Entanglement of Formation is Non-monotonic with Concurrence-A simple proof

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In this paper we explore the non-monotonic nature of entanglement of formation with respect to concurrence for pure bipartite states. For pure bipartite system, one of the basic physical reason of this non-monotonicity character is due to the existence of incomparable states, i.e., the pure bipartite states which are not convertible to each other by LOCC with certainty.
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Concurrence is an widely applicable [1] measure for computing entanglement [2, 3] of different physical systems[4]. It is originally proposed to compute entanglement of formation [5, 6] for 2×2 systems. For such a system, Entanglement of Formation is proposed to be a function of concurrence and that function is monotonic in nature. Afterwards, the measure is generalized for pure as well as mixed bipartite states in arbitrary dimensions [7, 8, 9] and also for multi-partite composite systems [10, 11, 12].

Now, for pure bipartite states, entropy of entanglement is the unique measure of entanglement [13]. Consequently, it is equal to both the entanglement of formation and distillable entanglement of any pure bipartite state. Although, both entanglement of formation and distillable entanglement are the most important measures of entanglement, but only for some special class of states they are calculable (e.g., [14]). So, one has to probe the relations between some other measures of entanglement or non-local correlations with the above measures. Concurrence has some role with entanglement of formation. It is believed that entanglement of formation of a bipartite state is a monotonic function of concurrence [15]. It is strongly supported by the evidence of monotonicity character shown for the class of Isotropic states[16]. It appears to be surprising that for 2×2 states entanglement of formation is a monotonic function of concurrence. The generalization of this measure for higher dimensional systems and the corresponding scheme for computing entanglement of formation would not suggest us the non-monotonic character of entanglement of formation with concurrence. There is no definite physical reason for such behaviour. Here, we investigate a possible root of this feature with the existence of incomparable pair of entangled states in pure bipartite systems. Further, we have found quite similar behaviour of negativity with entanglement of formation. Thus, our result would also establish the fact that both the concurrence and negativity have limited scope beyond some lower dimensional bipartite systems and are inadequate measures of entanglement.

The notion of incomparability evolve through the inter conversion of pure bipartite states under LOCC with certainty [17]. Two pure bipartite states are said to be comparable if one of them can be deterministically transformed into the other by LOCC [17]. To respect the physical restriction of non-increase of entanglement under LOCC, for a pair of comparable states $(|\Psi\rangle,\ |\Phi\rangle)$, $E(|\Psi\rangle) \geq E(|\Phi\rangle)$. But for incomparable pair of pure bipartite states, the relation is not straight forward. Recently, we have shown that if a pair of pure bipartite states of same Schmidt rank have the same entropy of entanglement, then either they are incomparable or locally connected. There exists an infinite number of mutually incomparable states of same Schmidt rank all having the same entanglement [18]. Therefore, entropy of entanglement as the unique measure for pure bipartite states, fails to express all the non-local character present within the system. Here, we shall establish the direct relation between the existence of incomparable states and the non-monotonic character of entanglement of formation with respect to concurrence. Let us first consider the following definition of generalized concurrence.

The generalized definition of concurrence for a mixed bipartite state ρ_{AB} in the joint space $H_A \otimes H_B$ of two finite dimensional Hilbert spaces H_A, H_B shared between two parties A, B is defined by,

$$C(\rho_{AB}) = \sqrt{2(1 - Tr{\rho_A}^2)} \tag{1}$$

where the reduced density matrix ρ_A is obtained by tracing over the subsystem B. Here it is to be noticed that the quantity $Tr{\rho_A}^2$ gives a measure of purity of the reduced state ρ_A. We assume that a pure $m \times n$ $(m \geq n)$

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bipartite state has the standard Schmidt form $|\Psi\rangle_{AB} = \sum_i \sqrt{\mu_i}|a_i\rangle_A|b_i\rangle_B$, with real valued Schmidt coefficients $\{\mu_i \in [0,1]\ ;\ \ i = 1, 2, \cdots, k(k \leq \min\{m, n\})\}$ and $\{|a_i\rangle_A\}$, $\{|b_i\rangle_B\}$ are the orthonormal bases for subsystems H_A and H_B respectively. The integer k is the number of non-zero Schmidt terms of the state $|\Psi\rangle_{AB}$. Then the concurrence $C(|\Psi\rangle_{AB})$ for this state is given by,

$$C^2(|\Psi\rangle_{AB}) = 4\sum_{i<j}\mu_i\mu_j = 2(1 - \sum_{i=1}^{k}\mu_i^2) \tag{2}$$

which varies smoothly from 0 for pure product states to $2\frac{k-1}{k}$ for maximally entangled pure states of Schmidt rank k. Then, we have the following result for any pair of comparable pure bipartite states.

