External leg amputation in conformal invariant three-point function

Indrajit Mitra*

Department of Theoretical Physics,
Indian Association for the Cultivation of Science,
2A & 2B, Raja S. C. Mullick Road, Kolkata 700032, India
and
Department of Physics, University of Calcutta,
92 A.P.C. Road, Kolkata 700009, India†

Abstract

Amputation of external legs is carried out explicitly for the conformal invariant three-point function involving two spinors and one vector field. Our results are consistent with the general result that amputing an external leg in a conformal invariant Green function replaces a field by its conformal partner in the Green function. A new star-triangle relation, involving two spinors and one vector field, is derived and used for the calculation.

Keywords: Conformal invariant Green function; amputation; star-triangle relation
PACS number: 11.10.-z

1 Introduction

This work is concerned with the amputation of external legs in conformal invariant Green functions in Euclidean space with general number of dimensions. Various aspects of CFT (conformal field theory) in D dimensions have been reviewed in Refs. [1], [2] and [3]. We consider conformal invariant Green functions involving spinors and vector fields, which are relevant for the infrared limit of massless QED_3 [4, 5], and for conformal QED_4 [6, 7]. Some other areas which use D-dimensional CFT's with fields of non-zero spin are $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, conformal windows of gauge theories, and unparticle physics [8, 9].

A conformal invariant Green function includes the external legs, but amputed Green functions are easier to calculate, because we do not have to integrate (in position space)

*E-mail: indrajit.mitra@saha.ac.in, imphys@caluniv.ac.in
†Present address

over the extra vertices. This provides a motivation for studying amputation. Moreover, the conformal partial wave expansion [1, 2, 3, 10, 11] involves amputed Green function. This expansion expresses the contribution of the various quasi-primary fields to the product of two field operators at arbitrary separation. From this, one can find the contributions of the quasi-primary fields to the four-point function. A recent work which uses the conformal partial wave expansion and the amputed three-point function is Ref. [9].

Formally, amputation of an external leg in a Green function in D-dimensional CFT replaces a field of scale dimension d by its conformal partner, having scale dimension $D-d$ [1, 2, 3, 10, 11]. However, only an explicit calculation can determine the coefficient which comes with the amputed Green function. The simplest calculation is that for scalar field theory. Compared to this, the case of massless Yukawa theory [1] is more involved, as there are more than one invariant structures for a given Green function. But we find that amputation in the case involving spinors and vector field is much more complicated. For amputing one of the two invariant structures in this case, we have to derive and make use of a new star-triangle relation.

Indeed, the techniques of calculation developed in this paper can be useful in other areas which involve evaluation of massless Feynman integrals, like $\mathcal{N}=4$ Yang-Mills theory [12]. The star-triangle relation involving scalar fields [13, 14] (referred to as the D'EPP formula in Ref. [9]) has wide-ranging applications: see Ref. [15] and references therein. It is expected that the analogous relation, involving two spinors and one vector field, as derived by us, will also find its applications.

The paper is organized as follows. In Sec. 2 we introduce amputed Green function in CFT through the example of massless scalar field theory. In Sec. 3, we introduce amputation of spinor leg through massless Yukawa field theory. In Sec. 4, we give the structures $C_{1\mu}$ and $C_{2\mu}$ of the conformal spinor-spinor-vector Green function and state how spinor leg amputation for these structures turns out to be different from the Yukawa case. The star-triangle relation with two spinors and one vector field is derived in Sec. 5. Spinor leg amputation of $C_{1\mu}$ and $C_{2\mu}$ is carried out in Secs. 6 and 7. Vector leg amputation of $C_{1\mu}$ and $C_{2\mu}$ is carried out in Sec. 8. In Sec. 9, checks on these results are performed. In Sec. 10, we present our conclusions.

2 Amputation in scalar field theory

In this section, we explain the aim of our work by reviewing the simplest example of scalar field theory. The two-point function and its inverse for a conformal scalar of scale dimension d are given by

$$G_d(x_{12}) = \frac{N}{(x_{12}^2)^d}\,, \quad G_d^{-1}(x_{12}) = \frac{\pi^{-D}}{N}\,\frac{\Gamma(d)\Gamma(D-d)}{\Gamma(D/2-d)\Gamma(d-D/2)}\,\frac{1}{(x_{12}^2)^{D-d}} \tag{1}$$

where $x_{ab} \equiv x_a - x_b$ and N is an arbitrary constant. Together they satisfy

$$\int d^D x_2 \; G_d(x_{12}) G_d^{-1}(x_{23}) = \int d^D x_2 \; G_d^{-1}(x_{12}) G_d(x_{23}) = \delta^{(D)}(x_{13})\,. \tag{2}$$

Eq. (1) shows that G_d^{-1} is the two-point function of a scalar field of scale dimension $D-d$. In Appendix A, we indicate how to arrive at G_d^{-1} from G_d. A field of the same spin but of scale dimension $D-d$ is called the *conformal partner* [3] or *shadow operator* [10] of the field of scale dimension d. Both these fields have the same set of values for the Casimir operators of the conformal group.

Consider next a three-point function $\langle\phi_d(x_1)\phi_l(x_2)\phi_\Delta(x_3)\rangle$ of three scalar fields of scale dimensions d, l and Δ. The three-point function with the ϕ_d-leg amputated is defined by

$$\langle\phi_d(x_1)\phi_l(x_2)\phi_\Delta(x_3)\rangle \equiv \int d^Dy\; G_d(x_1-y)\langle\tilde{\phi}_d(y)\phi_l(x_2)\phi_\Delta(x_3)\rangle\,, \tag{3}$$

with ˜ on a field denoting amputation. Using Eq. (2), this definition can also be written as

$$\int d^Dx_1\; G_d^{-1}(x_{41})\langle\phi_d(x_1)\phi_l(x_2)\phi_\Delta(x_3)\rangle \equiv \langle\tilde{\phi}_d(x_4)\phi_l(x_2)\phi_\Delta(x_3)\rangle\,. \tag{4}$$

Next, using the conformal transformation properties of the left-hand side of Eq. (4), it can be shown that [3] *the amputed three-point function is again a three-point function but with* ϕ_d *replaced by its conformal partner*. [In Appendix B of the present work, we demonstrate this result for spinor and vector field.] Thus,

$$\int d^Dx_1\; G_d^{-1}(x_{41})\langle\phi_d(x_1)\phi_l(x_2)\phi_\Delta(x_3)\rangle \sim \langle\phi_{D-d}(x_4)\phi_l(x_2)\phi_\Delta(x_3)\rangle\,, \tag{5}$$

where $\sim$ means upto some coefficient. Now, the structure of $\langle\phi_d(x_1)\phi_l(x_2)\phi_\Delta(x_3)\rangle$ is known in CFT: it is given by

$$C^{d,l,\Delta}(x_1x_2x_3) = \frac{1}{x_{12}^{d+l-\Delta}x_{13}^{d+\Delta-l}x_{23}^{l+\Delta-d}}\,. \tag{6}$$

The non-trivial part in determining the coefficient on the right-hand side of Eq. (5) is therefore the evaluation of the integral $\int d^Dx_1\;(x_{14}^2)^{-(D-d)}C^{d,l,\Delta}(x_1x_2x_3)$ occuring on the left-hand side. This can be done by using the star-triangle relation of Eq. (71). We then find that

$$\begin{aligned}\int d^Dx_1\; G_d^{-1}(x_{41})\,C^{d,l,\Delta}(x_1x_2x_3) \;&=\; \frac{\pi^{-D/2}}{N}\,\frac{\Gamma(d)\Gamma(\frac{D-d-l+\Delta}{2})\Gamma(\frac{D-d-\Delta+l}{2})}{\Gamma(\frac{D}{2}-d)\Gamma(\frac{d+l-\Delta}{2})\Gamma(\frac{d+\Delta-l}{2})}\\ &\quad\times C^{D-d,l,\Delta}(x_4x_2x_3)\,,\end{aligned} \tag{7}$$

where G_d^{-1} is given in Eq. (1). Thus the result given in Eq. (5) is explicitly realized in Eq. (7). [Let us note that that Eq. (2.11) of Ref. [9] can be reproduced from our Eq. (7) by relabelling the scale dimensions and the coordinates appropriately.] *The aim of the present work is to derive similar amputation equations for the spinor-spinor-vector Green function* which is relevant to QED.

