h_1 h_2 = 0$; because $Z(h_1 h_2) = Z_1 \cup Z_2 = X$. By [7, Exercise 7M4], the structure space of $\mathcal{M}(X, \mathcal{A})$ is Hausdorff. \square

We now show that the hull-kernel topology of the structure space of $\mathcal{M}(X, \mathcal{A})$ can be identified with the Stone-topology on the set of all \mathcal{A} -ultrafilters on X . We now focus on a measure-theoretic analog of [7, Theorem 6.5].

Let \hat{X} be an enlargement of the set X , with the intention that it will serve as an index set for the family of all \mathcal{A} -ultrafilters on X . For each $p \in \hat{X}$, let the corresponding \mathcal{A} -ultrafilter be denoted by \mathcal{U}^p with the stipulation that for $p \in X$, $\mathcal{U}^p = \mathcal{U}_p = \{A \in \mathcal{A} : p \in A\}$. For each $A \in \mathcal{A}$, let $\bar{A} = \{p \in \hat{X} : A \in \mathcal{U}^p\}$. Then $\{\bar{A} : A \in \mathcal{A}\}$ is a base for the closed sets of some topology, *viz* the Stone-topology on \hat{X} . We shall simply write \hat{X} to denote the set \hat{X} with this Stone-topology. Observing that for all measurable sets $A, B \in \mathcal{A}$, $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$, and $\bar{A} \cap \hat{X} = A$, it is not hard to establish the following theorem which is a measure-theoretic analog of [7, Theorem 6.5(b)].

THEOREM 2.9. *For any $A \in \mathcal{A}$, $\bar{A} = cl_{\hat{X}} A$. In particular $cl_{\hat{X}} X = \hat{X}$.*

Given a maximal ideal M in $\mathcal{M}(X, \mathcal{A})$, $Z[M]$ is an \mathcal{A} -ultrafilter on X , and therefore there exists a unique point $p \in \hat{X}$ such that $Z[M] = \mathcal{U}^p$. In this way, we obtain a map $\psi : \text{Max}(\mathcal{M}) \rightarrow \hat{X}$, where $\text{Max}(\mathcal{M})$ is the set of all maximal ideals in $\mathcal{M}(X, \mathcal{A})$, such that $\psi(M) = p$. By Theorem 2.7, ψ is a bijection. Furthermore for any $f \in \mathcal{M}(X, \mathcal{A})$ and $M \in \text{Max}(\mathcal{M})$, we have the following equivalence:

$$\begin{aligned} f \in M &\Leftrightarrow Z(f) \in Z(M) && \text{(Theorem 2.5)} \\ &\Leftrightarrow Z(f) \in \mathcal{U}^p && \text{where } \psi(M) = p \\ &\Leftrightarrow p \in cl_{\hat{X}} Z(f) \end{aligned}$$

Thus we can write $\psi(\mathcal{B}_f) = cl_{\hat{X}} Z(f) = \overline{Z(f)}$ for any $f \in \mathcal{M}(X, \mathcal{A})$, where $\mathcal{B}_f = \{M \in \text{Max}(\mathcal{M}) : f \in M\}$. Therefore ψ induces a bijection between the basic closed sets of the structure space $\text{Max}(\mathcal{M})$ of $\mathcal{M}(X, \mathcal{A})$ and the basic closed sets \bar{A} of the space \hat{X} with the Stone-topology. Furthermore we observe that for $f \in \mathcal{M}(X, \mathcal{A})$, $\hat{X} \setminus cl_{\hat{X}} Z(f) = cl_{\hat{X}}(X \setminus Z(f)) = cl_{\hat{X}} Z(g)$ for some $g \in \mathcal{M}(X, \mathcal{A})$. This

shows that $cl_{\hat{X}}Z(f), f \in \mathcal{M}(X, \mathcal{A})$ are all clopen sets in the space \hat{X} . So, we can write:

THEOREM 2.10. *The structure space $\text{Max}(\mathcal{M})$ of $\mathcal{M}(X, \mathcal{A})$ is homeomorphic to the space \hat{X} , under the map $\psi : M \mapsto p$, where $Z[M] = \mathcal{U}^p$. Furthermore \hat{X} is a compact Hausdorff zero-dimensional space.*

Let us write for each $p \in \hat{X}$, $Z^{-1}[\mathcal{U}^p] = M^p$. Thus $\{M^p : p \in \hat{X}\}$ is the complete list of maximal ideals of $\mathcal{M}(X, \mathcal{A})$. The following theorem is an analog of the Gelfand-Kalmogoroff theorem in rings of continuous functions for the maximal ideals of $\mathcal{M}(X, \mathcal{A})$. It is a consequence of the arguments above.

THEOREM 2.11. *For each $p \in \hat{X}$, $M^p = \{f \in \mathcal{M}(X, \mathcal{A}) : p \in cl_{\hat{X}}Z(f)\}$.*

Our next goal is to characterize those measurable spaces (X, \mathcal{A}) for which the σ -algebras are finite in terms of the ideals of the ring $\mathcal{M}(X, \mathcal{A})$. But first we need the following subsidiary result.

LEMMA 2.12. *Let \mathcal{A} be an infinite σ -algebra on X . Then there exists a countably infinite family $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint nonempty members of \mathcal{A} .*

PROOF. An element $E \in \mathcal{A}$ is called an atom if $E \neq \emptyset$ and E does not properly contain any nonempty member of \mathcal{A} . If there are infinitely many atoms of \mathcal{A} , then there is no more to prove because any two distinct atoms are pairwise disjoint. Assume therefore that there are only finitely many atoms of \mathcal{A} , say $A_1, A_2, A_3, \dots, A_n$. Let $A = \cup_{i=1}^n A_i$. We choose any nonempty set B_0 from \mathcal{A} , such that $B_0 \cap A = \emptyset$. Since B_0 is not an atom, we can choose a nonempty set B_1 from \mathcal{A} such that $B_1 \subsetneq B_0$. We continue the process and having chosen B_n , let B_{n+1} be a strictly smaller member of $\mathcal{A} \setminus \{\emptyset\}$, contained in B_n . In this way by induction, we construct a strictly decreasing chain $\{B_n\}_{n=0}^{\infty}$ of members of \mathcal{A} . Finally for each n , let $E_n = B_n \setminus B_{n+1}$. Then $\{E_n : n = 0, 1, 2, \dots\}$ is a pairwise disjoint family of nonempty members of \mathcal{A} . \square

In light of Lemma 2.12, the notion of a σ -algebra being compact from [6] (the collection of elements whose join is the top element has a finite subcollection whose join is the top element) is the equivalent to a σ -algebra being finite. The following theorem is then an extension of [6, Proposition 15] with more equivalences and an alternative proof.

