

Generation of exactly solvable non-Hermitian potentials with real energies

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Abstract

A series of exactly solvable non-trivial complex potentials (possessing real spectra) are generated by applying the Darboux transformation to the excited eigenstates of a non-Hermitian potential $V(x)$. This method yields an infinite number of non-trivial partner potentials, defined over the whole real line, whose spectra are nearly exactly identical to the original potential.

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I. Introduction

Ever since it was conjectured that non-Hermitian Hamiltonians exhibiting symmetry under the combined transformation of *parity* ($\mathcal{P} : x \rightarrow -x$), and *time reversal* ($\mathcal{T} : i \rightarrow -i$) possess a real bound state spectrum [1], provided the eigenstates are also simultaneous eigenstates of \mathcal{PT} , such systems have been widely studied [2]. However, \mathcal{PT} invariance alone is neither necessary nor sufficient to ensure the reality of the spectrum. It is observed that eigenenergies are real for unbroken \mathcal{PT} symmetry, whereas they occur as complex conjugate pairs for the spontaneously broken case. The latter case corresponds to the case when the Hamiltonian respects \mathcal{PT} invariance, but eigenfunctions do not. Fairly recently it was shown that the existence of real or complex conjugate pairs of energy eigenvalues is attributed to the so-called *pseudo-Hermiticity* of these non-Hermitian Hamiltonians [3] :

$$\eta H \eta^{-1} = H^\dagger \quad (1)$$

where η is some Hermitian linear automorphism. The eigenstates corresponding to real eigenvalues are η -orthogonal and eigenstates corresponding to complex eigenvalues have zero η -norm. Several non-Hermitian Hamiltonians, whether possessing \mathcal{PT} invariance or not, have been identified as pseudo-Hermitian under $\eta = e^{-\theta p}$ and $e^{-\phi(x)}$. It is worth mentioning here that the usual norm, $\langle \Psi_n | \Psi_n \rangle$, in Hermiticity is positive definite, whereas for non-Hermitian Hamiltonians it is indefinite, being given by

$$\langle \Psi_m | \eta \Psi_n \rangle = \epsilon_n \delta_{m,n} \quad (2)$$

where $\epsilon = \pm 1$.

Efforts have always been made to find exactly solvable models for the one-dimensional Schrödinger equation. Extensive work has been done on this topic for Hermitian potentials. Similar work has been done for complex \mathcal{PT} symmetric potentials as well [4, 5, 6]. The aim of the present work is to generate a series of exactly solvable non-Hermitian Hamiltonians with real spectra, applying the Darboux transformation [7, 8, 9]. In the Hermitian case, the Darboux transformation is equivalent to supersymmetric quantum mechanics (SUSYQM). Also, the method works only when applied to the lowest state. For such Hamiltonians, the technique of SUSYQM was generalized to generate superpotentials by higher excited states [9]. The domain is split up and the partner potentials are defined on specific intervals, depending on the number of singularities. However, for complex potentials, things are different. Nevertheless, this straightforward method yields an infinite number of exactly solvable, non-trivial partner potentials, whose spectra are exactly identical except for the m^{th} state, provided one starts with the eigenfunction ψ_m , corresponding to the m^{th} real eigenvalue E_m . Moreover, since the new potentials so constructed, do not have any singularity on the real line, they are defined on the entire domain $(-\infty, +\infty)$. As explicit examples, a few non-trivial partners are constructed for two non-Hermitian potentials with real bound state spectra, viz.,

(i) the \mathcal{PT} symmetric oscillator

$$V(x) = (x - i\epsilon)^2 + \frac{\alpha^2 - \frac{1}{4}}{(x - i\epsilon)^2} \quad (3)$$

(ii) the \mathcal{PT} symmetric version of the generalised Ginocchio potential

$$V(x) = \frac{\gamma^4}{\gamma^2 + \sinh^2 u} \left[s(s+1) + 1 - \gamma^2 - \frac{5\gamma^2(1-\gamma^2)^2}{4(\gamma^2 + \sinh^2 u)^2} \right. \\ \left. - \frac{3(1-\gamma^2)(3\gamma^2-1)}{4(\gamma^2 + \sinh^2 u)} - \left(\alpha^2 - \frac{1}{4} \right) \coth^2 u \right] \quad (4)$$

which is an example of an implicit potential, u being a function of r (as explained in detail in Section IV later on).