3 Amputation of spinor leg

The fermion two-point function $\langle\psi_d(x_1)\bar{\psi}_d(x_2)$ and its inverse in CFT are given by

$$\begin{aligned}S_d(x_{12}) \;&=\; N\frac{\not{x}_{12}}{(x_{12}^2)^{d+1/2}}\,,\\ S_d^{-1}(x_{12}) \;&=\; -\frac{\pi^{-D}}{N}\,\frac{\Gamma(d+1/2)\Gamma(D-d+1/2)}{\Gamma(D/2-d+1/2)\Gamma(d-D/2+1/2)}\,\frac{\not{x}_{12}}{(x_{12}^2)^{D-d+1/2}}\end{aligned} \tag{8}$$

which satisfy Eq. (2) with G_d replaced by S_d (see Appendix A). Here N is again an arbitrary constant. It will be instructive to first consider the Yukawa ($\bar{\psi}\gamma_5\psi\phi$) theory ($D$ is even and $\gamma_5 = i^{D/2}\gamma_1\gamma_2\cdots\gamma_D$). There are two conformal-invariant structures [3] for $\langle\psi_d(x_1)\bar{\psi}_l(x_2)\phi_\Delta(x_3)\rangle$:

$$C_+^{d,l,\Delta}(x_1x_2x_3) = \frac{\not{x}_{13}}{x_{13}^{d-l+\Delta+1}}\gamma_5\frac{\not{x}_{32}}{x_{23}^{l-d+\Delta+1}}\frac{1}{x_{12}^{l+d-\Delta}}\,, \tag{9}$$

$$C_-^{d,l,\Delta}(x_1x_2x_3) = \frac{\not{x}_{12}}{x_{12}^{l+d-\Delta+1}}\gamma_5\frac{1}{x_{13}^{d-l+\Delta}}\frac{1}{x_{23}^{l-d+\Delta}} \tag{10}$$

with $\gamma_5 C_\pm \gamma_5 = \pm C_\pm$. Corresponding to Eq. (7), we now have

$$\int d^Dx_1\ S_d^{-1}(x_{41})\,C_\pm^{d,l,\Delta}(x_1x_2x_3) = K_\pm C_\mp^{D-d,l,\Delta}(x_4x_2x_3) \tag{11}$$

$$\int d^Dx_2\ C_\pm^{d,l,\Delta}(x_1x_2x_3)S_l^{-1}(x_{25}) = K'_\pm C_\mp^{d,D-l,\Delta}(x_1x_5x_3) \tag{12}$$

the integrals being evaluated by using the star-triangle relation for the Yukawa theory given in Eq. (73). Here S_d^{-1} and S_l^{-1} are as given in Eq. (8), and

$$\begin{aligned} K_+ &= -\frac{\pi^{-D/2}}{N}\frac{\Gamma(d+\frac{1}{2})\Gamma(\frac{D-d+l-\Delta+1}{2})\Gamma(\frac{D-d-l+\Delta}{2})}{\Gamma(\frac{D}{2}-d+\frac{1}{2})\Gamma(\frac{d-l+\Delta+1}{2})\Gamma(\frac{d+l-\Delta}{2})}\,, \\ K_- &= \frac{\pi^{-D/2}}{N}\frac{\Gamma(d+\frac{1}{2})\Gamma(\frac{D-d+l-\Delta}{2})\Gamma(\frac{D-d-l+\Delta+1}{2})}{\Gamma(\frac{D}{2}-d+\frac{1}{2})\Gamma(\frac{d-l+\Delta}{2})\Gamma(\frac{d+l-\Delta+1}{2})}\,, \\ K'_+ &= -K_+|_{d\leftrightarrow l}\,, \quad K'_- = -K_-|_{d\leftrightarrow l}\,. \end{aligned} \tag{13}$$

For the case $D = 4$, these results are given in a different form in Appendix 6 of Ref. [1]. It may be noted that amputation again replaces d by $D-d$ (or l by $D-l$) in Eqs. (11) and (12) in accordance with the general result. An additional feature is that C_+ goes over to C_- and vice versa in these equations. This is consistent with the counting of the number of gamma matrices on each side of Eq. (11) and Eq. (12). The point is that we must have either odd or even number of gamma matrices on each side of an equation (since the product of odd (even) number of gamma matrices has a zero (non-zero) trace for even D). That amputation of *one* spinor leg gives back a standard structure is a special feature of the Yukawa theory. We will see that this feature is not present when we have a vector field coupling to the spinors.

4 Spinor-spinor-vector Green function

The Green function $\langle\psi_d(x_1)\bar{\psi}_l(x_2)\Phi_\mu^\Delta(x_3)\rangle$ has two conformal invariant structures [3]:

$$C_{1\mu}^{d,l,\Delta}(x_1x_2x_3) = \frac{\not{x}_{13}\gamma_\mu\not{x}_{32}}{x_{12}^{d+l-\Delta}x_{13}^{d-l+\Delta+1}x_{23}^{l-d+\Delta+1}}\,, \tag{14}$$

$$C_{2\mu}^{d,l,\Delta}(x_1x_2x_3) = \frac{\not{x}_{12}}{x_{12}^{d+l-\Delta+2}}\left(\frac{x_{13\mu}}{x_{13}^{d-l+\Delta+1}x_{23}^{l-d+\Delta-1}} - \frac{x_{23\mu}}{x_{13}^{d-l+\Delta-1}x_{23}^{l-d+\Delta+1}}\right) \tag{15}$$

$$= \frac{\not{x}_{12}}{x_{12}^{d+l-\Delta+2}x_{13}^{d-l+\Delta-1}x_{23}^{l-d+\Delta-1}}\lambda_\mu^{x_3}(x_1x_2)\,, \tag{16}$$

where

$$\lambda_\mu^{x_3}(x_1x_2) = \frac{x_{13\mu}}{x_{13}^2} - \frac{x_{23\mu}}{x_{23}^2}\,. \tag{17}$$

[These structures are also given in Refs. [2], [6] and [7], but only for the case $\Delta = 1$.]

To ampute ψ_d (say), we have to proceed as in Eq. (11). But we will now come across an important difference: *the amputation of one spinor leg will not give back either $C_{1\mu}$ or $C_{2\mu}$ (or a linear combination of them).* At least for even D, this can be understood from the fact that each $\int d^Dx_1\ (\not{x}_{41}/(x_{14}^2)^{D-d+1/2})C_{i\mu}$ (with $i = 1, 2$) is a product of even number of gamma matrices, while both (14) and (15) have odd number of gamma matrices. In order to be consistent with the general result, the structures resulting from amputing ψ_d will still be conformal invariant with the expected values of scale dimensions (i.e. $D - d$, l and Δ). [We explicitly check the conformal invariance of such structures are in Appendix C.] But these structures are non-standard in the sense that they do not have any symmetry under the interchange of the two fermions and hermitian conjugation. [On the other hand, the standard structures in Eqs. (14) and (15), and also those in Eqs. (9) and (10), are invariant when $x_1 \leftrightarrow x_2$, $d \leftrightarrow l$ and hermitian conjugation are performed together. Recall that the Euclidean gamma matrices are all hermitian.] However, when *both* ψ_d and ψ_l are amputated, we get back linear combinations of $C_{1\mu}$ and $C_{2\mu}$: see Secs. 6 and 7.

