Entanglement teleportation via Bell Mixture

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Abstract

We investigate the teleportation of the bipartite entangled states through two equally noisy quantum channels, namely mixture of Bell states. There is a particular mixed state channel for which all pure entanglement in a known Schmidt basis remain entangled after teleportation and it happens till the channel state remains entangled. Werner state channel lacks both these features. The relation of these noisy channels with violation of Bell's inequality and 2-E inequality is studied.

1 Introduction

Quantum teleportation is a well known phenomenon, proposed by Bennett *et al.* (BBCJPW) [1], in which an unknown state can be sent exactly, without being physically transported, to a distant party by local operations and clasical communication. For this purpose an entangled state (in fact a maximally entangled state) is required as teleportation channel between the sender and the receiver [2]. Quite recently, transfer of entanglement has been studied in the context of quantum teleportation of an unknown *entangled* state

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through two equally noisy quantum channels [3]. Here by teleportation of entanglement we mean after teleportation the final two qubit state remains entangled, whatever the reduced density matrices of the final two qubit state be. An arbitrary two-qubit state ρ_{12} was taken as the unknown entangled state which was to be sent, perhaps inexactly, to B_1B_2 where there were two noisy channels, one between A_1 and B_1 and another between A_2 and B_2. The same entangled state, namely the Werner state [4], was taken for both channels and it was shown that entanglement of the unknown state can be completely lost during the teleportation even when the channel is quantum correlated. At this point one should note the point that the Werner channel would stop teleporting entanglement depends solely on the maximally entangled fraction (MEF) [5] of the channel and irrelevant of whether the entangled state to be teleported is known or unknown. For a given Werner channel, it starts teleporting pure entanglement (i.e final state remain entangled after being teleported) if it is sufficiently entangled i.e MEF of the Werner state is greater than $1/\sqrt{3}$ ($MEF > 1/\sqrt{3}$). There is no Werner state channel which can teleport all the entanglement known or unknown. In this regard we search for some mixed channel which can teleport all *pure* entanglement where Schmidt basis of states is known. Here by "all entanglement", we mean a class of states whose entanglement ranges continuously from 0 to 1, with respect to some measure.

In this paper, we study the teleportation of pure entanglement in the above sense where Bell mixtures are considered as cahnnel state. We provide sufficient conditions, the channel state has to satisfy for teleporting at least one pure entanglement as well as full range of pure entanglement. We also study how this sufficient condition is related to violation of Bell's inequality [6] and 2-E inequality [7]. it is shown that there really exist such mixed state channels (mixture of two Bell states) which can teleport all *pure* entanglement from the plane spanned by an arbitrary but fixed Schmidt basis. Another important feature of this channel is that,unlike the Werner channel, it can teleport entanglement as long as it itself remains entangled.

2 Bennett protocol and mixture of Bell states

Consider the following situation. A_1 and B_1 are sharing a state which is a mixture of two Bell states

$$wP[|\Psi^+\rangle] + (1-w)P[|\Psi^-\rangle],$$

where $|\Psi^\pm\rangle = (1/\sqrt{2})(|01\rangle \pm |10\rangle)$ and $0 \leq w \leq 1$. A_2 and B_2 are also sharing the same state. Now A_1, A_2 are sharing an entangled state of two particle 1 and 2 respectively, which they want to telport through these noisy channels to B_1, B_2. Let the state shared by A_1, A_2 be $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$ (with $|\alpha|^2 + |\beta|^2 = 1$). Now A_1, B_1 follow a teleportation protocol (from now on which we call as $\mathcal{P}_{|\Psi^+\rangle}$) and A_1 wants to teleport the state of 1 to B_1 [8] by that protocol. The same protocol is used by A_2 and B_2. Finally the state shared between B_1, B_2 is given by

$$\begin{bmatrix} |\alpha|2 & 0 & 0 & (2w-1)^2\alpha\beta^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (2w-1)^2\alpha^*\beta & 0 & 0 & |\beta|^2 \end{bmatrix}$$

which is inseparable whenever $w \neq 1/2$. Interestingly, only for $w = 1/2$ the channel itself becomes separable. Note that each of the reduced density matrices of the final state is same as the corresponding ones of the initial state. Thus the mapping $\tilde{V}$ from Hilbert-Schmidt space of particle 1(2) to that of particle $B_1(B_2)$ is given by $\tilde{V} : I \mapsto I$, $\tilde{V} : \sigma_z \mapsto \sigma_z$. We provide an example of a mixed state as channel which can teleport any pure entanglement where the Schmidt basis of such states is known, unless the channel itself becomes disentangled.