THEOREM 2.13. *The statements written below are equivalent.*

- (i) $\mathcal{M}(X, \mathcal{A}) = \mathcal{M}^*(X, \mathcal{A}) = \{f \in \mathcal{M}(X, \mathcal{A}) : f \text{ is bounded on } X\}$.
- (ii) *Each ideal of $\mathcal{M}(X, \mathcal{A})$ is fixed.*
- (iii) *Each maximal ideal of $\mathcal{M}(X, \mathcal{A})$ is fixed.*
- (iv) *Each ideal of $\mathcal{M}^*(X, \mathcal{A})$ is fixed.*
- (v) *Each maximal ideal of $\mathcal{M}^*(X, \mathcal{A})$ is fixed.*
- (vi) \mathcal{A} is a finite σ -algebra on X .

PROOF. (i) \Leftrightarrow (vi): If (vi) is false, then by Lemma 2.12, there exists a countably infinite family $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint nonempty sets in \mathcal{A} . The function $f : X \mapsto \mathbb{R}$, given by: $f(E_n) = n$ for $n \in \mathbb{N}$ and $f(X \setminus \cup_{n=1}^{\infty} E_n) = 0$ is clearly an unbounded measurable function by the Pasting Lemma (Theorem 2.2). Thus $f \in \mathcal{M}(X, \mathcal{A}) \setminus \mathcal{M}^*(X, \mathcal{A})$ and so (i) is false.

Conversely, if (i) is false, then there exists a $g \in \mathcal{M}(X, \mathcal{A})$ such that $g \geq 0$ and g is unbounded above on X . Consequently there exists a countably infinite set of points $\{x_1, x_2, x_3, \dots\}$ in X for which $f(x_1) < f(x_2) < f(x_3) < \dots < f(x_n) < \dots$. Let $F_n = \{x \in X : f(x) < f(x_{n+1})\}$. Then $F_1 \subsetneq F_2 \subsetneq \dots$ is a strictly increasing sequence of nonempty members of \mathcal{A} . This renders (vi) false.

(ii) \Leftrightarrow (vi): It is trivial that (vi) \Leftrightarrow (ii). Conversely, if (vi) is false, and $\{E_n\}_{n=1}^\infty$ is the guaranteed collection of pairwise disjoint nonempty sets, then

$$I = \{f \in \mathcal{M}(X, \mathcal{A}) : f(E_n) = 0 \text{ for all but finitely many } n\text{'s in } \mathbb{N}\}$$

is a free ideal of $\mathcal{M}(X, \mathcal{A})$. Therefore the statement (ii) is false.

(ii) \Leftrightarrow (iv): If (ii) is true, then (iv) follows from the equivalence of (i) and (ii). Conversely, assume that (iv) is true and I is an ideal of $\mathcal{M}(X, \mathcal{A})$. Then $I \cap \mathcal{M}^*(X, \mathcal{A})$ is an ideal of $\mathcal{M}^*(X, \mathcal{A})$ and is fixed. Now with each f in I , we can associate a multiplicative unit

$$u_f = \frac{1}{1 + |f|}$$

of $\mathcal{M}(X, \mathcal{A})$ such that $u_f \cdot f \in \mathcal{M}^*(X, \mathcal{A})$. This implies that $\bigcap_{f \in I} Z(f) = \bigcap_{f \in I} Z(u_f \cdot f) \supseteq \bigcap_{g \in I \cap \mathcal{M}^*(X, \mathcal{A})} Z(g) \neq \emptyset$. This proves that I is a fixed ideal of $\mathcal{M}(X, \mathcal{A})$.

Altogether the statements (i), (ii), (iv), and (vi) are equivalent. The equivalence of (ii) and (iii) (respectively (iv) and (v)) follows from Zorn's Lemma. \square

DEFINITION 2.14. For each g in $\mathcal{M}(X, \mathcal{A})$ and each positive unit u of this ring, set $m(g, u) = \{f \in \mathcal{M}(X, \mathcal{A}) : |f - g| \leq u\}$. Then there exists a unique topology on $\mathcal{M}(X, \mathcal{A})$ which we call the m -topology in which for each g , $\{m(g, u) : u \text{ is a positive unit of } \mathcal{M}(X, \mathcal{A})\}$ is a neighbourhood base of it (compare this to [7, Exercise 2N]).

It is easy to prove that $\mathcal{M}(X, \mathcal{A})$ with the m -topology is a topological ring, by using some routine arguments and the fact that a continuous function of a real-valued measurable function is measurable. Furthermore it is not at all difficult to check that the set of all multiplicative units of the ring $\mathcal{M}(X, \mathcal{A})$ is an open set in this m -topology. It follows that if I is a (proper) ideal of $\mathcal{M}(X, \mathcal{A})$, then its closure is also a (proper) ideal. Thus every maximal ideal is closed.

THEOREM 2.15. *Each ideal in $\mathcal{M}(X, \mathcal{A})$ is closed in the m -topology.*

PROOF. By Theorem 2.6, we can write

$$\begin{aligned} I &= \{f \in \mathcal{M}(X, \mathcal{A}) : Z(f) \in Z[I]\} \\ &= \{f \in \mathcal{M}(X, \mathcal{A}) : Z(f^n) \in Z[I]\} \\ &= \{f \in \mathcal{M}(X, \mathcal{A}) : f^n \in I \text{ for some } n \in \mathbb{N}\}, \end{aligned}$$

Thus I is the intersection of all prime ideals of $\mathcal{M}(X, \mathcal{A})$ containing it (see [7, Corollary 0.18]). As $\mathcal{M}(X, \mathcal{A})$ is Von Neumann regular, each of its prime ideals is maximal, and hence I is the intersection of all maximal ideals containing it. As remarked in the comments preceding the theorem, each maximal ideal of $\mathcal{M}(X, \mathcal{A})$ is closed; hence I is a closed subset of $\mathcal{M}(X, \mathcal{A})$. \square

It was proved by Hewitt in [10, Theorem 3] that $C(X)$ with the m -topology is first-countable if and only if X is pseudocompact. We now give a characterization for $\mathcal{M}(X, \mathcal{A})$ with the m -topology to be first-countable.

THEOREM 2.16. *The m -topology of $\mathcal{M}(X, \mathcal{A})$ is first-countable if and only if \mathcal{A} is finite.*

PROOF. Let \mathcal{A} be finite. Then by Theorem 2.13, $\mathcal{M}(X, \mathcal{A}) = \mathcal{M}^*(X, \mathcal{A})$. So $\mathcal{M}(X, \mathcal{A})$ is a Banach space with the sup norm, and in particular, it is metrizable, and hence its metric topology is first countable. Furthermore, we observe that the m -topology is this norm topology, since if $u > 0$ is a unit in $\mathcal{M}^*(X, \mathcal{A})$, then as $\mathcal{M}^*(X, \mathcal{A}) = \mathcal{M}(X, \mathcal{A})$, $1/u$ is in $\mathcal{M}^*(X, \mathcal{A})$, and so there is a $\lambda > 0$, such that $u(x) > \lambda$ for all $x \in X$. Then $m(f, u) \subseteq U(f, \lambda)$, where $U(f, \lambda) = \{g \in \mathcal{M}(X, \mathcal{A})$

$|f(x) - g(x)| \leq \lambda\}$ is a closed base element of the norm topology. Furthermore $U(f, \lambda) = m(f, \lambda)$, where $\lambda(x) = \lambda$ for all $x \in X$.