5 Star-triangle relation with two spinors and one vector field

The star-triangle relation which we are going to prove, and which will be later used for amputing $C_{1\mu}^{d,l,\Delta}$, is:

$$\begin{aligned}
&\int d^Dx_4\ \frac{\not{x}_{14}}{(x_{14}^2)^{\delta_1+1/2}}\,\gamma_\nu\,\frac{\not{x}_{42}}{(x_{24}^2)^{\delta_2+1/2}}\,\frac{g_{\mu\nu}(x_{34})}{(x_{34}^2)^{\delta_3}}\\
=\ & \pi^{D/2}\frac{\Gamma(D/2-\delta_1+1/2)\Gamma(D/2-\delta_2+1/2)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1+1/2)\Gamma(\delta_2+1/2)\Gamma(\delta_3+1)}\\
&\times\Bigg((\delta_3-1)\frac{\not{x}_{13}\gamma_\mu\not{x}_{32}}{(x_{12}^2)^{D/2-\delta_3}(x_{13}^2)^{D/2-\delta_2+1/2}(x_{23}^2)^{D/2-\delta_1+1/2}}\\
&+(D-2\delta_3)\,\frac{\not{x}_{12}}{(x_{12}^2)^{D/2-\delta_3+1}(x_{13}^2)^{D/2-\delta_2-1/2}(x_{23}^2)^{D/2-\delta_1-1/2}}\lambda_\mu^{x_3}(x_1x_2)\Bigg)
\end{aligned} \tag{18}$$

where Eq. (72) holds. The vector field has the propagator corresponding to scale dimension δ_3 (see Eq. (36)) with

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\,. \tag{19}$$

Eq. (18) can be viewed as *analogous to the more familiar star-triangle relations* given by Eqs. (71) and (73), as follows. The left-hand side of Eq. (18) represents the propagation of two conformal spinors and one conformal vector field from the external points x_a ($a = 1, 2, 3$) to the internal vertex x_4 with an interaction γ_ν. The right-hand side is a linear combination of the two available structures (14) and (15).

A check for Eq. (18) can be performed for the case $\delta_3 = 1$. In this case, the vector field propagator on the left-hand side is $g_{\mu\nu}(x_{34})/x_{34}^2 = \partial_{x_3}^\mu \partial_{x_3}^\nu \ln|x_{34}|$, that is, longitudinal in x_3. On the right-hand side, only the second term remains, and this term is also longitudinal in x_3 as follows. Since Eq. (72) now gives $D/2 - \delta_1 - 1/2 = -(D/2 - \delta_2 - 1/2) = n/2$ (say), the coordinate x_3 now occurs in the combination $(x_{13}/x_{23})^n \lambda_\mu^{x_3}(x_1 x_2)$, which equals $-(1/n)\partial_{x_3}^\mu (x_{13}/x_{23})^n$.

A relation previously derived in Refs. [16] and [17] also involved two spinors and one vector field. But *there are two important differences between that relation and the relation (18) above.* Firstly, the previous relation had a covariant gauge propagator, while the vector field propagator in Eq. (18) is invariant under the standard transformation law for a conformal vector. This is necessary for the amputation of the spinor leg and also the vector leg in the structure $C_{1\mu}^{d,l,\Delta}$ [18]. The other difference is that here we have completely general values for the scale dimensions δ_1, δ_2 and δ_3; this will also be necessary for the present purpose.

The derivation of Eq. (18), which is to be presented now, *will be along the same lines as followed in Refs. [16] and [17].* We are thus going to use the operator algebraic method due to Isaev [15] which reduces Feynman integrals to products of position and momentum operators $\hat{q}_i$ and $\hat{p}_i$ taken between position eigenstates. As explained in Sec. 2 of Ref. [16], this method involves starting from the "$\hat{p}\hat{q}\hat{p}$" form and passing to the "$\hat{q}\hat{p}\hat{q}$" form, using the key relation given by Eq. (60). In our case, the idea is to split the left-hand side of Eq. (18) into a longitudinal part and a transverse part, and tackle them as in Sec. 4 of Ref. [16] and Sec. 2 of Ref. [17] respectively. In view of the general values of the scale dimensions, the starting "$\hat{p}\hat{q}\hat{p}$" forms are somewhat different from that in these references. The starting forms are

$$\Gamma_\mu^{\rm long} \equiv (\gamma_\lambda\gamma_\nu\gamma_\rho \hat{p}_\lambda \hat{p}^{-2\alpha-1} \hat{q}_\rho \hat{q}^{-2(\alpha+\beta)-1} \hat{p}^{-2\beta}) \hat{p}_\nu \hat{p}_\mu \hat{p}^{-2}\,, \tag{20}$$

$$\Gamma_\mu^{\rm tr} \equiv (\gamma_\lambda\gamma_\nu\gamma_\rho \hat{p}_\lambda \hat{p}^{-2\alpha-1} \hat{q}_\rho \hat{q}^{-2(\alpha+\beta)-1} \hat{p}^{-2\beta})(\delta_{\nu\mu} - \hat{p}_\nu \hat{p}_\mu \hat{p}^{-2})\,. \tag{21}$$

[These are, however, quite similar to the "$\hat{p}\hat{q}\hat{p}$" form for the three-point function of the Yukawa theory: see Eq. (5) of Ref. [16].] *A new element in this calculation is that* $\Gamma_\mu^{\rm long}$ *and* $\Gamma_\mu^{\rm tr}$ *have to be taken in a precise proportion* to result in the structure (19). To determine this proportion, we use the relation

$$\frac{g_{\mu\nu}(x)}{r^n} = \frac{n-2}{n}\left(\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right) + \frac{2D-n-2}{n-2}\frac{\partial_\mu\partial_\nu}{\partial^2}\right)\frac{1}{r^n}\,. \tag{22}$$

[This formula can be derived by first evaluating $\partial_\mu\partial_\nu(1/r^{n-2})$, and hence $\partial^2(1/r^{n-2})$. Here $r = \sqrt{x_\mu x_\mu}$.] Since $\hat{p}^{-2\beta}$ in Eqs. (20) and (21) goes as $r^{-(D-2\beta)}$ in position space, we need to consider the case $n = D - 2\beta$ in Eq. (22). Thus we have to start with the "$\hat{p}\hat{q}\hat{p}$" form

$$\Gamma_\mu = \Gamma_\mu^{\rm tr} + \frac{D+2\beta-2}{D-2\beta-2}\Gamma_\mu^{\rm long}\,. \tag{23}$$

Next, we have to express the position-space matrix elements (that is, between $\langle x|$ and $|y\rangle$) of the right-hand sides of Eqs. (20) and (21) in terms of the matrix elements of $\hat{p}_\lambda \hat{p}^{-2\alpha-1}$,

$\hat{p}^{-2\beta}$ and $\hat{p}_\nu \hat{p}_\mu \hat{p}^{-2}$ from the Appendix of Ref. [16] (see Eqs. (14) and (15) of Ref. [16]). Then we can write down the matrix element of the right-hand side of Eq. (23) using Eq. (22) for $n = D - 2\beta$. This leads to

$$\langle x|\Gamma_\mu|y\rangle = \frac{i\Gamma(D/2-\alpha+1/2)\Gamma(D/2-\beta+1)}{\pi^D 2^{2\alpha+2\beta-1}(D-2\beta-2)\Gamma(\alpha+1/2)\Gamma(\beta)} \int d^D z \, \frac{\not{x}-\not{z}}{|x-z|^{D-2\alpha+1}}\gamma_\nu \frac{\not{z}}{|z|^{2(\alpha+\beta)+1}} \frac{g_{\mu\nu}(y-z)}{|y-z|^{D-2\beta}} \,. \tag{24}$$

On the other hand, we can put Γ_μ in the "$\hat{q}\hat{p}\hat{q}$" form and then take the matrix element. This involves a long calculation, given in Appendix D (the important points are highlighted after Eqs. (68) and (70).) It leads to

$$\langle x|\Gamma_\mu|y\rangle = \frac{i\Gamma(D/2-\alpha-\beta+1/2)}{\pi^{D/2}2^{2\alpha+2\beta}\Gamma(\alpha+\beta+1/2)} \frac{x^2(\not{x}-\not{y})\gamma_\mu\not{y} + \frac{4\beta}{D-2\beta-2}((x-y)^2 y_\mu + y^2(x-y)_\mu)\not{x}}{x^{2\beta+2}|x-y|^{D-2\alpha-2\beta+1}y^{2\alpha+1}} . \tag{25}$$

The right-hand sides of Eqs. (24) and (25) are now to be equated. After that, we let $x = x_1 - x_2$ and $y = x_3 - x_2$, and also change to a new integration variable x_4 defined by $z = x_4 - x_2$. We also define δ_1, δ_2 and δ_3 by $D/2 - \alpha = \delta_1$, $\alpha + \beta = \delta_2$ and $D/2 - \beta = \delta_3$. This leads us to the relation given in Eq. (18).

