

ON GENERALIZED QUASI-EINSTEIN GRW SPACE-TIMES

UDAY CHAND DE AND SAMEH SHENAWY

ABSTRACT. Recently, it is proven that generalized Robertson-Walker space-times in all orthogonal subspaces of Gray's decomposition but one(unrestricted) are perfect fluid space-times. GRW space-times in the unrestricted subspace are identified by having constant scalar curvature. Generalized quasi-Einstein GRW space-times have a constant scalar curvature. It is shown that generalized quasi-Einstein GRW space-times reduce to Einstein space-times or perfect fluid space-times.

1. INTRODUCTION

The warped product $M = I \times_f M^*$ of an open connected interval $(I, -dt^2)$ of \mathbb{R} and a Riemannian manifold M^* with warping function $f : I \rightarrow \mathbb{R}^+$ is called a generalized Robertson-Walker space-time(or GRW space-times) [12, 15]. This family of Lorentzian space-times broadly extends the classical Robertson-Walker space-times, Friedmann cosmological models, Einstein-de Sitter space-times and many others [2, 15]. The classical Robertson-Walker spacetime is regarded as cosmological models since it is spatially homogeneous and spatially isotropic whereas GRW space-times serve as inhomogeneous extension of Robertson-Walker space-times that admit an isotropic radiation [2](see also [4, 15]). A Lorentzian manifold is called a perfect fluid space-time if the Ricci tensor Ric takes the form

$$\text{Ric}(X, Y) = ag(X, Y) + bA(X)A(Y)$$

where a, b are scalars and A is a 1-form metrically equivalent to a unit time-like vector field [13, 14]. Perfect fluid space-times in the language of differential geometry are called quasi-Einstein spaces where A is metrically equivalent to a unit space-like vector field. Recently, in [14], it is proven that a perfect fluid space-time with divergence-free conformal curvature tensor is a GRW space-time with Einstein fibers given that the scalar curvature is constant. Many sufficient conditions on perfect fluid space-times to be a GRW space-time are derived.

Gray presented an invariant orthogonal decomposition of the covariant derivative of the Ricci tensor [5](see also [10]). Recently, Carlo Mantica et al proved that the Ricci tensor of a generalized Robertson-Walker space-time in all classes of Gray's decomposition but $\mathcal{A} \oplus \mathcal{B}$ is either Einstein or takes the form of a perfect fluid whereas $\mathcal{A} \oplus \mathcal{B}$ is not restricted [13]. The class $\mathcal{A} \oplus \mathcal{B}$ is characterized by $\nabla R = 0$ i.e. the scalar curvature is constant. Now, the following question arises.

Does the Ricci tensor of all GRW space-times in $\mathcal{A} \oplus \mathcal{B}$ reduce to be Einstein or take the form of a perfect fluid?

2000 *Mathematics Subject Classification.* Primary 53C25; Secondary 83F05.

Key words and phrases. Quasi-Einstein manifolds, Perfect fluid space-times, Generalized Robertson-Walker space-times, Einstein manifolds.

In this work, we get a partial positive answer. A (pseudo-)Riemannian manifold (M, g) is called a generalized quasi-Einstein manifold if its Ricci curvature satisfies

$$(1.1) \quad \text{Ric}(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)],$$

where α, β and γ are non-zero constants, A and B are 1-forms corresponding to two orthonormal vector field [1, 3, 6–9]. If $\gamma = 0$, then M reduces to a quasi-Einstein manifold. It is clear that generalized quasi-Einstein space-times are generally imperfect fluid space-times with constant scalar curvature $R = n\alpha + \beta$. However, we prove that generalized quasi-Einstein GRW space-times are either Einstein or perfect fluid space-times.