Theorem: *For comparable pure bipartite states, entanglement of formation is monotone with concurrence.*

Let $|\Psi\rangle$, $|\Phi\rangle$ be any two comparable pure bipartite states of Schmidt rank m and n respectively, with Schmidt vectors, $\lambda_{|\Psi\rangle} = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $\lambda_{|\Phi\rangle} = (\beta_1, \beta_2, \cdots, \beta_n)$, where $\alpha_i \geq \alpha_{i+1} \geq 0$ and $\beta_i \geq \beta_{i+1} \geq 0$ and $\sum_{i=1}^{m}\alpha_i = 1 = \sum_{i=1}^{n}\beta_i$. Suppose, $|\Psi\rangle \longrightarrow |\Phi\rangle$ is possible under deterministic LOCC, then, we must have $m \geq n$ with $E(|\Psi\rangle) \geq E(|\Phi\rangle)$ and $E(|\Psi\rangle) = E(|\Phi\rangle)$ if and only if $\lambda_{|\Psi\rangle} \equiv \lambda_{|\Phi\rangle}$[18]. We rewrite the Schmidt vector of $|\Phi\rangle$ as $\lambda_{|\Phi\rangle} = (\beta_1, \beta_2, \cdots, \beta_m)$ with $\beta_i = 0\ \ \forall\ \ i = n+1, \ldots, m$. Now from Nielsen's criteria[17] for deterministic transformation of pure bipartite states under LOCC, $|\Psi\rangle \longrightarrow |\Phi\rangle$ implies $\lambda_{|\Psi\rangle} \prec \lambda_{|\Phi\rangle}$ (where, $\prec$ is the symbol for majorization of two real vectors), explicitly,

$$\textstyle\sum_{i=1}^{k}\alpha_i \leq \sum_{i=1}^{k}\beta_i, \quad \forall\ \ k = 1, 2, \ldots, m-1. \tag{3}$$

Then there exists $\{\epsilon_k > 0\ \ ;\ \ k = 1, 2, \ldots, m-1\}$ such that

$$\sum_{i=1}^{k}\beta_i = \sum_{i=1}^{k}\alpha_i + \epsilon_k, \ \ \forall\ k = 1, 2, \ldots, m-1. \tag{4}$$

Subtracting each equation from its next one we have

$$\beta_i = \alpha_i + \epsilon_i - \epsilon_{i-1}, \ \ \forall\ k = 1, 2, \ldots, m \tag{5}$$

assuming $\epsilon_0 = 0 = \epsilon_m$.

Comparing the values of concurrences for those two states, we have

$$\begin{aligned} C^2(|\Psi\rangle) - C^2(|\Phi\rangle) &= 2(1 - \textstyle\sum_{i=1}^{m}{\alpha_i}^2) - 2(1 - \sum_{i=1}^{m}{\beta_i}^2) \\ &= 2\textstyle\sum_{i=1}^{m}({\beta_i}^2 - {\alpha_i}^2) \\ &= 2\textstyle\sum_{i=1}^{m}(\beta_i + \alpha_i)(\beta_i - \alpha_i) \\ &= 2\textstyle\sum_{i=1}^{m}(2\alpha_i + \epsilon_i - \epsilon_{i-1})(\epsilon_i - \epsilon_{i-1}) \\ &= 2\{2\textstyle\sum_{i=1}^{m}\alpha_i(\epsilon_i - \epsilon_{i-1}) + \sum_{i=1}^{m}(\epsilon_i - \epsilon_{i-1})^2\} \\ &= 2\{2(\textstyle\sum_{i=1}^{m-1}\alpha_i\epsilon_i - \sum_{i=1}^{m-1}\alpha_{i+1}\epsilon_i) \\ &\qquad + \textstyle\sum_{i=1}^{m}(\epsilon_i - \epsilon_{i-1})^2\} \qquad [\because\ \epsilon_0 = \epsilon_m = 0] \\ &= 2\{2\textstyle\sum_{i=1}^{m-1}(\alpha_i - \alpha_{i+1})\epsilon_i + \sum_{i=1}^{m}(\epsilon_i - \epsilon_{i-1})^2\} \\ &\geq 0 \qquad [\because\ \ \alpha_i \geq \alpha_{i+1}\ \ \forall\ \ i = 1, 2, \ldots, m-1]. \end{aligned} \tag{6}$$