6 Spinor leg amputation in $C^{d,l,\Delta}_{1\mu}(x_1x_2x_3)$

In this Section and the next, we are going to evaluate

$$\int d^D x_1 \, d^D x_2 \, \frac{\not{x}_{41}}{(x_{14}^2)^{D-d+1/2}} C^{d,l,\Delta}_{i\mu}(x_1x_2x_3) \frac{\not{x}_{25}}{(x_{25}^2)^{D-l+1/2}} , \qquad i = 1,2. \tag{26}$$

The integration over x_1 amputes $\psi_d(x_1)$, while that over x_2 amputes $\bar{\psi}_l(x_2)$.

We consider $C_{1\mu}$ in this Section. From Eqs. (26) and (14), we see that the x_1 integration can be done by using the star-triangle relation of Eq. (73). The integration over x_2 then involves

$$\not{x}_{23}\gamma_\mu \not{x}_{32} = x_{23}^2 \gamma_\nu g_{\mu\nu}(x_{23}) \,. \tag{27}$$

Consequently, this integral is of the form

$$\int d^D x_2 \, \frac{\not{x}_{42}}{x_{24}^{D-d+l-\Delta+1}} \gamma_\nu \frac{\not{x}_{25}}{(x_{25}^2)^{D-l+1/2}} \frac{g_{\mu\nu}(x_{23})}{x_{23}^{d+l+\Delta-D}} \tag{28}$$

which can be evaluated by using the star-triangle relation of Eq. (18). We thus get

$$\begin{aligned} &\int d^D x_1 \, d^D x_2 \; S_d^{-1}(x_{41}) \, C^{d,l,\Delta}_{1\mu}(x_1x_2x_3) \, S_l^{-1}(x_{25}) \\ = \;& F(d,l,\Delta) \times (d+l-\Delta)\Bigg(\frac{d+l+\Delta-D-2}{2} C^{D-d,D-l,\Delta}_{1\mu}(x_4x_5x_3) \\ &+ (2D-d-l-\Delta) C^{D-d,D-l,\Delta}_{2\mu}(x_4x_5x_3)\Bigg) \end{aligned} \tag{29}$$

where S_d^{-1} and S_l^{-1} are as in Eq. (8), and the coefficient $F(d,l,\Delta)$ is given by

$$F(d,l,\Delta)=\frac{\pi^{-D}}{2N^2}\frac{\Gamma(d+\frac{1}{2})\Gamma(l+\frac{1}{2})\Gamma(\frac{D-d-l+\Delta}{2})\Gamma(\frac{2D-d-l-\Delta}{2})}{\Gamma(\frac{D}{2}-d+\frac{1}{2})\Gamma(\frac{D}{2}-l+\frac{1}{2})\Gamma(\frac{d+l-\Delta+2}{2})\Gamma(\frac{d+l+\Delta-D+2}{2})}\,. \tag{30}$$

7 Spinor leg amputation in $C_{2\mu}^{d,l,\Delta}(x_1x_2x_3)$

Using Eq. (15), we write down the integral (26) for $C_{2\mu}$ in full. There are two terms. On interchanging the integration variables x_1, x_2 in the second term, we find that the integral under consideration is

$$\int d^Dx_1\,d^Dx_2\;\frac{\not{x}_{41}}{(x_{14}^2)^{D-d+1/2}}\;\frac{\not{x}_{12}}{x_{12}^{d+l-\Delta+2}}\;\frac{x_{13\mu}}{x_{13}^{d-l+\Delta+1}}\;\frac{1}{x_{23}^{l-d+\Delta-1}}\;\frac{\not{x}_{25}}{(x_{25}^2)^{D-l+1/2}}$$
$$+\Big(\text{hermitian conjugate}, x_4\leftrightarrow x_5, d\leftrightarrow l\Big) \tag{31}$$

Let us evaluate the first term in (31). First we perform the x_2 integration using Eq. (73). Then the remaining x_1 integral is of the form

$$\int d^Dx_1\;\frac{\not{x}_{41}}{x_{14}^{2(D-d)+1}}\;\frac{\not{x}_{13}x_{13\mu}}{x_{13}^{d+l+\Delta-D+2}}\;\frac{1}{x_{15}^{D+d-l-\Delta+1}}$$
$$=\;\frac{1}{D-d-l-\Delta}\bigg(\Big(\frac{1}{2(D-d)-1}\frac{\partial}{\partial\not{x}_4}\frac{\partial}{\partial x_{3\mu}}I\Big)\not{x}_{43}+\frac{\partial}{\partial x_{3\mu}}I\bigg)\,, \tag{32}$$

where

$$I\;=\;\int d^Dx_1\;\frac{1}{x_{14}^{2(D-d)-1}x_{13}^{d+l+\Delta-D}x_{15}^{D+d-l-\Delta+1}}\,. \tag{33}$$

The right-hand side of Eq. (32) is obtained by writing $\not{x}_{13}=\not{x}_{43}-\not{x}_{41}$ on the left-hand side. Now I can be evaluated by using Eq. (71). After some algebra, the first term in (31) is found to be

$$\frac{\Big((l-d+\Delta-1)x_{35}^2x_{34\mu}+(2d-D+1)x_{34}^2x_{35\mu}\Big)\frac{\not{x}_{45}}{x_{45}^2}+\frac{l-d+\Delta-1}{2D-d-l-\Delta}\not{x}_{43}\gamma_\mu\not{x}_{35}}{x_{34}^{l-d+\Delta+1}x_{35}^{d-l+\Delta+1}x_{45}^{2D-d-l-\Delta}} \tag{34}$$

multiplied with a coefficient which is symmetric in d and l. Then adding the second term in (31), we finally arrive at

$$\int d^Dx_1\,d^Dx_2\;S_d^{-1}(x_{41})\,C_{2\mu}^{d,l,\Delta}(x_1x_2x_3)\,S_l^{-1}(x_{25})$$
$$=\;F(d,l,\Delta)\times\bigg((\Delta-1)C_{1\mu}^{D-d,D-l,\Delta}(x_4x_5x_3)$$
$$+\frac{(2D-d-l-\Delta)(d+l-\Delta-D+2)}{2}C_{2\mu}^{D-d,D-l,\Delta}(x_4x_5x_3)\bigg) \tag{35}$$

where the coefficient $F(d,l,\Delta)$ is given by Eq. (30).

For the special case of $\Delta=1$, the amputation of spinor legs in $C_{2\mu}$ can be done much more easily. The reason is that in this case the x_3-dependent part in $C_{2\mu}$ equals $(\partial/\partial x_{3\mu})(x_{23}/x_{13})^{d-l}$, and we need to just apply Eq. (73).

8 Vector leg amputation in $C_{1\mu}^{d,l,\Delta}(x_1x_2x_3)$ and $C_{2\mu}^{d,l,\Delta}(x_1x_2x_3)$

The vector field two-point function and its inverse are given by

$$\begin{aligned} D_{\mu\nu}(x_{12}) &= N\frac{g_{\mu\nu}(x_{12})}{(x_{12}^2)^{\Delta}}\,, \\ D_{\mu\nu}^{-1}(x_{12}) &= \frac{\pi^{-D}}{N}\,\frac{\Delta}{D-\Delta-1}\,\frac{\Gamma(\Delta-1)\Gamma(D-\Delta+1)}{\Gamma(\frac{D}{2}-\Delta)\Gamma(\Delta-\frac{D}{2})}\,\frac{g_{\mu\nu}(x_{12})}{(x_{12}^2)^{D-\Delta}}\,. \end{aligned} \tag{36}$$

They satisfy

$$\int d^Dx_2\; D_{\mu\nu}(x_{12})D_{\nu\rho}^{-1}(x_{23}) = \int d^Dx_2\; D_{\mu\nu}^{-1}(x_{12})D_{\nu\rho}(x_{23}) = \delta_{\mu\rho}\delta^{(D)}(x_{13})\,. \tag{37}$$