Suppose instead that \mathcal{A} is infinite. Then by Theorem 2.12, there is a countable family $\{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint non-empty sets A_n from \mathcal{A} . We claim that $\mathcal{M}(X, \mathcal{A})$ with the m -topology is not first-countable at the constant functions $\mathbf{0}$. Suppose toward a contradiction, that $\mathbf{0}$ has a countable base $\{m(\mathbf{0}, u_i)\}_{i \in \mathbb{N}}$, where $u_i(x) > 0$ for all $x \in X$. To obtain a contradiction, we construct a positive unit u in $\mathcal{M}(X, \mathcal{A})$, such that $m(\mathbf{0}, u_i) \not\subseteq m(\mathbf{0}, u)$ for any $i \in \mathbb{N}$. Indeed, let $u : X \rightarrow \mathbb{R}$ be defined as follows:

$$u(x) = \begin{cases} \frac{1}{2}u_n(x) & \text{if } x \in A_n \text{ for some } n \in \mathbb{N} \\ 1 & \text{if } x \in (X - \bigcup_{n=1}^{\infty} A_n) \end{cases}$$

By the pasting lemma (Lemma 2.2), u is measurable. But for each n , $m(\mathbf{0}, u_n) \not\subseteq m(\mathbf{0}, u)$, since $\frac{2}{3}u_n \in m(\mathbf{0}, u_n)$, but $\frac{2}{3}u_n \notin m(\mathbf{0}, u)$. \square

3. Residue class rings of $\mathcal{M}(X, \mathcal{A})$ modulo ideals

In this section, we consider the ordering of a quotient ring of measurable functions by an absolutely convex ideal. In what follows we denote $I(a)$ to be the residue class $I + a$ in R/I which contains a . Also, let 0 be the identity element I of R/I . An ideal I of a lattice-ordered ring R is called absolutely convex if whenever $|a| < |b|$ and $b \in I$ then $a \in I$. We begin by recalling the following well-known results (see [7, §5.3]).

PROPOSITION 3.1. *If I is an absolutely convex ideal in a lattice ordered ring R , then*

- (1) R/I is a lattice ordered ring, using the following ordering: $I(a) \geq 0$ if there exists an $x \in R$ such that $x \geq 0$ and $a \equiv x \pmod{I}$.
- (2) $I(a) \geq 0$ if and only if $a \equiv |a| \pmod{I}$
- (3) $I(|a|) = |I(a)|$ for each $a \in R$.

Note that $\mathcal{M}(X, \mathcal{A})$ is a lattice-ordered ring with the natural order for each $f, g \in \mathcal{M}(X, \mathcal{A})$, $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$.

PROPOSITION 3.2. *Each ideal in $\mathcal{M}(X, \mathcal{A})$ is absolutely convex.*

PROOF. If $f, g \in \mathcal{M}(X, \mathcal{A})$, $g \in I$, and $|f| \leq |g|$, then $Z(g) \subseteq Z(f)$; consequently by Theorem 2.3, f is a multiple of g , hence $f \in I$. \square

The following theorem follows immediately from Propositions 3.1 and 3.2.

THEOREM 3.3. *If I is an ideal of $\mathcal{M}(X, \mathcal{A})$, then the quotient ring $\mathcal{M}(X, \mathcal{A})/I$ is a lattice ordered ring.*

The following theorem provides useful description of non-negative elements of the quotient ring $\mathcal{M}(X, \mathcal{A})/I$. Compare with [7, §5.4(a)].

THEOREM 3.4. *Let I be an ideal of $\mathcal{M}(X, \mathcal{A})$ and $f \in \mathcal{M}(X, \mathcal{A})$. Then $I(f) \geq 0$ if and only if there exists $E \in Z[I]$ such that $f \geq 0$ on E .*

PROOF. Let $I(f) \geq 0$. Then it follows from Proposition 3.1 that $I(f) = |I(f)| = I(|f|)$. Consequently $f - |f| \in I$, and hence $E = Z(f - |f|) \in Z[I]$. It is clear that $f \geq 0$ on E .

Conversely, if $E \in Z[I]$ and $f \geq 0$ on E , then it is clear that $E \subseteq Z(f - |f|)$. Since $Z[I]$ is a \mathcal{A} -filter on X , it follows that $Z(f - |f|) \in Z[I]$. Hence by Proposition 2.6, we can write $f - |f| \in I$. Since $|f| \geq 0$, we have that $I(|f|) \geq 0$. But by Proposition 3.1, $I(|f|) = I(f)$. So $I(f) \geq 0$. \square

The following result gives a characterization of maximal ideals of $\mathcal{M}(X, \mathcal{A})$.

THEOREM 3.5. *For an ideal I in $\mathcal{M}(X, \mathcal{A})$, the following statements are equivalent:*

- (i) *The ideal I is a maximal ideal of $\mathcal{M}(X, \mathcal{A})$.*
- (ii) *Given $f \in \mathcal{M}(X, \mathcal{A})$, there exists $E \in Z[I]$ on which f does not change its sign.*
- (iii) *The residue class ring $\mathcal{M}(X, \mathcal{A})/I$ is totally ordered.*

PROOF. (i) \Rightarrow (ii): Suppose (i) holds and let $f \in \mathcal{M}(X, \mathcal{A})$. Then since $(f \vee 0)(f \wedge 0) = 0$ and each maximal ideal is prime, it follows that $f \vee 0 \in I$ or $f \wedge 0 \in I$. Consequently $Z(f \vee 0) \in Z[I]$ or $Z(f \wedge 0) \in Z[I]$. We note that $f \leq 0$ on $Z(f \wedge 0)$ and $f \geq 0$ on $Z(f \vee 0)$.

(ii) \Rightarrow (iii): Suppose (ii) is true. Let $f \in \mathcal{M}(X, \mathcal{A})$. Then there is an $E \in Z[I]$ on which $f \geq 0$ or $f \leq 0$. This implies in view of Theorem 3.4 that $I(f) \geq 0$ or $I(f) \leq 0$ in $\mathcal{M}(X, \mathcal{A})/I$. Thus $\mathcal{M}(X, \mathcal{A})/I$ is totally ordered.