Remark 1. *It is noted that any vector field orthogonal to a time-like vector field is space-like. Thus the generators couldn't be time-like. Now, one may assume that one of the generators is time-like and the other is space-like. Generally, the results of this article still hold in this case with minor changes.*

2. NOTES ON GENERALIZED QUASI-EINSTEIN MANIFOLDS

Let M be a generalized quasi-Einstein (pseudo-)Riemannian manifold i.e.

$$(2.1) \quad R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma(A_i B_j + A_j B_i),$$

where $A_i A^i = B_j B^j = 1$ and $A_i B^i = 0$. The trace of this equations gives

$$R = n\alpha + \beta.$$

It is noted that

$$\begin{aligned} A^i A^j R_{ij} &= \alpha + \beta, \\ A^i B^j R_{ij} &= \gamma, \\ B^i B^j R_{ij} &= \alpha, \end{aligned}$$

and consequently $\beta = (A^i A^j - B^i B^j) R_{ij}$. Hence we can state the following result.

Proposition 1. *Let M be a generalized quasi-Einstein manifold with generators A and B . Then the scalar curvature is*

$$R = (n - 1) B^i B^j R_{ij} + A^i A^j R_{ij}.$$

Now assume that A is an eigenvector of the Ricci tensor with eigenvalue ξ i.e. $A^i R_{ij} = \xi A_j$. A contraction of Equation (2.1) with A^i yields

$$A^i R_{ij} = \alpha A^i g_{ij} + \beta A^i A_i A_j + \gamma(A^i A_i B_j + A^i A_j B_i)$$

implies $(\xi - \alpha - \beta) A_j = \gamma B_j$. Thus $\gamma = 0$ and $\xi = \alpha + \beta$. If B is an eigenvector of the Ricci tensor with eigenvalue ϕ , then

$$B^i R_{ij} = \alpha B^i g_{ij} + \beta B^i A_i A_j + \gamma(B^i A_i B_j + B^i A_j B_i)$$

infers

$$(\phi - \alpha) B_j = \gamma A_j.$$

Consequently, $\phi = \alpha$ and $\gamma = 0$. Conversely, assume that M is a quasi-Einstein manifold with generator A . Then

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j$$

yields

$$A^i R_{ij} = \alpha A^i g_{ij} + \beta A^i A_i A_j = (\alpha + \beta) A_j.$$

Also,

$$B^i R_{ij} = \alpha B^i g_{ij} + \beta B^i A_i A_j = \alpha A_j.$$

This leads to the following.

Theorem 1. *Let M be a generalized quasi-Einstein manifold. Then, M reduces to a quasi-Einstein manifold if and only if one of the generators is an eigenvector of the Ricci tensor.*

The covariant derivative of the Ricci tensor of a generalized quasi-Einstein manifold is given by

$$\begin{aligned} \nabla_k R_{ij} &= \beta (\nabla_k A_i) A_j + \beta A_i (\nabla_k A_j) \\ &\quad + \gamma [(\nabla_k A_i) B_j + A_i (\nabla_k B_j) + (\nabla_k A_j) B_i + A_j (\nabla_k B_i)] \end{aligned}$$

and hence

$$\begin{aligned} \nabla_i R_{kj} &= \beta (\nabla_i A_k) A_j + \beta A_k (\nabla_i A_j) \\ &\quad + \gamma [(\nabla_i A_k) B_j + A_k (\nabla_i B_j) + (\nabla_i A_j) B_k + A_j (\nabla_i B_k)] \end{aligned}$$

Thus, the Codazzi deviation tensor \mathcal{D} is

$$\begin{aligned} \mathcal{D}_{kij} &= \nabla_k R_{ij} - \nabla_i R_{kj} \\ &= \beta (\nabla_k A_i) A_j + \beta A_i (\nabla_k A_j) - \beta (\nabla_i A_k) A_j - \beta A_k (\nabla_i A_j) \\ &\quad + \gamma [(\nabla_k A_i) B_j + A_i (\nabla_k B_j) + (\nabla_k A_j) B_i + A_j (\nabla_k B_i)] \\ &\quad - \gamma [(\nabla_i A_k) B_j + A_k (\nabla_i B_j) + (\nabla_i A_j) B_k + A_j (\nabla_i B_k)] \end{aligned}$$