The amputation equations are

$$\begin{aligned} \int d^Dx_3\, C_{1\mu}^{d,l,\Delta}(x_1x_2x_3)\, D_{\mu\nu}^{-1}(x_{34}) &= F'(d,l,\Delta)\times\Big((D-\Delta-1)C_{1\nu}^{d,l,D-\Delta}(x_1x_2x_4) \\ &\quad +(2\Delta-D)C_{2\nu}^{d,l,D-\Delta}(x_1x_2x_4)\Big)\,, \end{aligned} \tag{38}$$

which is obtained by using Eq. (18), and

$$\int d^Dx_3\, C_{2\mu}^{d,l,\Delta}(x_1x_2x_3)\, D_{\mu\nu}^{-1}(x_{34}) = F'(d,l,\Delta)\times(\Delta-1)C_{2\nu}^{d,l,D-\Delta}(x_1x_2x_4)\,, \tag{39}$$

which is obtained by using Eq. (74). Here the coefficient $F'(d,l,\Delta)$ is given by

$$F'(d,l,\Delta) = \frac{\pi^{-D/2}}{N}\,\frac{\Delta}{D-\Delta-1}\,\frac{\Gamma(\Delta-1)\Gamma(\frac{D-d+l-\Delta+1}{2})\Gamma(\frac{D-l+d-\Delta+1}{2})}{\Gamma(\frac{D}{2}-\Delta)\Gamma(\frac{d-l+\Delta+1}{2})\Gamma(\frac{l-d+\Delta+1}{2})}\,. \tag{40}$$

9 Checking results by amputation in $C_{1\mu}-C_{2\mu}$

First consider the case $\Delta = D-1$, which is the scale dimension of the current. In this case we have the relation

$$\frac{\partial}{\partial x_{3\mu}}\Big(C_{1\mu}^{d,l,D-1}(x_1x_2x_3) - C_{2\mu}^{d,l,D-1}(x_1x_2x_3)\Big) = 0\,. \tag{41}$$

[The case $d=l$ of Eq. (41) is consistent with the fact that both $C_{1\mu}$ and $C_{2\mu}$ satisfy

$$\frac{\partial}{\partial x_{3\mu}}C_{1\mu}^{d,d,D-1}(x_1x_2x_3) = \frac{\partial}{\partial x_{3\mu}}C_{2\mu}^{d,d,D-1}(x_1x_2x_3) = -\frac{2\pi^{D/2}}{\Gamma(D/2)}(\delta(x_{13})-\delta(x_{23}))\frac{\not{x}_{12}}{x_{12}^{2d+1}}\,, \tag{42}$$

which is the Ward identity in position space.] Now from Eq. (26), we see that $\partial/\partial x_{3\mu}$ commutes with the operation of spinor leg amputation (also, Δ stays $D-1$ after this amputation). Thus, *the combination $C_{1\mu}-C_{2\mu}$ should continue to be of this form after the* *spinor legs are amputed.* Indeed, by putting $\Delta = D-1$ in Eqs. (29) and (35) and taking the difference, we obtain

$$\begin{aligned} & \int d^D x_1\, d^D x_2\; S_d^{-1}(x_{41}) \Big(C_{1\mu}^{d,l,D-1}(x_1x_2x_3) - C_{2\mu}^{d,l,D-1}(x_1x_2x_3) \Big) S_l^{-1}(x_{25}) \\ = \; & \frac{1}{2} F(d,l,D-1) \times (d+l-1)(d+l-D-1) \\ & \times \Big(C_{1\mu}^{D-d,D-l,D-1}(x_4x_5x_3) - C_{2\mu}^{D-d,D-l,D-1}(x_4x_5x_3) \Big) \end{aligned} \tag{43}$$

This serves as a check on the coefficients obtained in Eqs. (29) and (35).

Next, for any d, l and Δ, the two structures satisfy the relation

$$\mathrm{Tr}\Big[\not{x}_{12} \Big(C_{1\mu}^{d,l,\Delta}(x_1x_2x_3) - C_{2\mu}^{d,l,\Delta}(x_1x_2x_3) \Big) \Big] = 0 \,. \tag{44}$$

Now vector leg amputation involves only x_3. So $C_{1\mu} - C_{2\mu}$ *should stay in this combination after the vector leg is amputed.* From Eqs. (38) and (39), we indeed find that

$$\begin{aligned} & \int d^D x_3 \, \Big(C_{1\mu}^{d,l,\Delta}(x_1x_2x_3) - C_{2\mu}^{d,l,\Delta}(x_1x_2x_3) \Big) D_{\mu\nu}^{-1}(x_{34}) \\ = \; & F'(d,l,\Delta) \times (D-\Delta-1) \Big(C_{1\nu}^{d,l,D-\Delta}(x_1x_2x_4) - C_{2\nu}^{d,l,D-\Delta}(x_1x_2x_4) \Big) \,, \end{aligned} \tag{45}$$

which checks the coefficients in the amputation equations.

10 Conclusion

We have derived a new star-triangle relation involving two spinors and one vector field, with general values of the scale dimensions. The relation has been applied to amputation of conformal invariant three-point function involving these fields. The star-triangle relation can be of general use in conformal field theoretical context.

Amputed Green functions are the coefficients of the various quasi-primary fields in the conformal partial wave expansion of the product of two field operators, and thereby enter the partial wave expansion of the four-point function. Two examples of application of amputation equations in these context are the solution of the Thirring model in Ref. [1] and the calculation of various unparticle processes in the Sommerfield model in Ref. [9]. We have extended the existing work on amputation of conformal Green functions to the next complicated case involving two spinors and a vector field, With the infrared limit of massless QED_3 as one of the possible areas of application.

A general discussion of conformal partial wave expansion for the case of more than one independent invariant structure is given in Ref. [2]. (That discussion does not deal with spinor legs, but the presence of more than one structure applies to our work also.) In this case, the proper choice of linear combination of the independent structures is determined by the orthogonality condition, which involves amputed Green functions [19]. The partial wave expansion then takes a diagonal form. Amputation equations will thus be necessary for finding explicit realization of such a situation.

Acknowledgement

The author completed writing up the first version of this work during a visit to IMSc, Chennai. He thanks IMSc for kind hospitality and H. S. Sharatchandra for useful discussion during the visit.

Appendices

A Inverse of two-point function

The inverses of the two-point functions are given in Eqs. (1), (8) and (36). Once we settle for a value of N in any of these equations, the coefficient of the inverse two-point function gets fixed by an integral like $\int d^D x_2\; x_{12}^{-2d} x_{23}^{-2(D-d)}$ (this is for the scalar field). In this Appendix, we indicate how such integrals can be evaluated.

For the scalar field, we have

$$\int d^D x_2\; \frac{1}{x_{12}^{2d}}\, \frac{1}{x_{23}^{2(D-d)}} = \int d^D x_2\; \frac{\langle x_1|\hat{p}^{-2(D/2-d)}|x_2\rangle}{a(D/2-d)}\, \frac{\langle x_2|\hat{p}^{-2(d-D/2)}|x_3\rangle}{a(d-D/2)}\,, \tag{46}$$

using $\langle x|\hat{p}^{-2\alpha}|y\rangle = a(\alpha)|x-y|^{-(D-2\alpha)}$, with $a(\alpha)$ given in the Appendix of Ref. [16]. Using $\int d^D x_2\; |x_2\rangle\langle x_2| = 1$, we can then evaluate the right-hand side of Eq. (46).

For the spinor field, we can similarly evaluate $\int d^D x_2\; (\not{x}_{12}/x_{12}^{2d+1})(\not{x}_{23}/x_{23}^{2(D-d)+1})$ by using the expression for $\langle x|\hat{p}_i\hat{p}^{-2\alpha}|y\rangle$ given in the Appendix of Ref. [16]. Finally, for the vector field we need to evaluate the integral $\int d^D x_2\; (g_{\mu\nu}(x_{12})/x_{12}^{2\Delta})(g_{\nu\rho}(x_{23})/x_{23}^{2(D-\Delta)})$. Here the method is to use Eq. (22), and convert this integral to a differential operator acting on the integral given in Eq. (46).

B General treatment of amputation of spinor leg and vector leg

Here we demonstrate the result that amputation replaces a spinor or vector field by its its conformal partner in a Green function using the specific case of the three-point function. This extends the demonstration for scalar field given in Ref. [3].