In each case, the shift from the real axis to the complex plane ensures the removal of singularities from the real line.

II. Darboux Transformation

To make this work self contained we first give a brief review of the Darboux transformation [7]. Though it is applicable to any general differential equation, in this section we shall assume the potential to be real. We start with a particle moving in the potential $V_-(x)$ in the m^{th} state, (i.e., m is the quantum number equal to the number of nodes, of the m^{th} eigenfunction $\psi_m(x)$ of the starting potential $V_-(x)$). (It must be kept in mind that for complex potentials with real energies, m denotes the m^{th} energy level; it has nothing to do with the number of nodes of the eigenfunction, as there are none on the real line.) If the energy scale is adjusted so that the m^{th} energy eigenvalue is exactly zero ($E_m^{(-)} = 0$), then the Schrödinger equation reads

$$H_- \psi_m = \left(-\frac{d^2}{dx^2} + V_-(x) \right) \psi_m = 0 \quad (5)$$

where the Hamiltonian H_- is given by

$$H_- = -\frac{d^2}{dx^2} + V_-(x) \quad (6)$$

(The units used are $\hbar = 2m = 1$ for convenience).

Equation (5) has solution

$$V_-(x) = \frac{\psi_m''}{\psi_m} \quad (7)$$

which is regular everywhere, such that

$$H_- = \left(-\frac{d^2}{dx^2} + \frac{\psi_m''}{\psi_m} \right) \quad (8)$$

Thus if the general solution $\psi = \psi(x)$ of the Schrödinger equation [9]

$$\frac{d^2\psi}{dx^2} + [\epsilon - V_-(x)] \psi = 0 \quad (9)$$

is known for all values of ϵ , and for a particular value of $\epsilon = E_m$, the particular solution is ψ_m , then the general solution of the equation

$$\frac{d^2\phi}{dx^2} + [E - V_+(x)] \phi = 0 \quad (10)$$

with

$$V_+(x) = \psi_m(x) \frac{d^2}{dx^2} \left(\frac{1}{\psi_m(x)} \right) = 2 \left(\frac{\psi_m'}{\psi_m} \right)^2 - \left(\frac{\psi_m''}{\psi_m} \right) \quad (11)$$

$$E = \epsilon - E_m \quad (12)$$

for $E \neq 0$ is

$$\begin{aligned} \phi_n(x) &= \psi_m(x) \left\{ \frac{\psi_n(x)}{\psi_m(x)} \right\}' \\ &= \psi_n'(x) - \left(\frac{\psi_m'(x)}{\psi_m(x)} \right) \psi_n(x) \end{aligned} \quad (13)$$

Since for Hermitian Hamiltonians the Darboux transformation is equivalent to the intertwining method of SUSY, we shall seek a similar attempt for non-Hermitian Hamiltonians, by defining two intertwining operators A and B :

$$A = \frac{d}{dx} + W_m \quad (14)$$

$$B = -\frac{d}{dx} + W_m \quad (15)$$

where

$$W_m = -\frac{\psi'_m}{\psi_m} \quad (16)$$

Then

$$H_- = BA = \left(-\frac{d^2}{dx^2} + V_-(x) \right) \quad (17)$$

Now let us construct a partner Hamiltonian H_+ by

$$H_+ = AB = \left(-\frac{d^2}{dx^2} + V_+(x) \right) \quad (18)$$

where

$$V_+ = V_- - 2W_m \quad (19)$$

such that

$$V_{\pm} = W_m^2 \pm W'_m \quad (20)$$

Evidently if ψ_n is an eigenfunction of H_- with energy eigenvalue E_n^- , then $\phi_n = A\psi_n$ is also an eigenfunction of H_+ with the same energy eigenvalue E_n^- , for specific values of n .