(iii) \Rightarrow (i): Suppose (iii) is true. Let $g, h \in \mathcal{M}(X, \mathcal{A})$ such that $gh \in I$. By the condition (iii), we can write either $I(|g| - |h|) \geq 0$ or $I(|g| - |h|) \leq 0$. Without loss of generality, $I(|g| - |h|) \geq 0$. It follows from Theorem 3.4 that there is an $E \in Z[I]$ such that $|g| - |h| \geq 0$ on E . This implies that $E \cap Z(g) \subseteq Z(h)$ and hence $E \cap Z(gh) \subseteq Z(h)$. As $E \in Z[I]$ and $gh \in I$, the last relation implies that $Z(h) \in Z[I]$, hence $h \in I$ by Proposition 2.6. If we assume that $I(|g| - |h|) \leq 0$, we could have obtained analogously that $g \in I$. Thus either $g \in I$ or $h \in I$. Hence I is a prime ideal and therefore maximal ideal in $\mathcal{M}(X, \mathcal{A})$. \square

DEFINITION 3.6. A totally ordered field F is called archimedean if given $\alpha \in F$, there is an $n \in \mathbb{N}$ such that $n > \alpha$. Otherwise F is called non-archimedean.

So, if F is non archimedean, then there is an element $\alpha \in F$ such that $\alpha > n$ for each $n \in \mathbb{N}$. Such an α is called an infinitely large member of F . The reciprocal of an infinitely large member is called an infinitely small member of F . Thus a non archimedean totally ordered field is characterized by the presence of infinitely large (equivalently infinitely small) members in it.

If M is a maximal ideal of $\mathcal{M}(X, \mathcal{A})$ and $\pi : \mathcal{M}(X, \mathcal{A}) \rightarrow \mathcal{M}(X, \mathcal{A})/M$ given by $f \mapsto M(f)$ is the canonical map, then M and $\mathcal{M}(X, \mathcal{A})/M$ are called real if the set of images of the constant functions under π is all of $\mathcal{M}(X, \mathcal{A})/M$ (in which case, $\mathcal{M}(X, \mathcal{A})/M$ is isomorphic to \mathbb{R}), and hyperreal otherwise. The residue class field $\mathcal{M}(X, \mathcal{A})/M$ is archimedean if and only if it is real, since an ordered field is archimedean if and only if it is isomorphic to subfield of \mathbb{R} ([7, §0.21]), and identity is the only non-zero homomorphism of \mathbb{R} into itself ([7, §0.22]).

The following result relates infinitely large members in the residue class fields of $\mathcal{M}(X, \mathcal{A})$ modulo hyperreal maximal ideals of $\mathcal{M}(X, \mathcal{A})$ and unbounded functions in $\mathcal{M}(X, \mathcal{A})$.

THEOREM 3.7. *For a given maximal ideal M in $\mathcal{M}(X, \mathcal{A})$ and an $f \in \mathcal{M}(X, \mathcal{A})$, the following statements are equivalent:*

- (i) *$|M(f)|$ is an infinitely large member of the residue class field $\mathcal{M}(X, \mathcal{A})/M$.*
- (ii) *f is unbounded on every set in the \mathcal{A} -ultrafilter $Z[M]$.*
- (iii) *For each $n \in \mathbb{N}$, $E_n = \{x \in X : |f(x)| \geq n\} \in Z[M]$.*

PROOF. (i) \Leftrightarrow (ii) are equivalent because $|M(f)|$ is not infinitely large means there is an $n \in \mathbb{N}$ such that $|M(f)| \leq M(n)$ (n stands for the constant function with value n on X). By Theorem 3.4, this is the case when and only when $|f| \leq n$ on some $E \in Z[M]$.

(iii) \Rightarrow (i) is a straightforward consequence of Theorem 3.4.

(i) \Rightarrow (iii): If (i) hold, then $|M(f)| \geq M(n), \forall n \in \mathbb{N}$, i.e. $M(|f|) \geq M(n)$ for all $n \in \mathbb{N}$. It follows from Theorem 3.4 that there is an $E \in Z[M]$ for which $|f| \geq n$ on E . Such an E is contained in E_n , and hence $E_n \in Z[M]$. \square

THEOREM 3.8. *An $f \in \mathcal{M}(X, \mathcal{A})$ is unbounded on X if and only if there is a maximal ideal M in $\mathcal{M}(X, \mathcal{A})$ for which $|M(f)|$ is infinitely large in $\mathcal{M}(X, \mathcal{M})/M$.*

PROOF. If f is unbounded on X , then $E_n = \{x \in X : |f(x)| \geq n\} \neq \emptyset$ for each $n \in \mathbb{N}$. Therefore $\{E_n : n \in \mathbb{N}\}$ is a family of \mathcal{A} -measurable sets with the finite intersection property and is therefore extendable to an \mathcal{A} -ultrafilter \mathcal{U} on X . Clearly $\mathcal{U} = Z[M]$ for a unique maximal ideal M in $\mathcal{M}(X, \mathcal{A})$. Thus $E_n \in Z[M]$ for every $n \in \mathbb{N}$. Hence by Theorem 3.7, $|M(f)|$ is infinitely large.

Conversely if $|M(f)|$ is infinitely large, then by Theorem 3.7, f is unbounded on every members in $Z[M]$; in particular, f is unbounded on X . \square

The following theorem is a measure-theoretic analog of [7, Theorem 5.14].

THEOREM 3.9. *For any maximal ideal M in $\mathcal{M}(X, \mathcal{A})$, the following statements are equivalent:*

- (i) M is a real maximal ideal.
- (ii) $Z[M]$ is closed under countable intersection.
- (iii) $Z[M]$ has the countable intersection property.

PROOF. (i) \Rightarrow (ii) Suppose (ii) is false. This means that there is a countable collection of functions $(f_n)_{n \in \mathbb{N}}$ in M such that $\bigcap Z(f_n) \notin Z[M]$. Then g , defined by $g(x) = \sum_{n \in \mathbb{N}} |f_n(x)| \wedge 3^{-n}$ for each $x \in X$, is a member of $\mathcal{M}(X, \mathcal{A})$, since whenever a series of real-valued measurable functions is uniformly convergent, then its limit function is measurable and also real-valued. Since $g \geq 0$ by construction, it follows that $M(g) \geq 0$. But $Z(g) = \bigcap Z(f_n) \notin Z[M]$. So $g \notin M$ making $M(g)$ strictly positive. For each $k \in \mathbb{N}$, $Z(f_1) \cap Z(f_2) \cap \dots \cap Z(f_k) \in Z[M]$, and on this set, $g \leq \sum_{n=k}^{\infty} 3^{-n} = 2^{-1}3^{-k}$. Consequently, by Theorem 3.4, it follows that $M(g) \leq M(2^{-1}3^{-k})$ for each $k \in \mathbb{N}$. Thus $M(g)$ is infinitely small, and M is not real.

(ii) \Rightarrow (iii) is trivial since $\emptyset \notin Z[M]$.

(iii) \Rightarrow (i) Suppose (i) is false, that is M is hyperreal. Then $\mathcal{M}(X, \mathcal{A})/M$ is non-archimedean. Consequently, there exists $f \in M$, such that $f \geq 0$ and $M(f)$ is infinitely large. Hence by Theorem 3.7, for each $n \in \mathbb{N}$, $E_n = \{x \in X : |f(x)| \geq n\} \in Z[M]$. But then $\bigcap Z[M] \subseteq \bigcap E_n = \emptyset$, in which case (iii) is false. \square

The following example gives a measurable space with infinite σ -algebra for which every real maximal ideal is fixed.