For amputation of the *spinor leg*, we want to show that (compare with Eq. (4) for the scalar field)

$$\langle\tilde{\psi}_d(x_4)\bar{\psi}_l(x_2)\Phi^{\Delta}_{\mu}(x_3)\rangle \equiv \int d^D x_1\; S_d^{-1}(x_4-x_1)\langle\psi_d(x_1)\bar{\psi}_l(x_2)\Phi^{\Delta}_{\mu}(x_3)\rangle \tag{47}$$

is a three-point function with dimensions $D-d$, l and Δ for the three fields. For this, we need to check that it satisfies the invariance condition for the three-point function with these scale dimensions under conformal inversion and under scale transformation.

Let us consider first conformal inversion: $x_\mu \to Rx_\mu = x_\mu/x^2$. Under this operation, the various (Euclidean) fields transform as [3]

$$\psi'_d(x) = \frac{\not{x}}{(x^2)^{d+1/2}}\,\psi_d(Rx), \qquad \bar{\psi}'_l(x) = \bar{\psi}_l(Rx)\,\frac{\not{x}}{(x^2)^{l+1/2}}\,, \tag{48}$$

$$\Phi'^{\Delta}_{\mu}(x) = \frac{g_{\mu\nu}(x)}{(x^2)^{\Delta}}\Phi^{\Delta}_{\nu}(Rx)\,. \tag{49}$$

So the invariance condition $\langle\psi_d(x_1)\bar{\psi}_l(x_2)\Phi^{\Delta}_{\mu}(x_3)\rangle = \langle\psi'_d(x_1)\bar{\psi}'_l(x_2)\Phi'^{\Delta}_{\mu}(x_3)\rangle$ implies that

$$\langle\psi_d(x_1)\bar{\psi}_l(x_2)\Phi^{\Delta}_{\mu}(x_3)\rangle = \frac{g_{\mu\nu}(x_3)}{(x_1^2)^{d+1/2}(x_2^2)^{l+1/2}(x_3^2)^{\Delta}}\,\not{x}_1\langle\psi_d(Rx_1)\bar{\psi}_l(Rx_2)\Phi^{\Delta}_{\nu}(Rx_3)\rangle\not{x}_2\,. \tag{50}$$

Similarly, the condition $\langle\psi_d(x_1)\bar{\psi}_d(x_2)\rangle = \langle\psi'_d(x_1)\bar{\psi}'_d(x_2)\rangle$ implies that

$$S_d^{-1}(x_4 - x_1) = \frac{1}{(x_1^2 x_4^2)^{D-d+1/2}}\not{x}_4 S_d^{-1}(Rx_4 - Rx_1)\not{x}_1\,, \tag{51}$$

since S_d^{-1} is the two-point function of a spinor of dimension $D-d$ (see Eq. (8)). We insert Eqs. (50) and (51) on the right-hand side of Eq. (47), and then let $x_1 \to Rx_1$ (so $x_1^2 \to 1/x_1^2$ and $d^Dx_1 \to d^Dx_1\,(x_1^2)^{-D}$). Comparing the resulting expression with Eq. (47) again, we get

$$\langle\tilde{\psi}_d(x_4)\bar{\psi}_l(x_2)\Phi^{\Delta}_{\mu}(x_3)\rangle = \frac{g_{\mu\nu}(x_3)}{(x_4^2)^{D-d+1/2}(x_2^2)^{l+1/2}(x_3^2)^{\Delta}}\not{x}_4\langle\tilde{\psi}_d(Rx_4)\bar{\psi}_l(Rx_2)\Phi^{\Delta}_{\nu}(Rx_3)\rangle\not{x}_2\,. \tag{52}$$

Comparing this with Eq. (50) leads to the desired conclusion.

For the scale transformation $x_\mu \to \lambda x_\mu$, we proceed along similar lines, using $\psi'_d(x) = \lambda^d\psi_d(\lambda x)$, $\bar{\psi}'_l(x) = \lambda^l\bar{\psi}_l(\lambda x)$, and $\Phi'^{\Delta}_{\mu}(x) = \lambda^{\Delta}\Phi^{\Delta}_{\mu}(\lambda x)$. Amputation of $\bar{\psi}_l$ can be handled similarly.

For amputation of the *vector leg*, we have to show that

$$\langle\psi_d(x_1)\bar{\psi}_l(x_2)\tilde{\Phi}^{\Delta}_{\nu}(x_4)\rangle \equiv \int d^Dx_3\ \langle\psi_d(x_1)\bar{\psi}_l(x_2)\Phi^{\Delta}_{\mu}(x_3)\rangle D^{-1}_{\mu\nu}(x_{34}) \tag{53}$$

is a three-point function with dimensions d, l and $D-\Delta$ for the three fields. The condition $\langle\Phi^{\Delta}_{\mu}(x_1)\Phi^{\Delta}_{\nu}(x_2)\rangle = \langle\Phi'^{\Delta}_{\mu}(x_1)\Phi'^{\Delta}_{\nu}(x_2)\rangle$ implies that

$$D^{-1}_{\mu\nu}(x_{34}) = \frac{g_{\mu\rho}(x_3)g_{\nu\sigma}(x_4)}{(x_3^2x_4^2)^{D-\Delta}}\,D^{-1}_{\rho\sigma}(Rx_{34}) \tag{54}$$

since $D^{-1}_{\mu\nu}$ is the two-point function of a vector field of dimension $D-\Delta$. We insert Eqs. (50) and (54) in the right-hand side of Eq. (53), then let $x_3 \to Rx_3$ and follow the procedure adopted for ψ_d.

C Conformal invariance of structures obtained by amputing one spinor leg

In this Appendix, we directly check that the structures obtained by amputing *one* spinor leg in $C_{1\mu}$ are indeed conformal invariant, albeit non-standard, structures. Amputing *only* $\psi_d(x_1)$ in $C^{d,l,\Delta}_{1\mu}(x_1x_2x_3)$ by using $S^{-1}_d(x_{14})$ (see Sec. 6), we obtain the structure $(\not{x}_{42}/x_{24}^{D-d+l-\Delta+1})\gamma_\nu(g_{\mu\nu}(x_{23})/x_{23}^{d+l+\Delta-D})(1/x_{34}^{D-d-l+\Delta})$. This should be a conformal invariant structure for $\langle\psi_{D-d}(x_4)\bar{\psi}_l(x_2)\Phi^\Delta_\mu(x_3)\rangle$. Equivalently,

$$\frac{\not{x}_{12}}{x_{12}^{d+l-\Delta+1}}\,\gamma_\nu\frac{g_{\mu\nu}(x_{23})}{x_{23}^{l-d+\Delta}}\,\frac{1}{x_{13}^{d-l+\Delta}} \tag{55}$$

should be a conformal invariant structure for $\langle\psi_d(x_1)\bar{\psi}_l(x_2)\Phi^\Delta_\mu(x_3)\rangle$.

Our aim is to check the invariance condition expressed in Eq. (50) for this structure. This amounts to showing that the expression (55) equals the expression

$$\frac{g_{\mu\nu}(x_3)}{x_{12}^{d+l-\Delta+1}x_{23}^{l-d+\Delta}x_{13}^{d-l+\Delta}}\,\not{x}_1\left(\frac{\not{x}_1}{x_1^2}-\frac{\not{x}_2}{x_2^2}\right)\gamma_\rho g_{\nu\rho}(Rx_{23})\not{x}_2 \tag{56}$$

(where we used $Rx_{12}=x_{12}/(|x_1||x_2|)$). The equality of the two expressions can be shown by using $g_{\nu\rho}(Rx_{23})=g_{\nu\lambda}(x_3)g_{\lambda\kappa}(x_{23})g_{\kappa\rho}(x_2)$, $g_{\mu\nu}(x_3)g_{\nu\lambda}(x_3)=\delta_{\mu\lambda}$, and $\gamma_\rho g_{\kappa\rho}(x_2)=-\not{x}_2\gamma_\kappa\not{x}_2/x_2^2$.

Similarly, by amputing *only* $\bar{\psi}_l$ in $C^{d,l,\Delta}_{1\mu}(x_1x_2x_3)$ we get another conformal-invariant structure for $\langle\psi_d(x_1)\bar{\psi}_l(x_2)\Phi^\Delta_\mu(x_3)$, namely,

$$\gamma_\nu\frac{g_{\mu\nu}(x_{13})}{x_{13}^{d-l+\Delta}}\,\frac{\not{x}_{12}}{x_{12}^{d+l-\Delta+1}}\,\frac{1}{x_{23}^{l-d+\Delta}}\,. \tag{57}$$

The structures (55) and (57), being products of even number of gamma matrices, are independent of $C_{1\mu}$ and $C_{2\mu}$, which are products of odd number of gamma matrices. The former are, however, not invariant under interchange of the two fermions and simultaneous hermitian conjugation. But they are valid structures if the two fermions in the Green function are not identical.