$$H_+A\psi_n = (AB)A\psi_n = A(H_-\psi_n) = E_n^-(A\psi_n) \quad (21)$$

Thus, in case of Hermitian SUSY QM, the potentials V_+ and V_- are isospectral except for the m lowest states of V_- , for which there is no corresponding state of V_+ . Consequently, the ground state of V_+ is $E_0^+ = E_m^-$. All higher states have identical energies. Furthermore, W_m is the superpotential, and $B = A^\dagger$. V_{\pm} are called SUSY- m partner potentials. Since $\psi'_m \neq 0$ at x_j , W_m has singularities at the nodes x_j , $j = 1, 2, 3, \dots$ of ψ_m . For $m = 0$, the usual SUSY partners are defined on $(-\infty, +\infty)$, for $m = 1$, there are two separated potential wells, each of them on a semi-infinite domain, for $m = 2$ there is one infinite potential well on a finite domain between nodes x_1 and x_2 , and two binding potential wells on the two semi-infinite domains $(-\infty, x_1]$ and $[x_2, +\infty)$, and so on [9].

However, the scenario is different in case of non-Hermitian quantum mechanics. If there exists a linear, invertible Hermitian operator η such that [3]

$$B = A^\# = \eta^{-1}A^\dagger\eta \quad (22)$$

then the partner Hamiltonians can be written as

$$H_+ = AA^\# \quad H_- = A^\#A \quad (23)$$

and A and B are mutual pseudo-adjoints, so that V_{\pm} can be termed as pseudo-supersymmetric partners. Moreover, the new potentials are defined on the entire domain as the complex potentials have no singularities on the real axis. Thus, if one of the partner systems is exactly solvable, this SUSY induced formalism enables one to solve the other non-trivial partner as well.

It may be worth mentioning here that if V_{\pm} are isospectral, so are

$$v_{\pm} = V_{\pm} - \beta_m \quad (24)$$

where, β_m is some n -independent arbitrary constant (real or imaginary). The last expression ensures the eigenspectrum to be real under \mathcal{PT} invariance as explained below. If the eigen energies of the complex potential $V_{\pm}(x)$ are $E_n^{\pm} + \beta_m$, where E_n^{\pm} are some n -dependent real constants, and β_m are as defined above, then the eigen energies of $v_{\pm}(x)$ are E_n^{\pm} . Thus eq. (24) ensures the eigenspectrum to be real under \mathcal{PT} invariance, provided the complex potentials v_{\pm} are \mathcal{PT} symmetric.

III. \mathcal{PT} symmetric oscillator

We start with the widely studied \mathcal{PT} symmetric oscillator [10]

$$V(x) = z^2 + \frac{\alpha^2 - \frac{1}{4}}{z^2} \quad (25)$$

$$z = x - i\epsilon \quad (26)$$

which is known to possess the double set of eigenfunctions

$$\psi_{nq} = N_{nq} e^{-\frac{z^2}{2}} z^{-q\alpha + \frac{1}{2}} L_n^{(-q\alpha)}(z^2) \quad (27)$$

with eigenvalues

$$E_{nq} = 4n + 2 - 2q\alpha \quad (28)$$

where $L_n^{(\sigma)}$ are the associated Laguerre polynomials given by [11]

$$L_n^{(\sigma)}(y) = \frac{\Gamma(n + \sigma + 1)}{\Gamma(n + 1) \Gamma(\sigma + 1)} M(-n, \sigma + 1, y) \quad (29)$$

and $q = \pm 1$ is the *quasi-parity*.

Starting with the eigenfunction ψ_{mq} , corresponding to the m^{th} eigen value which is real, the pseudo-superpotential term W_{mq} is calculated to be

$$\begin{aligned} W_{mq} &= -\frac{\psi'_{mq}}{\psi_{mq}} \\ &= -z + \frac{(-q\alpha + 2m + 3/2)}{z} - \frac{2(m+1)}{z} \frac{L_{m+1}^{(-q\alpha)}(z^2)}{L_m^{(-q\alpha)}(z^2)} \end{aligned} \quad (30)$$

for $m = 0, 1, 2, \dots$, where prime denotes differentiation with respect to x .