Now, we have the following cases

$$\begin{aligned} A^j \mathcal{D}_{kij} &= \beta (\nabla_k A_i - \nabla_i A_k) + \gamma (\nabla_k B_i - \gamma \nabla_i B_k) \\ B^j \mathcal{D}_{kij} &= \nabla_k A_i - \nabla_i A_k \end{aligned}$$

Thus, for a Codazzi Ricci tensor, the generators are both closed. In this case,

$$\begin{aligned} 0 &= \mathcal{D}_{kij} \\ &= \beta A_i (\nabla_k A_j) - \beta A_k (\nabla_i A_j) + \gamma A_i (\nabla_k B_j) \\ &\quad + \gamma (\nabla_k A_j) B_i - \gamma A_k (\nabla_i B_j) - \gamma (\nabla_i A_j) B_k \\ &= \nabla_j [\beta A_i A_k + \gamma A_i B_k + \gamma A_k B_i] \\ &\quad - 2\beta A_k (\nabla_j A_i) - 2\gamma (\nabla_j A_i) B_k - 2\gamma A_k (\nabla_j B_i) \\ &= \nabla_j R_{ik} - 2(\beta A_k + \gamma B_k) \nabla_j A_i - 2\gamma A_k \nabla_j B_i. \end{aligned}$$

Thus we have the following.

Proposition 2. *Let M be a generalized quasi-Einstein manifold. Assume that M is Einstein-like of class \mathcal{B} (i.e. the Ricci tensor is a Codazzi tensor). Then A and B are closed. Moreover,*

$$\nabla_j R_{ik} = 2(\beta A_k + \gamma B_k) \nabla_j A_i + 2\gamma A_k \nabla_j B_i.$$

A contraction of \mathcal{D}_{kij} by g^{ij} and then by the generators A^k and B^k infers

$$\begin{aligned} 0 &= (\beta A^i + \gamma B^i) \nabla_i A_k + (\beta A_k + \gamma B_k) (\nabla_i A^i) + \gamma A_k (\nabla_i B^i) + \gamma A^i \nabla_i B_k, \\ 0 &= \beta (\nabla_i A^i) + \gamma \nabla_i B^i, \\ 0 &= \gamma \nabla_i A^i. \end{aligned}$$

Thus $\nabla_i A^i = \nabla_i B^i = 0$.

Assume that M is Einstein-like of class \mathcal{P} (that is, the Ricci tensor is a parallel, $\nabla_k R_{ij} = 0$). Then,

$$\begin{aligned} 0 &= \beta (\nabla_i A_k) A_j + \beta A_k (\nabla_i A_j) + \gamma (\nabla_i A_k) B_j \\ &\quad + \gamma A_k (\nabla_i B_j) + \gamma (\nabla_i A_j) B_k + \gamma A_j (\nabla_i B_k) \end{aligned}$$

Contractions by A^k and B^k imply

$$\begin{aligned} 0 &= \beta \nabla_i A_j + \gamma \nabla_i B_j, \\ 0 &= \gamma \nabla_i A_j. \end{aligned}$$

Assume that A is not parallel, then $\beta = \gamma = 0$. Thus we conclude.

Theorem 2. *Let M be a Ricci-symmetric generalized quasi-Einstein manifold. Then, M is Einstein if the generator A is not covariantly constant.*

3. GENERALIZED QUASI-EINSTEIN GRW SPACE-TIMES

A Lorentzian manifold M is a GRW space-time if and only if M has a unit time-like vector field u_i such that

$$\nabla_k u_j = \varphi (g_{kj} + u_k u_j),$$

which is also an eigenvector of the Ricci tensor i.e. $R_{ij} u^i = \xi u_j$ for some scalar functions φ and ξ [11–13]. We say that u is a nontrivial torse-forming vector field if $\varphi \neq 0$. This characterization is an alternative of Chen's theorem in [2]. If M is a generalized quasi-Einstein manifold, then