By amputing one spinor leg in $C_{2\mu}$ also, we get non-standard structures, which are more complicated than the two structures (55) and (57).

D Some steps in the derivation of the star-triangle relation of Sec. 5

Here we show how to express Γ_μ as given by Eqs. (20), (21) and (23) in the "$\hat{q}\hat{p}\hat{q}$" form, and arrive at Eq. (25). For $\Gamma^{\rm long}_\mu$, we follow Sec. 4 of Ref. [16]. Thus, we write

$$\langle x|\Gamma^{\rm long}_\mu|y\rangle=-i\frac{\partial^y_\mu}{(\partial^2)^y}\langle x|\Gamma'|y\rangle\,, \tag{58}$$

$$\Gamma'=\gamma_\lambda\gamma_\nu\gamma_\rho\hat{p}_\lambda\hat{p}^{-2\alpha-1}\hat{q}_\rho\hat{q}^{-2(\alpha+\beta)-1}\hat{p}^{-2\beta}\hat{p}_\nu\,. \tag{59}$$

[In $\langle x|\Gamma^{\rm long}_\mu|y\rangle$, we insert $\int d^Dz\,|z\rangle\langle z|$ just after $\hat{p}_\nu$. Now $\langle z|\hat{p}_\mu\hat{p}^{-2}|y\rangle=-i\partial^y_\mu/(\partial^2)^y\delta^{(D)}(y-z)$, and $\partial^y_\mu/(\partial^2)^y$ can be taken outside the integral over z.]

In going from "$\hat{p}\hat{q}\hat{p}$" to the "$\hat{q}\hat{p}\hat{q}$" form, the essential idea is to move $\hat{q}_\mu$ (or $\hat{p}_\mu$) through powers of $\hat{p}^2$ (or $\hat{q}^2$) by using $[\hat{q}_\mu,\hat{p}^{2\alpha}]=i2\alpha\hat{p}^{2\alpha-2}\hat{p}_\mu$ (or $[\hat{p}_\mu,\hat{q}^{2\alpha}]=-i2\alpha\hat{q}^{2\alpha-2}\hat{q}_\mu$), so that one can use the key relation

$$\hat{p}^{-2\alpha}\hat{q}^{-2(\alpha+\beta)}\hat{p}^{-2\beta}=\hat{q}^{-2\beta}\hat{p}^{-2(\alpha+\beta)}\hat{q}^{-2\alpha}\,, \tag{60}$$

at an intermediate stage. Eq. (60) is the star-triangle relation of Eq. (71) in the operator form. We thus follow the steps in Eqs. (19)-(23) of Ref. [16]. In the present case, this leads to

$$\begin{aligned}\Gamma' &= \gamma_\rho \hat{q}^{-2\beta}\hat{p}^{-2(\alpha+\beta)+1}\hat{q}^{-2\alpha-1}\hat{q}_\rho + i2\beta\gamma_\lambda\gamma_\nu\gamma_\rho\hat{q}^{-2\beta-2}\hat{q}_\lambda\hat{p}^{-2(\alpha+\beta)-1}\hat{p}_\nu\hat{q}^{-2\alpha-1}\hat{q}_\rho \\ &+i(D-2\alpha-2\beta-1)\gamma_\lambda\hat{q}^{-2\beta}\hat{p}^{-2(\alpha+\beta)-1}\hat{p}_\lambda\hat{q}^{-2\alpha-1} \\ &-2\beta(D-2\alpha-2\beta-1)\gamma_\lambda\hat{q}^{-2\beta-2}\hat{q}_\lambda\hat{p}^{-2(\alpha+\beta)-1}\hat{q}^{-2\alpha-1}\,. \end{aligned} \tag{61}$$

Then we use the various position space matrix elements listed in the Appendix of Ref. [16] to arrive at

$$\begin{aligned}\langle x|\Gamma'|y\rangle &= \frac{\Gamma(D/2-\alpha-\beta+1/2)}{\pi^{D/2}2^{2\alpha+2\beta-1}\Gamma(\alpha+\beta+1/2)} \\ &\times\frac{(D/2-\beta-1)x^2\not{y}-(D/2-\alpha-1/2)x^2\not{x}+2\beta x\cdot y\not{x}}{x^{2\beta+2}|x-y|^{D-2\alpha-2\beta+1}y^{2\alpha+1}}\,. \end{aligned} \tag{62}$$

For $\Gamma_\mu^{\rm tr}$, given in Eq. (21), we follow Sec. 2 of Ref. [17], and first split it into two parts:

$$\Gamma_\mu^{\rm tr} = \Gamma_\mu^{\rm tr(1)} + \Gamma^{\rm tr(2)}\,, \tag{63}$$

$$\Gamma_\mu^{\rm tr(1)} = 2\gamma_\lambda\hat{p}_\lambda\hat{p}^{-2\alpha-1}\hat{q}_\nu\hat{q}^{-2(\alpha+\beta)-1}\hat{p}^{-2\beta}\mathcal{P}_{\mu\nu}\,, \tag{64}$$

$$\Gamma_\mu^{\rm tr(2)} = -\gamma_\lambda\gamma_\rho\gamma_\nu\hat{p}_\lambda\hat{p}^{-2\alpha-1}\hat{q}_\rho\hat{q}^{-2(\alpha+\beta)-1}\hat{p}^{-2\beta}\mathcal{P}_{\mu\nu} \tag{65}$$

with $\mathcal{P}_{\mu\nu}=\delta_{\mu\nu}-\hat{p}_\mu\hat{p}_\nu\hat{p}^{-2}$. Following the steps in Eqs. (6)-(12) of Ref. [17], we have

$$\Gamma_\mu^{\rm tr(1)} = 2\gamma_\lambda(\hat{q}^{-2\beta}+i2\beta\hat{q}_\lambda\hat{q}^{-2\beta-2})\hat{p}^{-2(\alpha+\beta)-1}\hat{q}^{-2\alpha-1}\hat{q}_\nu\mathcal{P}_{\mu\nu} \tag{66}$$

$$\begin{aligned}\Gamma_\mu^{\rm tr(2)} &= (-\gamma_\lambda\gamma_\rho\gamma_\nu\hat{q}_\rho\hat{q}^{-2\beta}\hat{p}_\lambda\hat{p}^{-2(\alpha+\beta)-1}\hat{q}^{-2\alpha-1} \\ &+i(D-2\alpha-2\beta-1)\gamma_\nu\hat{q}^{-2\beta}\hat{p}^{-2(\alpha+\beta)-1}\hat{q}^{-2\alpha-1})\mathcal{P}_{\mu\nu}\,. \end{aligned} \tag{67}$$

These equations give

$$\begin{aligned}\langle x|\Gamma_\mu^{\rm tr}|y\rangle &= \frac{i\Gamma(D/2-\alpha-\beta+1/2)}{\pi^{D/2}2^{2\alpha+2\beta-1}\Gamma(\alpha+\beta+1/2)} \\ &\times\left(\delta_{\mu\nu}-\frac{\partial_\mu^y\partial_\nu^y}{(\partial^2)^y}\right)\frac{\frac{1}{2}x^2(\not{x}-\not{y})\gamma_\nu\not{y}+\frac{\beta}{D/2-\alpha-\beta-1/2}(x-y)^2\not{x}y_\nu}{x^{2\beta+2}|x-y|^{D-2\alpha-2\beta+1}y^{2\alpha+1}}\,. \end{aligned} \tag{68}$$