The isospectral partner potentials for various values of m are obtained from the formula

$$v_{+q}^{(m)}(x) = W_{mq}^2 + W'_{mq} - \beta_{mq} \quad (31)$$

with

$$\beta_{mq} = 2q\alpha - 2(2m + 1) \quad (32)$$

The potential $v_{+q}^{(m)}(x)$ shares all the energy levels of the (energy shifted) \mathcal{PT} symmetric oscillator given in eq.(3), with $V(x) = v_-(x) = W_{mq}^2 - W'_{mq} - \beta_{mq}$, except for the m^{th} state of v_- which has no counterpart in v_+ . Using the explicit forms of the Laguerre polynomials [11], the partners for $m = 0, 1, 2$ are obtained below :

(i) $m = 0$

$$v_{+q}^{(0)} = z^2 + \frac{\alpha^2 - 2q\alpha + 3/4}{z^2} + 2 \quad (33)$$

This is analogous to the Hermitian case of a satellite potential.

(ii) $m = 1$

This gives the first non-trivial partner.

$$v_{+q}^{(1)} = z^2 + \frac{\alpha^2 - 2q\alpha + 3/4}{z^2} + \frac{4}{(-q\alpha + 1) - z^2} + \frac{8z^2}{\{(-q\alpha + 1) - z^2\}^2} + 2 \quad (34)$$

with ground state (GS) wave function

$$\phi_{0q} = \frac{2}{-q\alpha + 1 - z^2} e^{-\frac{z^2}{2}} z^{-q\alpha + \frac{3}{2}} \quad (35)$$

and GS energy

$$E_{0q} = 2 - 2q\alpha \quad (36)$$

It can be shown that because of normalization criterion, ϕ_{1q} has to be excluded from the spectrum. All other states ($n = 1, 2, 3 \dots$) can be obtained from (13) as

$$\phi_{(n+1)q}(x) = \frac{f_1 f'_{(n+1)} - f'_1 f_{(n+1)}}{f_1} \psi_0 \quad (37)$$

where f_n stands for $L_n^{-q\alpha}(z^2)$, and prime denotes differentiation with respect to x . Since the functions $L_n^{-q\alpha}(z^2)$ are well defined over the entire real line, singularities do not appear for $\phi_{nq}(x)$. Also, the eigenfunctions are well behaved and normalizable.

(iii) Similarly for $m = 2$

$$v_{+q}^{(2)} = z^2 + \frac{\alpha^2 - 2q\alpha + 3/4}{z^2} - \frac{4[3z^2 - (-q\alpha + 2)]}{L_2^{-q\alpha}(z^2)} + \frac{8z^2[z^2 - (-q\alpha + 2)]^2}{\{L_2^{-q\alpha}(z^2)\}} + 2 \quad (38)$$

As is observed, there are two partner potentials, characterized by the quasi-parity $q = \pm 1$, for each m value. Also, the \mathcal{PT} invariance of the partner potentials depends on the parameter α . For real α , the partners retain their \mathcal{PT} symmetry, whereas for imaginary α , we obtain new complex potentials with real energies. Moreover, since there are no singularities on the real axis, the new potentials so constructed are defined on the entire domain $(-\infty, +\infty)$. We have plotted the real and imaginary parts of the partner potential $v_{+q}^{(1)}$ (34) and the ground state wave function ϕ_{0q} (35) of the same in Fig. 1 and 2 respectively.