EXAMPLE 3.10. Let \mathcal{A} be the σ -algebra of all Lebesgue measurable sets in \mathbb{R} . Then $(\mathbb{R}, \mathcal{A})$ is a measurable space. Let M be a real maximal ideal of $\mathcal{M}(X, \mathcal{A})$. Let $i \in \mathcal{M}(X, \mathcal{A})$ be the constant function. Because M is real, there is an $r \in \mathbb{R}$ such that $M(i) = M(r)$ and hence $Z(i - r) \in Z[M]$. But $Z(i - r)$ is a one-point set. So $Z[M]$ is fixed, i.e. M is fixed.

In light of Theorem 3.9 and Example 3.10, a maximal ideal in $\mathcal{M}(\mathbb{R}, \mathcal{A})$ is real if and only if it is fixed. We will see in Example 5.2 that there exists a measurable space (X, \mathcal{A}) such that $\mathcal{M}(X, \mathcal{A})$ has a real free maximal ideal.

DEFINITION 3.11. An \mathcal{A} -ultrafilter \mathcal{U} on X is a real \mathcal{A} -ultrafilter if it is closed under countable intersection (or equivalently which has countable intersection property).

Example 3.10 raises the following questions.

QUESTION 3.12. Can we characterize the measurable spaces (X, \mathcal{A}) for which each real \mathcal{A} -ultrafilter on X is fixed?

QUESTION 3.13. If X is a real compact space and $\mathcal{B}(X)$ is the σ -algebra of all Borel subsets of X , does the measure space $(X, \mathcal{B}(X))$ satisfy the property that each real $\mathcal{B}(X)$ -ultrafilter on X is fixed?

4. Ideals in intermediate rings of measurable functions

By an intermediate ring (of measurable functions), we mean any ring $\mathcal{N}(X, \mathcal{A})$ lying between $\mathcal{M}^*(X, \mathcal{A})$ and $\mathcal{M}(X, \mathcal{A})$. Let $\Omega(X, \mathcal{A})$ stand for the aggregate of all these intermediate rings.

DEFINITION 4.1. Let $E \in \mathcal{A}$. We say that $f \in \mathcal{N}(X, \mathcal{A})$ is E -regular if there exist $g \in \mathcal{N}(X, \mathcal{A})$ such that $fg|_{E^c} = 1$, where $E^c = X \setminus E$.

It is clear that f is E -regular if and only if f^2 is E -regular if and only if $|f|$ is E -regular.

DEFINITION 4.2. For $f \in \mathcal{N}(X, \mathcal{A})$ define

$$\mathcal{Z}_{\mathcal{N}}(f) = \{E \in \mathcal{A} : f \text{ is } E^c\text{-regular}\}$$

and for any $S \subseteq \mathcal{N}(X, \mathcal{A})$ and any $\mathfrak{F} \subseteq \mathcal{A}$, let

$$\mathcal{Z}_{\mathcal{N}}[S] = \bigcup_{f \in S} \mathcal{Z}_{\mathcal{N}}(f) \quad \text{and} \quad \mathcal{Z}_{\mathcal{N}}^{-1}[\mathfrak{F}] = \{f \in \mathcal{N}(X, \mathcal{A}) : \mathcal{Z}_{\mathcal{N}}(f) \subseteq \mathfrak{F}\}.$$

The following facts are measure theoretic analogs of results in [13] and [17, Lemma 3.1]. We omit proofs as they are straightforward.

THEOREM 4.3. *Let $\mathcal{N}(X, \mathcal{A})$ be an intermediate ring of measurable functions.*

- (i) *If I is an ideal in $\mathcal{N}(X, \mathcal{A})$, then $\mathcal{Z}_{\mathcal{N}}[I]$ is an \mathcal{A} -filter on X .*
- (ii) *For any \mathcal{A} -filter \mathfrak{F} on X , $I = \mathcal{Z}_{\mathcal{N}}^{-1}[\mathfrak{F}]$ is an ideal in $\mathcal{N}(X, \mathcal{A})$.*
- (iii) *For $f \in \mathcal{N}(X, \mathcal{A})$, $\cap \mathcal{Z}_{\mathcal{N}}(f) = Z(f)$.*

For any measurable set E , let $\langle E \rangle$ be the principal \mathcal{A} -filter whose intersection is E .

LEMMA 4.4. *If $E \in \mathcal{A}$, then there exists $f \in \mathcal{N}(X, \mathcal{A})$ such that $E = Z(f)$ and $\mathcal{Z}_{\mathcal{N}}(f) = \langle Z(f) \rangle$.*

PROOF. Take $f = \chi_{E^c}$. Then $Z(f) = E$ and surely f is invertible on E^c . This means that $E \in \mathcal{Z}_{\mathcal{N}}(f)$. Hence $\langle E \rangle \subseteq \mathcal{Z}_{\mathcal{N}}(f)$. Conversely if $F \in \mathcal{Z}_{\mathcal{N}}(f)$, then $F \supseteq Z(f)$ by Theorem 4.3(iii), which implies $F \in \langle E \rangle$. Thus $\mathcal{Z}_{\mathcal{N}}(f) \subseteq \langle E \rangle$. \square

It is easy to see that for any ideal I in $\mathcal{N}(X, \mathcal{A})$, and any \mathcal{A} -filter \mathfrak{F} on X ,

$$(4.1) \quad \mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{Z}_{\mathcal{N}}[I]] \supseteq I \quad \text{and} \quad \mathcal{Z}_{\mathcal{N}}\mathcal{Z}_{\mathcal{N}}^{-1}[\mathfrak{F}] \subseteq \mathfrak{F}.$$

Compare this with (2.1) which gives equality when $\mathcal{N}(X, \mathcal{A})$ include all measurable functions.

In an intermediate ring of continuous functions $A(X)$, if M is a maximal ideal, then $\mathcal{Z}_A(M)$ need not be a z -ultrafilter on X , or even a prime z -filter (see [15, p. 154]). With intermediate rings of measurable functions, we have the following.