Now we can put Eqs. (58), (62) and (68) together to obtain an expression for the position space matrix element of Γ_μ of Eq. (23). *A crucial step is to perform ∂_ν^y in Eq. (68), so that $\partial_\mu^y/(\partial^2)^y$ in Eq. (68) and in Eq. (58) can be taken together.* (Whereas in Refs. [16] and [17], the transverse and longitudinal parts were kept separate.) We then end up with

$$\frac{\partial_\mu^y}{(\partial^2)^y}\frac{(2\alpha-1)(D-2\alpha-1)x^2+2\beta((D-2\beta-2)y^2-2(2\alpha-1)x\cdot y)}{|x-y|^{D-2\alpha-2\beta+1}y^{2\alpha+1}}\,. \tag{69}$$

in the expression for $\langle x|\Gamma_\mu|y\rangle$. Now we use the relation

$$(\partial^2)^y\frac{1}{|x-y|^m y^n} = \frac{n(n-D+2)x^2+(m+n-D+2)((m+n)y^2-2nx\cdot y)}{|x-y|^{m+2}y^{n+2}}\,. \tag{70}$$

The expression (69) is therefore equal to $\partial_\mu^y(1/(|x-y|^{D-2\alpha-2\beta-1}y^{2\alpha-1}))$. Note that *the conformal invariant propagator $g_{\mu\nu}(x)/r^{D-2\beta}$ ensures that $\Gamma_\mu^{\rm tr}$ and $\Gamma_\mu^{\rm long}$ are added in the precise proportion so that $1/(\partial^2)^y$ can be taken care of.* After this, it is straightforward to arrive at Eq. (25).

E Some important relations

The star-triangle relation involving three scalar fields is given by [13, 14]

$$\int d^D x_4 \; (x_{14}^2)^{-\delta_1}(x_{24}^2)^{-\delta_2}(x_{34}^2)^{-\delta_3} \;=\; \pi^{D/2}\frac{\Gamma(D/2-\delta_1)\Gamma(D/2-\delta_2)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\delta_3)} \times(x_{12}^2)^{-D/2+\delta_3}(x_{13}^2)^{-D/2+\delta_2}(x_{23}^2)^{-D/2+\delta_1}\,, \tag{71}$$

where

$$\delta_1+\delta_2+\delta_3 = D\,. \tag{72}$$

The star-triangle relation for the Yukawa theory, involving two spinors and one scalar field, is given by [14, 1, 16]

$$\begin{aligned}&\int d^D x_4 \; \frac{\not x_{14}}{(x_{14}^2)^{\delta_1+1/2}}\,\frac{\not x_{42}}{(x_{24}^2)^{\delta_2+1/2}}\,\frac{1}{(x_{34}^2)^{\delta_3}}\\ &= \pi^{D/2}\frac{\Gamma(D/2-\delta_1+1/2)\Gamma(D/2-\delta_2+1/2)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1+1/2)\Gamma(\delta_2+1/2)\Gamma(\delta_3)}\\ &\quad\times\frac{\not x_{13}}{(x_{13}^2)^{D/2-\delta_2+1/2}}\,\frac{\not x_{32}}{(x_{23}^2)^{D/2-\delta_1+1/2}}\,\frac{1}{(x_{12}^2)^{D/2-\delta_3}}\end{aligned} \tag{73}$$

where Eq. (72) holds again. An analogous relation involving two scalars and one vector field is [1]

$$\begin{aligned}&\int d^D x_4 \; (x_{14}^2)^{-\delta_1}(x_{24}^2)^{-\delta_2}(x_{34}^2)^{-\delta_3} g_{\mu\nu}(x_{14})\lambda_\nu^{x_4}(x_2x_3)\\ &= \pi^{D/2}(D-\delta_1-1)\frac{\Gamma(D/2-\delta_1)\Gamma(D/2-\delta_2)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)\Gamma(\delta_3+1)}\\ &\quad\times(x_{12}^2)^{-D/2+\delta_3+1}(x_{13}^2)^{-D/2+\delta_2+1}(x_{23}^2)^{-D/2+\delta_1}\;\lambda_\mu^{x_1}(x_2x_3)\end{aligned} \tag{74}$$

where $\delta_1+\delta_2+\delta_3 = D-1$. Eq. (74) can be obtained by using the identity [1]

$$g_{\mu\nu}(x_{14})\lambda_\nu^{x_4}(x_2x_3) = \frac{x_{12}^2}{x_{24}^2}\lambda_\mu^{x_1}(x_2x_4) - \frac{x_{13}^2}{x_{34}^2}\lambda_\mu^{x_1}(x_3x_4) \tag{75}$$

and the relation [1]

$$\begin{aligned}&\int d^D x_4 \; (x_{14}^2)^{-\delta_1}(x_{24}^2)^{-\delta_2}(x_{34}^2)^{-\delta_3}\lambda_\mu^{x_1}(x_2x_4)\\ &= \pi^{D/2}\frac{\Gamma(D/2-\delta_1)\Gamma(D/2-\delta_2+1)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1+1)\Gamma(\delta_2)\Gamma(\delta_3)}\\ &\quad\times(x_{12}^2)^{-D/2+\delta_3}(x_{13}^2)^{-D/2+\delta_2}(x_{23}^2)^{-D/2+\delta_1}\lambda_\mu^{x_1}(x_2x_3)\,,\end{aligned} \tag{76}$$

where Eq. (72) holds. [Eq. (76), in turn, follows from Eq. (71).]

References

[1] E. S. Fradkin and M. Ya. Palchik, Phys. Rep. **44** (1978) 249.

[2] E. S. Fradkin and M. Ya. Palchik, Phys. Rep. **300** (1998) 1.

[3] E. S. Fradkin and M. Ya. Palchik, Conformal Quantum Field Theory in D-Dimensions, Kluwer, Dordrecht, 1996.

[4] I. Mitra, R. Ratabole and H. S. Sharatchandra, Phys. Lett. **B611** (2005) 289 [arXiv:hep-th/0410120].

[5] I. Mitra, R. Ratabole and H. S. Sharatchandra, Phys. Lett. **B634** (2006) 557 [arXiv:hep-th/0510055].

[6] M. Ya. Palchik, J. Phys. **A16** (1983) 1523.

[7] E.S. Fradkin, A.A. Kozhevnikov, M.Ya. Palchik and A.A. Pomeransky, Commun. Math. Phys. **91** (1983) 529.

[8] H. Georgi, Phys. Rev. Lett. **98** (2007) 221601 [arXiv:hep-ph/0703260].

[9] H. Georgi and Y. Kats, JHEP **1002** (2010) 065 [arXiv:0904.1962v3 [hep-ph]].

[10] S. Ferrara, A. F. Grillo, G. Parisi and R. Gatto, Lett. Nuovo Cim. **4** (1972) 115.

[11] S. Ferrara and G. Parisi, Nucl. Phys. **B42** (1972) 281.

[12] G. Cvetic, I. Kondrashuk, A. Kotikov and I. Schmidt, Int. J. Mod. Phys. **A22** (2007) 1905 [arXiv:hep-th/0604112]; G. Cvetic and I. Kondrashuk, JHEP **0802** (2008) 023 [arXiv:hep-th/0703138]; G. Cvetic and I. Kondrashuk, Int. J. Mod. Phys. **A23** (2008) 4145 [arXiv:0710.5762 [hep-th]].

[13] M. D'Eramo, L. Peliti and G. Parisi, Lett. Nuovo Cim. **2** (1971) 878.

[14] K. Symanzik, Lett. Nuovo Cim. **3** (1972) 734.

[15] A. P. Isaev, Nucl. Phys. **B662** (2003) 461 [arXiv:hep-th/0303056].

[16] I. Mitra, J. Phys. **A41** (2008) 315401 [arXiv:0803.2630 [hep-th]].

[17] I. Mitra, J. Phys. **A42** (2009) 035404 [arXiv:0808.2448 [hep-th]].

[18] In the process of amputing the spinor leg in $C_{1\mu}^{d,l,\Delta}$, the structure $g_{\mu\nu}(x)/(x^2)^{\delta_3}$ automatically comes out (see Eq. (28)). So we need to have this propagator in the star-triangle relation instead of the covariant gauge propagator. It may be mentioned that the transverse part of the covariant gauge propagator can be directly related to the form $g_{\mu\nu}(x)/(x^2)^{\delta_3}$ only by a careful regularization: see Eqs. (2.31)-(2.33) of Ref. [2].

[19] See p. 17-19 and, in particular, Eq. (1.84) of Ref. [2]; here a dot on an internal line denotes amputation.