Thus for any pair of comparable pure bipartite states we find a direct relation between entropy of entanglement and concurrence, i.e., $E(|\Psi\rangle) \geq E(|\Phi\rangle)$ implies $C(|\Psi\rangle) \geq C(|\Phi\rangle)$. Also, for comparable states $E(|\Psi\rangle) = E(|\Phi\rangle)$ implies $\lambda_{|\Psi\rangle} \equiv \lambda_{|\Phi\rangle}$ which further imply, $C(|\Psi\rangle) = C(|\Phi\rangle)$. Thus for strict relation, $E(|\Psi\rangle) > E(|\Phi\rangle)$ we have $\lambda_{|\Psi\rangle} \prec \lambda_{|\Phi\rangle}$ with $\lambda_{|\Psi\rangle} \not\equiv \lambda_{|\Phi\rangle}$, i.e., for at least one value of i, $\epsilon_i > 0$ and ultimately it implies, $C(|\Psi\rangle) > C(|\Phi\rangle)$.

So, for comparable set of pure bipartite states, entanglement of formation(which is equal to the entropy of entanglement), is monotone with concurrence. This result is quite compatible with the case of 2×2 states where entanglement of formation is always a monotonic function of concurrence and all pure states of 2×2 system are comparable. From 3×3 composite systems the Hilbert space structure is so much complicated that one could really understand the full nature of entangled states. Recently, we found that in the neighborhood of one pure state of higher Schmidt rank (≥ 3) their exists an infinite number of other pure states of the same rank which are all incomparable with the state and all have the same value of entropy of entanglement [18]. This feature readily shows that entanglement of formation is not a monotone function of concurrence. To explain it, we first consider the following example.

Example: Consider a pair of pure bipartite states of 3×3 system represented by the corresponding Schmidt vectors as,

$$\begin{aligned} |\Psi_1\rangle &\equiv (.46, .306, .234), \\ |\Phi_1\rangle &\equiv (.43, .3646, .2054) \end{aligned} \tag{7}$$

The concurrences for these states are, $C(|\Psi_1\rangle) \simeq 1.280016$ and $C(|\Phi_1\rangle) \simeq 1.279955$. We compute the value of entanglement of formation for these states by their entropy of entanglement, which are, $E(|\Psi_1\rangle) \simeq 1.528432837$, $E(|\Phi_1\rangle) \simeq 1.52331025$. Next perturbing slightly the Schmidt coefficients, we choose another neighboring pair of pure bipartite states of 3×3 system as

$$\begin{aligned} |\Psi_2\rangle &= (.43, .3645, .2055), \\ |\Phi_2\rangle &= (.46, .3061, .2339) \end{aligned} \tag{8}$$

For this new pair the calculated values of concurrences are $C(|\Psi_2\rangle) \simeq 1.280019$ and $C(|\Phi_2\rangle) \simeq 1.27998716$. Whereas the values of the entropy of entanglement for these states are, $E(|\Psi_2\rangle) \simeq 1.523392983$, $E(|\Phi_2\rangle) \simeq 1.52839408$.

Both the pairs $(|\Psi_1\rangle,\ |\Phi_1\rangle)$ and $(|\Psi_2\rangle,\ |\Phi_2\rangle)$ are incomparable in nature. So, the implicit effect of presence of incomparability is obviously reflected here. For the first pair of pure bipartite states, $(|\Psi_1\rangle,\ |\Phi_1\rangle)$ we see $C(|\Psi_1\rangle) > C(|\Phi_1\rangle)$ and $E(|\Psi_1\rangle) > E(|\Phi_1\rangle)$, while for the second pair of pure bipartite states $(|\Psi_2\rangle,\ |\Phi_2\rangle)$, we see $C(|\Psi_2\rangle) > C(|\Phi_2\rangle)$, but $E(|\Psi_2\rangle) < E(|\Phi_2\rangle)$. We could set many such examples as there are infinite number of pure bipartite states of same Schmidt rank which are incomparable, having the same value of entropy of entanglement. Then with a slight perturbation we may construct infinite number of states which have approximately the same value of concurrence. For a better understanding we plot the graphs of concurrences for three equi-entangled classes. By an equi-entangled class we mean all states of Schmidt rank 3 having a specific value of entanglement. Here we present three different classes having entanglements 1.545 e-bit, 1.547 e-bit, 1.550 e-bit and plotted the graphs of concurrences of those states against the largest Schmidt coefficients of the states.