THEOREM 4.5. *Let $\mathcal{N}(X, \mathcal{A})$ be an intermediate ring of measurable functions. Then*

- (i) *If M is a maximal ideal in $\mathcal{N}(X, \mathcal{A})$, then $\mathcal{Z}_{\mathcal{N}}[M]$ is an \mathcal{A} -ultrafilter on X .*
- (ii) *If \mathcal{U} is an \mathcal{A} -ultrafilter on X , then $\mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}]$ is a maximal ideal in $\mathcal{N}(X, \mathcal{A})$.*

PROOF. (i): Let M be a maximal ideal in $\mathcal{N}(X, \mathcal{A})$. Then $\mathcal{Z}_{\mathcal{N}}[M]$ is an \mathcal{A} -filter on X (by Theorem 4.3(i)). Hence there exists an \mathcal{A} -ultrafilter \mathcal{U} on X such that $\mathcal{Z}_{\mathcal{N}}[M] \subseteq \mathcal{U}$. We claim that $\mathcal{Z}_{\mathcal{N}}[M] = \mathcal{U}$. So let us choose $E \in \mathcal{U}$. Then by Lemma 4.4, we can find an $f \in \mathcal{N}(X, \mathcal{A})$ such that $\mathcal{Z}_{\mathcal{N}}(f) = \langle E \rangle = \langle Z(f) \rangle$. By (4.1), we can write $M \subseteq \mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{Z}_{\mathcal{N}}[M]] \subseteq \mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}]$, and by Theorem 4.3(ii), $\mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}]$ is an ideal in $\mathcal{N}(X, \mathcal{A})$. This implies, in view of the maximality of M and also the properness of the ideal $\mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}]$, that $M = \mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}]$. Since $\mathcal{Z}_{\mathcal{N}}(f) \subseteq \mathcal{U}$, it follows that $f \in \mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}] = M$, and hence $\mathcal{Z}_{\mathcal{N}}(f) \subseteq \mathcal{Z}_{\mathcal{N}}[M]$. Since $\mathcal{Z}_{\mathcal{N}}(f) = \langle E \rangle$, we then have that $E \in \mathcal{Z}_{\mathcal{N}}[M]$. Thus $\mathcal{U} \subseteq \mathcal{Z}_{\mathcal{N}}[M]$. Hence $\mathcal{Z}_{\mathcal{N}}[M] = \mathcal{U}$.

(ii): By Theorem 2.7, $\mathcal{U} = Z[M']$ for some maximal ideal M' in $\mathcal{M}(X, \mathcal{A})$. It is clear that $M' \cap \mathcal{N}(X, \mathcal{A})$ is a prime ideal in $\mathcal{N}(X, \mathcal{A})$ which is extendable to a maximal ideal M of $\mathcal{N}(X, \mathcal{A})$. Enough to prove that $\mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}] \supseteq M$ (and this implies in view of the maximality of M that $\mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}] = M$). We argue by contradiction and assume that there exists an $f \in M$ such that $f \notin \mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{U}]$; this means that $\mathcal{Z}_{\mathcal{N}}(f)$ is not contained in \mathcal{U} . Consequently there exists $E \in \mathcal{Z}_{\mathcal{N}}(f)$ such that $E \notin \mathcal{U}$. Now $E \in \mathcal{Z}_{\mathcal{N}}(f)$ means that there exists $h \in \mathcal{N}(X, \mathcal{A})$ such that $fh|_{E^c} = 1$. On the other hand $E \notin \mathcal{U}$ implies that there exists $k \in M'$ such that $Z(k) \cap E = \emptyset$ and without loss of generality, we can assume that k is bounded on X and therefore $k \in M' \cap \mathcal{N}(X, \mathcal{A})$ and hence $k \in M$. Let $l = \chi_E$. Then $Z(l) \supseteq Z(k)$ implies by Theorem 2.3 that l is a multiple of k in the ring $\mathcal{M}(X, \mathcal{A})$. Since $k \in M'$, which is a maximal ideal in $\mathcal{M}(X, \mathcal{A})$, it follows that $l \in M'$. Also l is bounded on X , hence $l \in \mathcal{N}(X, \mathcal{A})$; therefore $l \in M' \cap \mathcal{N}(X, \mathcal{A})$, and consequently $l \in M$. Since $f \in M$, this implies that $f^2h^2 + l \in M$ (taking care of the fact that $h \in \mathcal{N}(X, \mathcal{A})$). Finally note that $f^2h^2 + l \geq 1$ on (X, \mathcal{A}) , and therefore $f^2h^2 + l \geq 1$ is bounded away from zero on X and hence $f^2h^2 + l \geq 1$ is a unit of $\mathcal{N}(X, \mathcal{A})$ — this is a contradiction. \square

Since, we have already established that if M is a maximal ideal of $\mathcal{N}(X, \mathcal{A})$, then $\mathcal{Z}_{\mathcal{N}}[M]$ is an \mathcal{A} -ultrafilter on X , the following fact is immediate:

THEOREM 4.6. *The map $\mathcal{Z}_{\mathcal{N}} : \text{Max}(\mathcal{N}) \mapsto \hat{X}$ described by $M \mapsto \mathcal{Z}_{\mathcal{N}}[M]$ is a bijection from the set of all maximal ideals of $\mathcal{N}(X, \mathcal{A})$ onto the set of all \mathcal{A} -ultrafilters on X .*

THEOREM 4.7. *The structure space $\text{Max}(\mathcal{N})$ of the intermediate ring $\mathcal{N}(X, \mathcal{A})$ is a (compact) Hausdorff space.*

PROOF. The proof is analogous to that of [15, Theorem 3.6]. Let $M_1, M_2 \in \text{Max}(\mathcal{N})$ with $M_1 \neq M_2$. By [7, Exercise 7M4], it suffices to find $h_1, h_2 \in \mathcal{N}(X, \mathcal{A})$ such that $h_1 \notin M_1$, $h_2 \notin M_2$, and $h_1h_2 = 0$, as $\{0\} = \cap \text{Max}(\mathcal{N})$. We observe that there exist $E_1 \in \mathcal{Z}_{\mathcal{N}}[M_1]$ and $E_2 \in \mathcal{Z}_{\mathcal{N}}[M_2]$ such that $E_1 \cap E_2 = \emptyset$, for otherwise $\mathcal{Z}_{\mathcal{N}}[M_1] \cup \mathcal{Z}_{\mathcal{N}}[M_2]$ is a family of members of \mathcal{A} with finite intersection property and hence there is an \mathcal{A} -filter \mathcal{F} on X such that $\mathcal{Z}_{\mathcal{N}}[M_1] \cup \mathcal{Z}_{\mathcal{N}}[M_2] \subseteq \mathcal{F}$; consequently $M_1 \cup M_2 \subseteq \mathcal{Z}_{\mathcal{N}}^{-1}[\mathcal{F}]$, which is a proper ideal of $\mathcal{N}(X, \mathcal{A})$, contradicting the maximality and distinctness of M_1 and M_2 . Now since $E_1 \in \mathcal{Z}_{\mathcal{N}}[M_1]$ and $E_2 \in \mathcal{Z}_{\mathcal{N}}[M_2]$ with $E_1 \cap E_2 = \emptyset$, there exists $f \in M_1$ and $g \in M_2$ such that $E_1 \in \mathcal{Z}_{\mathcal{N}}(f)$ and $E_2 \in \mathcal{Z}_{\mathcal{N}}(g)$. This means that, there exist $f_1, g_1 \in \mathcal{N}(X, \mathcal{A})$ such that $ff_1|_{E_1^c} = 1$ and $gg_1|_{E_2^c} = 1$. Since $ff_1 \in M_1$ and $gg_1 \in M_2$, it follows that $1 - ff_1 \notin M_1$ and $1 - gg_1 \notin M_2$. But we note that $(1 - ff_1)(1 - gg_1) = 0$. \square