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma (A_i B_j + A_j B_i).$$

A contraction by u^i yields

$$u^i R_{ij} = \alpha u^i g_{ij} + \beta u^i A_i A_j + \gamma (u^i A_i B_j + A_j u^i B_i)$$

which implies

$$(\xi - \alpha) u_j = \beta (u^i A_i) A_j + \gamma (u^i A_i) B_j + \gamma A_j (u^i B_i)$$

and hence

$$(\xi - \alpha) u_j = [\beta (u^i A_i) + \gamma (u^i B_i)] A_j + \gamma (u^i A_i) B_j$$

Two different contractions by the generators give

$$(\xi - \alpha - \beta) (u^i A_i) - \gamma (u^i B_i) = 0$$

and

$$(\xi - \alpha) (u^i B_i) - \gamma (u^i A_i) = 0.$$

Thus

$$\begin{aligned} (\xi - \alpha) u_j &= [\beta (u^i A_i) + (\xi - \alpha - \beta) (u^i A_i)] A_j + (\xi - \alpha) (u^i B_i) B_j \\ &= (\xi - \alpha) (u^i A_i) A_j + (\xi - \alpha) (u^i B_i) B_j \end{aligned}$$

and hence

$$(3.1) \quad (\xi - \alpha) [u_j - (u^i A_i) A_j - (u^i B_i) B_j] = 0.$$

It is clear that u_j is not a linear combination of A_j and B_j only since u^i is time-like whereas A^i and B^i are orthonormal space-like fields so $\xi = \alpha$. Therefore,

$$(3.2) \quad \beta (u^i A_i) + \gamma (u^i B_i) = 0$$

$$(3.3) \quad \gamma (u^i A_i) = 0$$

It is noted that $\gamma = \beta = 0$ if $(u^i A_i)$ is not zero. Suppose that $(u^i B_i)$ does not vanish. Then Equation (3.3) implies that either $\gamma = 0$ or $(u^i A_i) = 0$. The later case with Equation (3.2) yield $\gamma = 0$ i.e. M is quasi-Einstein if $(u^i B_i) \neq 0$. Now, assume that u^i is orthogonal to both the generators i.e. $(u^i A_i) = (u^i B_i) = 0$. The Ricci tensor of a GQE manifold is

$$R_{ij} = \xi g_{ij} + \beta A_i A_j + \gamma (A_i B_j + A_j B_i)$$

and so

$$\nabla_k R_{ij} = \beta A_j \nabla_k A_i + \beta A_i \nabla_k A_j + \gamma (B_j \nabla_k A_i + A_i \nabla_k B_j + A_j \nabla_k B_i + B_i \nabla_k A_j)$$

A contraction by u^i implies

$$\begin{aligned} u^i \nabla_k R_{ij} &= 0 \\ \nabla_k (u^i R_{ij}) - R_{ij} \nabla_k u^i &= 0 \end{aligned}$$

It is noted that u^i is an eigenvector of the Ricci tensor (i.e. $u^i R_{ij} = \xi u_j$) and $\nabla_k u^i = \varphi (\delta_k^i + u_k u^i)$. Thus

$$\begin{aligned} \nabla_k (\xi u_j) - R_{ij} \varphi (\delta_k^i + u_k u^i) &= 0 \\ \xi \nabla_k u_j - \varphi \delta_k^i R_{ij} - \varphi u_k u^i R_{ij} &= 0 \\ \xi \varphi (g_{kj} + u_k u_j) - \varphi R_{kj} - \varphi \xi u_k u_j &= 0 \\ \varphi (\xi g_{kj} - R_{kj}) &= 0 \end{aligned}$$

So M is Einstein if u is a nontrivial torse-forming vector field.