IV. The generalised Ginocchio potential

Our next attempt is to find the isospectral partners of the generalized Ginocchio potential given by [12]

$$V(r) = \frac{\gamma^4}{\gamma^2 + \sinh^2 u} \left[s(s+1) + 1 - \gamma^2 - \frac{5\gamma^2(1-\gamma^2)^2}{4(\gamma^2 + \sinh^2 u)^2} \right. \\ \left. - \frac{3(1-\gamma^2)(3\gamma^2-1)}{4(\gamma^2 + \sinh^2 u)} - \left(\alpha^2 - \frac{1}{4} \right) \coth^2 u \right] \quad (39)$$

This is an example of an implicit potential, as it is expressed in terms of a function $u(r)$ which is known only in the implicit form :

$$r = \frac{1}{\gamma^2} \left[\tanh^{-1} \left\{ (\gamma^2 + \sinh^2 u)^{-\frac{1}{2}} \sinh u \right\} \right. \\ \left. + (\gamma^2 - 1)^{\frac{1}{2}} \tan^{-1} \left\{ (\gamma^2 - 1)^{\frac{1}{2}} (\gamma^2 + \sinh^2 u)^{-\frac{1}{2}} \sinh u \right\} \right] \quad (40)$$

The monotonously increasing function $u(r)$ is the solution of the first order differential equation

$$\frac{du}{dr} = \frac{\gamma^2 \cosh u}{(\gamma^2 + \sinh^2 u)^{\frac{1}{2}}} \quad (41)$$

The bound state eigenfunctions are expressed in terms of the Jacobi polynomials as

$$\psi_0^{(n)}(r) = N_n (\gamma^2 + \sinh^2 u)^{\frac{1}{4}} (\sinh u)^{-q\alpha+1/2} (\cosh u)^{-\mu_n+q\alpha-1} f_n \quad (42)$$

where

$$f_n = P_n^{(\mu_n, -q\alpha)} (2 \tanh^2 u - 1) \quad (43)$$

are the Jacobi polynomials given by [11]

$$\mathcal{P}_n^{\alpha, \beta}(z) = \frac{\Gamma(n + \sigma + 1)}{\Gamma(n + 1) \Gamma(\alpha + 1)} F \left(-n, n + \alpha + \beta + 1, \alpha + 1, \frac{1 - z}{2} \right) \quad (44)$$

and N_n are normalization constants. The bound states are located at

$$E_n = -\gamma^4 \mu_n^2, \quad n = 0, 1, 2, \dots < \frac{1}{2} \left(s + q\alpha - \frac{1}{2} \right) \quad (45)$$

with

$$\mu_n = \frac{1}{\gamma^2} \left[- (2n - q\alpha + 1) + \left\{ (2n - q\alpha + 1)^2 (1 - \gamma^2) + \gamma^2 \left(s + \frac{1}{2} \right)^2 \right\}^{\frac{1}{2}} \right] \quad (46)$$

Analogous to the \mathcal{PT} symmetric oscillator, the presence of quasi-parity ($q = \pm 1$) gives rise to a double set of solutions.

Starting with the eigenfunction, ψ_{mq} , corresponding to real eigenvalue E_m , we obtain

$$\begin{aligned} W_{mq} &= -\frac{\psi'_{mq}}{\psi_{mq}} \\ &= -\frac{\gamma^2 \sinh u \cosh^2 u}{2(\gamma^2 + \sinh^2 u)^{\frac{3}{2}}} \\ &\quad + \frac{\gamma^2}{(\gamma^2 + \sinh^2 u)^{\frac{1}{2}}} \left\{ \left(\mu_m + \frac{1}{2} \right) \sinh u + \frac{(q\alpha - \frac{1}{2})}{\sinh u} \right\} - \frac{f'_m}{f_m} \end{aligned} \quad (47)$$

for $m = 0, 1, 2, \dots$, where prime denotes differentiation with respect to x .

Proceeding in a manner similar to that shown above, one can obtain new, exactly solvable, \mathcal{PT} symmetric potentials which share the same eigen energies as the generalised Ginocchio potential given in (39), with the help of

$$v_{+q}^{(m)}(x) = W_{mq}^2 + W'_{mq} - \beta_{mq} \quad (48)$$

where

$$\beta_{mq} = \gamma^4 \mu_m^2 \quad (49)$$

As an explicit example, we give the results for $m = 1$ only.