THEOREM 4.8. *Let $\mathcal{N}(X, \mathcal{A})$ be an intermediate ring. Then the bijection $\mathcal{Z}_{\mathcal{N}} : \text{Max}(\mathcal{N}) \mapsto \hat{X}$ by $M \mapsto \mathcal{Z}_{\mathcal{N}}[M]$ is a homeomorphism.*

PROOF. Since each of the two spaces $\text{Max}(\mathcal{N})$ and \hat{X} is already known to be a compact Hausdorff space, it suffices to check that $\mathcal{Z}_{\mathcal{N}}$ is a closed map. A typical

basic closed set in the space $\text{Max}(\mathcal{N})$ is a set of the form $N_f = \{M \in \text{Max}(\mathcal{N}) : f \in M\}$, for some $f \in \mathcal{N}(X, \mathcal{A})$. It is enough to show that $\mathcal{Z}_{\mathcal{N}}(N_f) = \cap\{\bar{E} : E \in \mathcal{Z}_{\mathcal{N}}(f)\}$, which is an intersection of a family of basic closed sets in \hat{X} , and is hence a closed set in \hat{X} . We see that if $M \in N_f$, then $f \in M$. Consequently $\mathcal{Z}_{\mathcal{N}}(f) \subseteq \mathcal{Z}_{\mathcal{N}}[M]$, meaning if $E \in \mathcal{Z}_{\mathcal{N}}(f)$, then E belongs to the \mathcal{A} -ultrafilter $\mathcal{Z}_{\mathcal{N}}[M]$, so that $\mathcal{Z}_{\mathcal{N}}[M] \in \bar{E}$. Thus $\mathcal{Z}_{\mathcal{N}}(N_f) \subseteq \cap\{\bar{E} : E \in \mathcal{Z}_{\mathcal{N}}(f)\}$. Conversely if $M \in \text{Max}(\mathcal{N})$ such that $\mathcal{Z}_{\mathcal{N}}[M] \in \bar{E}$ for every $E \in \mathcal{Z}_{\mathcal{N}}(f)$, then $E \in \mathcal{Z}_{\mathcal{N}}[M]$. Hence $E \in \mathcal{Z}_{\mathcal{N}}[M]$ for each $E \in \mathcal{Z}_{\mathcal{N}}(f)$; thus $\mathcal{Z}_{\mathcal{N}}(f) \subseteq \mathcal{Z}_{\mathcal{N}}[M]$, which implies that $f \in \mathcal{Z}_{\mathcal{N}}^{-1}\mathcal{Z}_{\mathcal{N}}[M] = M$ (as M is a maximal ideal of $\mathcal{N}(X, \mathcal{A})$), i.e. $M \in N_f$ and so $\mathcal{Z}_{\mathcal{N}}[M] \in \mathcal{Z}_{\mathcal{N}}(N_f)$. Hence $\cap\{\bar{E} : E \in \mathcal{Z}_{\mathcal{N}}(f)\} \subseteq \mathcal{Z}_{\mathcal{N}}[N_f]$. \square

We have previously observed that each ideal in $\mathcal{M}(X, \mathcal{A})$ is a z° -ideal (indeed $\mathcal{M}(X, \mathcal{A})$ is a Von-Neumann regular ring). We shall now show that this property characterizes $\mathcal{M}(X, \mathcal{A})$ among intermediate rings of real valued measurable functions on (X, \mathcal{A}) .

THEOREM 4.9. *An intermediate ring $\mathcal{N}(X, \mathcal{A})$ becomes identical to $\mathcal{M}(X, \mathcal{A})$ if and only if each ideal of $\mathcal{N}(X, \mathcal{A})$ is a z° -ideal.*

PROOF. Suppose each ideal of $\mathcal{N}(X, \mathcal{A})$ be a z° -ideal. We claim that for any $f \in \mathcal{M}(X, \mathcal{A})$, the function $\frac{1}{1+|f|}$ is a unit in $\mathcal{N}(X, \mathcal{A})$. It would follow that $|f| \in \mathcal{N}(X, \mathcal{A})$ and consequently $f \in \mathcal{N}(X, \mathcal{A})$. To prove the claim suppose toward a contradiction that there is an $f \in \mathcal{M}(X, \mathcal{A})$ such that $\frac{1}{1+|f|}$ is not a unit in $\mathcal{N}(X, \mathcal{A})$. Since $\frac{1}{1+|f|} \in \mathcal{N}(X, \mathcal{A})$, it follows that the principal ideal $(\frac{1}{1+|f|})$ in $\mathcal{N}(X, \mathcal{A})$ is a proper ideal and is hence a z° -ideal by hypothesis. But $\frac{1}{1+|f|}$ is clearly not a divisor of zero in $\mathcal{N}(X, \mathcal{A})$. Since each element of a z° -ideal in a reduced ring is a divisor of zero a fact easily verifiable, this is a contradiction. \square

COROLLARY 4.10. *An intermediate ring $\mathcal{N}(X, \mathcal{A})$ is Von-Neumann regular if and only if $\mathcal{N}(X, \mathcal{A}) = \mathcal{M}(X, \mathcal{A})$.*

PROOF. If $\mathcal{N}(X, \mathcal{A}) \subsetneq \mathcal{M}(X, \mathcal{A})$, then by Theorem 4.9 there exists an ideal I in $\mathcal{N}(X, \mathcal{A})$ which is not a z° -ideal. Since in a Von-Neumann regular ring each (proper) ideal is a z° -ideal, it follows that $\mathcal{N}(X, \mathcal{A})$ is not a regular ring. \square

We have observed earlier (vide Theorem 2.3) that for $f, g \in \mathcal{M}(X, \mathcal{A})$, $Z(f) \supseteq Z(g)$ if and only if f is a multiple of g . The following result indicates that this fact also characterizes $\mathcal{M}(X, \mathcal{A})$ among the intermediate rings.