We remember that Werner channels lack both these features: firstly Werner channels cannot teleport all entanglement and secondly it can teleport entanglement only when it is sufficiently entangled (when $MEF > 1/\sqrt{3}$).

It can be easily shown that all entanglement in a unknown Schmidt basis can not be teleported through this kind of channel.

3 Sufficient condition for teleportation of entanglement

Next we consider the most general channel state between A_i and B_i $(i = 1, 2)$ for which there exists a teleportation protocol which maps the basis operators of Hilbert-Schmidt space of one particle to the Hilbert-Schmidt space of another particle in the following way:

$$\tilde{V} : I \mapsto I,\ \tilde{V} : \sigma_i \mapsto \lambda_i \sigma_i\ (i = x, y, z),$$

where λ_i's are real numbers s.t. $|\lambda_i| \leq 1$. Now we show that when $\lambda_x^2 + \lambda_y^2 + \lambda_z^2 > 1$, the channel can teleport at least one entangled state. Consider the maximally entangled state $(1/\sqrt{2})(|00\rangle + |11\rangle)$, $|0\rangle, |1\rangle$ being eigenstates of σ_z. After the teleportation, the state becomes

$$\frac{1}{4}\begin{bmatrix} 1+\lambda_z^2 & 0 & 0 & \lambda_x^2+\lambda_y^2 \\ 0 & 1-\lambda_z^2 & \lambda_x^2-\lambda_y^2 & 0 \\ 0 & \lambda_x^2-\lambda_y^2 & 1-\lambda_z^2 & 0 \\ \lambda_x^2+\lambda_y^2 & 0 & 0 & 1+\lambda_z^2 \end{bmatrix}$$

which remains entangled if and only if

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 > 1.$$

Let us provide an example of a channel state and a teleportation protocol which realizes the map $\tilde{V}$. If one takes the teleportation protocol to be $\mathcal{P}_{|\Psi^+\rangle}$ [8] and takes the channel state as a mixture of four Bell states, then the map $\tilde{V}$ is realised through this teleportation process in the following way [9]: Let the channel state be $\rho_{\text{ch}} = w_1 P[\Psi^+] + w_2 P[\Psi^-] + w_3 P[\Phi^+] + w_4 P[\phi^-]$, ($w_i \geq 0$, $i = 1, 2, 3, 4$ and $\sum_{i=1}^4 w_i = 1$) and if one follows the above mentioned teleportation protocol, one realises the map $\tilde{V}$ with $\lambda_x = w_1 - w_2 + w_3 - w_4$, $\lambda_y = w_1 - w_2 - w_3 + w_4$, $\lambda_z = w_1 + w_2 - w_3 - w_4$.

It is clear that just nonzero entanglement of channel state ρ_{ch} does not guarantee entanglement teleportation (see for example, ref. [3]). In the absence of any necessary condition on the channel state for teleporting pure entanglement, we inquire for some sufficient physical condition on the channel state, namely its relation with violation of Bell's inequality [6] and 2–E inequality [7].

(1) The channel state (Bell Mixture) between A_i and B_i ($i = 1, 2$) can teleport at least one pure entanglement if it violates Bell's inequality. Following Horodecki's result [6] a two qubit state ρ violates Bell's inequality iff the sum of two greatest eigenvalues of $T_\rho^T T_\rho$ is greater than 1, where T_ρ is the T matrix of ρ in the standard Hilbert-Schmidt representation [6].

Now the state $\rho_{\rm ch}$ violates Bell's inequality iff

$$\max[[\{2(w_1 + w_3) - 1\}^2 + \{2(w_1 + w_4) - 1\}^2], [\{2(w_1 + w_3) - 1\}^2 + \{2(w_3 + w_4) - 1\}^2],$$

$$[2(w_4 + w_3) - 1^2 + 2(w_1 + w_4) - 1^2]] > 1,$$

i.e.

$$\max[(\lambda_x^2 + \lambda_y^2), (\lambda_z^2 + \lambda_y^2), (\lambda_x^2 + \lambda_z^2)] > 1,$$

which is stronger than $\lambda_x^2 + \lambda_y^2 + \lambda_z^2 > 1$.