Theorem 3. *Let M be a generalized quasi-Einstein GRW space-time. Then $u^i R_{ij} = \alpha u_j$ i.e. α is the eigenvalue of the eigenvector u^i and*

- (1) *M reduces to be Einstein space-time if u^i is orthogonal to both the generators provided $\varphi \neq 0$.*
- (2) *M reduces to be Einstein space-time if u^i is not orthogonal to first generator.*
- (3) *M reduces to be perfect fluid space-time if u^i is not orthogonal to the second generator.*

Corollary 1. *Let M be a generalized quasi-Einstein Lorentzian manifold admitting a unit time-like non-trivial torse-forming vector field. Then M reduces to an Einstein GRW space-time or a perfect fluid GRW space-time.*

REFERENCES

- [1] Chaki, M. C. *On generalized quasi Einstein manifolds*, Publ. Math. Debrecen 58 (2001): 683-691.
- [2] Chen, Bang-Yen. *A simple characterization of generalized Robertson-Walker spacetimes*, General Relativity and Gravitation 46, no. 12 (2014): 1833.
- [3] De, Avik, Ahmet Yildiz, and Uday Chand De. *On Generalized Quasi Einstein Manifolds*, Filomat 28, no. 4 (2014): 811-820.
- [4] Ehlers, Jurgen, P. Geren, and Rainer K. Sachs. "Isotropic Solutions of the Einstein-Liouville Equations." Journal of Mathematical Physics 9, no. 9 (1968): 1344-1349.
- [5] Gray, Alfred. "Einstein-like manifolds which are not Einstein." Geometriae dedicata 7, no. 3 (1978): 259-280.
- [6] Guler, Sinem, and Sezgin Altay Demirbag. *On Ricci symmetric generalized quasi Einstein spacetimes*, Miskolc Mathematical Notes 16, no. 2 (2015): 853-868.
- [7] Guler, Sinem, and Sezgin Altay Demirbag. *A Study of Generalized Quasi Einstein Spacetimes with Applications in General Relativity*, International Journal of Theoretical Physics 55, no. 1 (2016): 548-562.

- [8] Guler, Sinem, and Sezgin Altay Demirbag. *Riemannian Manifolds Satisfying Certain Conditions on Pseudo-Projective Curvature Tensor*, Filomat 30, no. 3 (2016): 721-731.
- [9] Mallick, Sahanous, and Uday Chand De. *On a Class of Generalized quasi-Einstein Manifolds with Applications to Relativity*, Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica 55, no. 2 (2016): 111-127.
- [10] Mantica, Carlo Alberto, and Luca Guido Molinari. *Riemann compatible tensors*, In Colloquium Mathematicum, vol. 128, pp. 197-210. Instytut Matematyczny Polskiej Akademii Nauk, 2012.
- [11] Mantica, Carlo Alberto, and Luca Guido Molinari. *On the Weyl and Ricci tensors of Generalized Robertson-Walker space-times*, Journal of Mathematical Physics 57, no. 10 (2016): 102502.
- [12] Mantica, Carlo Alberto, and Luca Guido Molinari. *Generalized Robertson-Walker spacetimes—a survey*, International Journal of Geometric Methods in Modern Physics 14, no. 03 (2017): 1730001.
- [13] Mantica, Carlo Alberto, Luca Guido Molinari, Young Jin Suh, and Sameh Shenawy. *Perfect-Fluid, Generalized Robertson-Walker Space-times, and Gray's Decomposition*, Journal of Mathematical Physics, revised.
- [14] Mantica, Carlo Alberto, Uday Chand De, Young Jin Suh, and Luca Guido Molinari. "Perfect fluid spacetimes with harmonic generalized curvature tensor." Osaka Journal of Mathematics 56, no. 1 (2019): 173-182.
- [15] Sanchez, Miguel. "On the geometry of generalized Robertson-Walker spacetimes: geodesics." General Relativity and Gravitation 30, no. 6 (1998): 915-932.

(U. C. De) DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF CALCUTTA, 35, BALLYGAUNGE CIRCULAR ROAD, KOLKATA 700019, WEST BENGAL, INDIA,
E-mail address: uc.de@yahoo.com

(S. Shenawy) BASIC SCIENCE DEPARTMENT, MODERN ACADEMY FOR ENGINEERING AND TECHNOLOGY, MAADI, EGYPT,
E-mail address: drshenawy@mail.com