THEOREM 4.11. *Let $\mathcal{N}(X, \mathcal{A}) (\subsetneq \mathcal{M}(X, \mathcal{A}))$ be an intermediate ring of measurable functions on the measurable space (X, \mathcal{A}) . Then there exist $g, h \in \mathcal{N}(X, \mathcal{A})$ such that $Z(g) \supseteq Z(h)$ but g is not a multiple of h in the ring $\mathcal{N}(X, \mathcal{A})$.*

PROOF. We can choose $f \in \mathcal{M}(X, \mathcal{A}) \setminus \mathcal{N}(X, \mathcal{A})$. Take $g = \frac{f}{1+|f|}$ and $h = \frac{1}{1+|f|}$. Then g and h are both bounded functions on X and hence $g, h \in \mathcal{N}(X, \mathcal{A})$. We observe that $Z(g) \supseteq Z(h) = \emptyset$. But we claim that g is not a multiple of h in this ring $\mathcal{N}(X, \mathcal{A})$. To prove this claim, suppose there exists $k \in \mathcal{N}(X, \mathcal{A})$ with the relation $g = hk$. This means

$$(4.2) \quad \frac{f}{1+|f|} = \frac{k}{1+|f|}.$$

Since all the functions are real valued, on multiplying both sides of (4.2) by $1+|f|$, we get $f = k$. But this is a contradiction since $f \notin \mathcal{N}(X, \mathcal{A})$ while $k \in \mathcal{N}(X, \mathcal{A})$. \square

5. P -spaces and continuous functions

The ring and lattice structures of $\mathcal{M}(X, \mathcal{A})$ share a number of properties possessed by the lattice ordered ring $C(Y)$ of all real valued continuous functions defined over a P -space Y . Here are some of the properties shared by both $\mathcal{M}(X, \mathcal{A})$ and $C(Y)$: every prime ideal is maximal (Corollary 2.4 and [7, §4J]); each ideal is a z° -ideal (a consequence of the previous property); each ideal is closed when the m -topology is imposed on the ring (Theorem 2.15 and [7, §7Q4]); an ideal is maximal if and only if its residue class ring is totally ordered (Theorem 3.5 and [7, §5P]); the structure space is a compact Hausdorff zero-dimensional space (Theorem 2.8 and [7, §7N] in light of the fact that βY is basically disconnected and in particular zero-dimensional); as well as other properties.

The following proposition shows that when Y is a P -space, then for some specific choice of the measurable space (X, \mathcal{A}) , the ring $\mathcal{M}(X, \mathcal{A})$ is identical to $C(Y)$.

THEOREM 5.1. *If X is a P -space, then the set $Z(X)$ of all zero-sets in X is a σ -algebra on X , and if $\mathcal{A} = Z(X)$, then $\mathcal{M}(X, \mathcal{A}) = C(X)$ and $\mathcal{M}^*(X, \mathcal{A}) = C^*(X)$.*

PROOF. Since the zero-sets and cozero-sets of the P -space X are one and the same (see [7, §J(3)]) and $Z(X)$ is closed under countable intersection, it follows clearly that $Z(X)$ is a σ -algebra on X .

If $f \in \mathcal{M}(X, \mathcal{A})$, then for any open set G in \mathbb{R} , $f^{-1}(G)$ is a member of $\mathcal{A} = Z(X)$, in particular $f^{-1}(G)$ is open in X . Hence $f \in C(X)$. Conversely, let $f \in C(X)$, and $a \in \mathbb{R}$, then $(-\infty, a]$ is a closed set and hence a zero set in \mathbb{R} . Since the preimage of a zero set under a continuous map is a zero set, it follows that $f^{-1}((-\infty, a])$ is a zero set in X and therefore a member of \mathcal{A} . Thus f turns out to be a measurable function i.e. $f \in \mathcal{M}(X, \mathcal{A})$. Hence $\mathcal{M}(X, \mathcal{A}) = C(X)$ and $\mathcal{M}^*(X, \mathcal{A}) = C^*(X)$ immediately follows. \square

Theorem 5.1 contributes to Question 3.12 raised earlier, by clarifying that it is possible for there to be a free real maximal ideal of a ring of measurable functions, as the next example shows.

EXAMPLE 5.2. There is a P -space X that is not realcompact given in [7, Exercise 9L]. Then $C(X)$ has a free real maximal ideal. By Theorem 5.1, $C(X) = \mathcal{M}(X, Z[X])$, and hence $\mathcal{M}(X, Z[X])$ has a free real maximal ideal.

A question that naturally arises from Theorem 5.1 is as follows.

QUESTION 5.3. Given a (possibly infinite) σ -algebra \mathcal{A} on a set X , does there exist a P -space Y such that the ring or equivalently the lattice structure of $\mathcal{M}(X, \mathcal{A})$ and $C(Y)$ are isomorphic?

One possible candidate for the space Y is the set X , equipped with the weak topology induced by $\mathcal{M}(X, \mathcal{A})$; as the characteristic functions of measurable sets are in $\mathcal{M}(X, \mathcal{A})$, this topology is the smallest topology containing \mathcal{A} . But we show by way of a counterexample below that for such a choice of Y , even if Y is a P -space, it may happen that $\mathcal{M}(X, \mathcal{A})$ is not isomorphic to $C(Y)$.

EXAMPLE 5.4. Let $X = [0, 1]$ and \mathcal{A} be the σ -algebra of all Lebesgue measurable sets on $[0, 1]$. Let τ be the smallest topology on X that contains \mathcal{A} (equivalently, the weak topology on X induced by $\mathcal{M}(X, \mathcal{A})$). Since every one-point set is Lebesgue measurable, all of the singleton sets are open. Hence (X, τ) is the discrete topological space. Then $C(X, \tau)$ consists of *all* real-valued functions on $[0, 1]$, and is hence distinct from $\mathcal{M}(X, \mathcal{A})$.

To see that $C(X, \tau)$ and $\mathcal{M}(X, \mathcal{A})$ are not even isomorphic, we look at their lattice structure. Since Y is discrete, it is in particular extremally disconnected,

meaning that every zero-set has an open closure. Consequently by Stone-Nakano's theorem ([7, §3N6]), $C(Y)$ is a conditionally complete lattice in the sense that each nonempty subset of $C(Y)$, with an upper bound in $C(Y)$ has a supremum also lying in $C(Y)$. But we show that the lattice $\mathcal{M}(X, \mathcal{A})$ is not conditionally complete and hence $\mathcal{M}(X, \mathcal{A})$ and $C(Y)$ are not isomorphic as lattices, and consequently not isomorphic as rings. Indeed for each point s lying on a fixed non Lebesgue measurable set S in $[0, 1]$, let $\chi_{\{s\}}$ be the characteristic function of $\{s\}$. Surely $\{\chi_{\{s\}} : s \in S\}$ is a subfamily of $\mathcal{M}(X, \mathcal{A})$, which is bounded above in this ring by the constant function 1. However $\sup\{\chi_{\{s\}} : s \in S\}$ does not exist in $\mathcal{M}(X, \mathcal{A})$, an easy verification.

This example raises the following questions.

QUESTION 5.5. Under what conditions on a σ -algebra \mathcal{A} on X , is the ring $\mathcal{M}(X, \mathcal{A})$ isomorphic to the ring $C(X, \tau)$, where τ is the smallest topology on X containing \mathcal{A} ?

QUESTION 5.6. Under what conditions is the weak topology on X , induced by $\mathcal{M}(X, \mathcal{A})$ a P -space?

6. Acknowledgement

The authors would like to thank Professor Alan Dow for suggesting us the proof of Lemma 2.12.

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