(2) The channel state can teleport some pure entangelement if it violates the 2-E inequality [7].

2-E inequality for bipartite density matrix is given by

$$S_2(\rho_{12}) \geq \max\{S_2(\rho_1), S_2(\rho_2)\}$$

where ρ_1 and ρ_2 are the reduced density matrices of ρ_{12} and

$$S_2(\rho) = -\ln{\rm Tr}\rho^2$$

One can easily check that the state $\rho_{\rm ch}$ violates the 2-E inequality [7] iff $\lambda_x^2 + \lambda_y^2 + \lambda_z^2 > 1$. Thus if the channel state $\rho_{\rm ch}$ violates 2-E inequality at least one entanglement can be teleported through this channel.

One should note that for Werner state channel (*i.e.*, $w_2 = w_3 = w_4$), violation of 2-E inequality is necessary and sufficient condition for teleporting at least one entanglement.

4 Teleportation of All Pure Entanglement

Now we consider the special case where any one of the λ's say, $\lambda_z = 1$ and the rest, *i.e.*, λ_x, $\lambda_y \neq 0$. And we want to teleport the state $\alpha|00\rangle + \beta|11\rangle$ (where $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle =$

$-|1\rangle$ and $|\alpha|^2 + |\beta|^2 = 1$) by using our teleportation protocol $\mathcal{P}_{|\Psi^+\rangle}$, given by the corresponding map $\tilde{V}$. So after teleportation, this state becomes

$$\begin{bmatrix} |\alpha|^2 & 0 & 0 & \frac{\alpha^*\beta(\lambda_x-\lambda_y)^2+\alpha\beta^*(\lambda_x+\lambda_y)^2}{4} \\ 0 & 0 & \frac{\mathrm{Re}\{\alpha\beta^*\}(\lambda_x^2-\lambda_y^2)}{2} & 0 \\ 0 & \frac{\mathrm{Re}\{\alpha\beta^*\}(\lambda_x^2-\lambda_y^2)}{2} & 0 & 0 \\ \frac{\alpha^*\beta(\lambda_x+\lambda_y)^2+\alpha\beta^*(\lambda_x-\lambda_y)^2}{4} & 0 & 0 & |\beta|^2 \end{bmatrix}$$

which is entangled for all nonzero $\alpha,\ \beta$. We now show that a mixture of two Bell states is the only solution of the map $\tilde{V}$ in this special case.

Consider the channel state as $\rho_{\text{ch}} = w_1 P[\Psi^+] + w_2 P[\Psi^-] + w_3 P[\Phi^+] + w_4 P[\Phi^-]$ where $w_1 + w_2 + w_3 + w_4 = 1$ and $w_1, w_2, w_3, w_4 \geq 0$. If we use the protocol $\mathcal{P}_{|\Psi^+\rangle}$, then the mapping for σ_z is $\sigma_z \mapsto (w_1 + w_2 - w_3 - w_4)\sigma_z$. So $\lambda_z = w_1 + w_2 - w_3 - w_4 = 1$ gives us $w_1 + w_2 = 1$ and $w_3 = w_4 = 0$. Again, for the same channel $\lambda_x (= w_1 - w_2 - w_3 + w_4)$ and $\lambda_y (= w_1 - w_2 + w_3 - w_4)$, both has to be nonzero. And so $w_1 \neq w_2$. Hence mixtures of two Bell states with unequal weights are the only solutions. The other feature to be noted is that after teleportation through this type of channel, the marginal density matrices of the pure states of type $\alpha|00\rangle + \beta|11\rangle$, remain unchanged, whereas, in case of Werner channel the marginal density matrices of no entangled state remain unchanged.

Thus a sufficient condition that a channel realising the map ($\tilde{V} : I \mapsto I, \sigma_i \mapsto \lambda_i \sigma_i (i = x, y, z)$) can teleport all pure entanglement (in a known basis) is that one of λ_i's, say $\lambda_z = 1$ and $\lambda_x, \lambda_y \neq 0$.

5 Discussion

In this paper, we investigated the effects of noisy channels on the entanglement in entanglement teleportation. Here we have shown that the channel composed of mixture of two Bell states with unequal weight can teleport a full range of pure entanglement (although non-universally, i.e in a fixed basis), whereas the Werner channel can not teleport all such entanglements. In this respect one can easily see that to teleport any pure entanglement, the channel should violate 2-E inequality.

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