$$\begin{aligned} v_{+q}^{(1)}(x) &= -\frac{\gamma^4}{\gamma^2 + \sinh^2 u} \left\{ s(s + 1) + \gamma^2(1 - 2\mu_1) + 2q\alpha - 2 \right. \\ &\quad - \left(\alpha^2 - 2q\alpha + \frac{3}{4} \right) \coth^2 u + \frac{7}{4} \frac{\gamma^2(1 - \gamma^2)^2}{(\gamma^2 + \sinh^2 u)^2} \\ &\quad \left. - \frac{(1 - \gamma^2)}{\gamma^2 + \sinh^2 u} \left[\gamma^2 \left(2\mu_1 - \frac{11}{4} \right) + \frac{9}{4} - 2q\alpha \right] \right\} \\ &\quad - 4\gamma^4 \frac{(\mu_1 - q\alpha + 2)}{\gamma^2 + \sinh^2 u} + \frac{4\gamma^4}{(f_1)^2} \frac{(\mu_1 - q\alpha + 2)^2}{(\gamma^2 + \sinh^2 u)} \tanh^2 u \operatorname{sech}^2 u \end{aligned}$$

$$-\frac{4\gamma^4 (\mu_1 - q\alpha + 2)}{f_1 (\gamma^2 + \sinh^2 u)} \left[-2\tanh^2 u + \frac{\gamma^2}{\gamma^2 + \sinh^2 u} \right] \quad (50)$$

where

$$f_1 = q\alpha - 1 + (\mu_1 - q\alpha + 2) \tanh^2 u \quad (51)$$

Thus one obtains two isospectral partners for each m , because of the presence of quasi-parity. If α is real, then the new potentials thus formed are \mathcal{PT} symmetric. However, for α pure imaginary, we get non-Hermitian non- \mathcal{PT} invariant potentials with real bound state spectra. Since neither f_1 or $(\gamma^2 + \sinh^2 u)$ has any root on the real axis (u being complex), the new potential so constructed is defined on the entire domain $(-\infty, +\infty)$, in this case as well.

V. Conclusions

To conclude, starting with the eigenfunction ψ_m corresponding to the m^{th} real eigenvalue E_m , of the potential $v_-(x)$, new, exactly solvable, non-trivial partners have been constructed for two \mathcal{PT} symmetric potentials, viz.,

- (i) the \mathcal{PT} symmetric oscillator,
 - (ii) the \mathcal{PT} symmetric version of the generalised Ginocchio potential
- the partners being related to the original potential $v_-(x)$ by

$$v_{\pm} = W_m^2 \pm W_m - \beta_m$$

where β_m is some n -independent constant and

$$W_m = -\frac{\psi'_m}{\psi_m}$$

Though $m = 0$ case gives the usual shape-invariant form, higher values of m generate new examples of non-shape invariant partners, as they do not obey the shape-invariance condition [13], which states that if the profiles of V_{\pm} are such that they satisfy the relationship

$$V_-(x, a_0) = V_+(x, a_1) + R(a_1) \quad (52)$$

where a_1 is some function of a_0 (say, $a_1 = f(a_0)$), then only the potentials V_{\pm} are termed as shape-invariant. However, the new exactly solvable potentials constructed in this work do not fall in this category. Moreover, the new potentials admit all the energies of the original potential except for the m^{th} state, which is missing in the partner potentials, unlike the absence of the lowest m states in the Hermitian case. Furthermore, contrary to the Hermitian case where the partners are defined on specific disjoint intervals depending on the level m , in this case the partners are defined on the entire domain $(-\infty, +\infty)$, as there are no singularities on the real line.

It will be interesting to investigate the effect of spontaneous \mathcal{PT} symmetry breaking on such partners. We propose to take up this study in the recent future.

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Figure Captions

Fig. 1 : Graph of the real (dotted curve) and imaginary (solid curve) parts of the potential $v_{+q}^{(1)}$ given in eq.(34) for $\alpha = 3/4$, $q = +1$, $\epsilon = 1$.

Fig. 2 : Graph of the real (dotted curve) and imaginary (solid curve) parts of the wave function ϕ_{0q} given in eq.(35) for $\alpha = 3/4$, $q = +1$, $\epsilon = 1$.