FIGURE

The figure shows that for the states $|\Psi_A\rangle,\ |\Phi_B\rangle,\ |\Psi_C\rangle,\ |\Psi_D\rangle$ represented by the points A, B, C, D, we have, $E(|\Psi_A\rangle) = E(|\Psi_B\rangle) < E(|\Psi_C\rangle) = E(|\Psi_D\rangle)$ and $C(|\Psi_A\rangle) = C(|\Psi_D\rangle)$, $C(|\Psi_B\rangle) = C(|\Psi_C\rangle)$. So for the pair of states $(|\Psi_A\rangle,\ |\Psi_C\rangle)$, we have, $C(|\Psi_A\rangle) > C(|\Psi_C\rangle)$ with $E(|\Psi_A\rangle) < E(|\Psi_C\rangle)$ and for the pair, $(|\Psi_B\rangle,\ |\Psi_D\rangle)$, $C(|\Psi_D\rangle) > C(|\Psi_B\rangle)$ with $E(|\Psi_D\rangle) > E(|\Psi_B\rangle)$. This does not contrary with the theorem, as states representing two distinct points(such as the points A and B) of an equi-entangled class, are necessarily incomparable to each other. Here we have plotted only three distinct equi-entangled classes. However, one could draw many such curves. It shows readily, the entanglement of formation is not a monotonic function of concurrence, even for the pure two-qutrit states. Clearly, this feature establishes the fact that the entanglement of formation, behaves completely random with the value of concurrence. This feature of entanglement with concurrence is also observed for other important measures, like, negativity.

Negativity [19] is a well known measure of entanglement, which is functionally related with concurrence for 2×2 and 2×3 systems and a lower bound of it. For a general bipartite mixed state ρ_{AB} it corresponds to the absolute value of the sum of negative eigenvalues of ρ^{T_A} (partial transpose of ρ_{AB} with respect to system A) [20]. It is defined as $N(\rho_{AB}) = \frac{\|\rho^{T_A}\|_1 - 1}{2}$. For the pure bipartite states, $|\Psi\rangle_{AB} = \sum_i \sqrt{\mu_i}|a_i\rangle_A|b_i\rangle_B$ with Schmidt vector $\lambda_{|\Psi\rangle_{AB}} = (\mu_1, \mu_2, \cdots, \mu_m)$, Negativity is given by $N(|\Psi\rangle_{AB}) = \frac{1}{2}((\sum_i \sqrt{\mu_i})^2 - 1)$. The measure is proposed to quantify the entanglement cost of preparing the state under PPT preserving operations [21]. Under numerical study we found, like concurrence it is also an inadequate measure of entanglement and behaves quite similarly with the entropy of entanglement. Considering different classes of equi-entangled states the monotonicity of entropy of entanglement with negativity also does not hold for general pure bipartite states. Thus, in essence, we found, both the negativity and concurrence have limited scope to demonstrate the entanglement behaviour of composite quantum systems.

In conclusion, we have established a physical reason for non-monotonicity of entanglement of formation with concurrence. Not merely the parametric behavior of entropy function for higher dimensional systems, but the incapability of transferring any pair of bipartite pure entangled states under deterministic LOCC has a fundamental role in such behaviour of two important measures of entanglement. Our proof is simple and the existence of so many incomparable states made it much easier to prove. We hope our result will provide a deep insight on the behaviour of different entanglement measures in composite quantum systems.

[1] M. A. Cirone, *Phys. Lett. A* **339** (2005) 269.
[2] W. K. Wootters, *Phys. Rev. Lett.* **80** (1998) 2245.
[3] W. K. Wootters, *Quantum Information and Computation* **1** (2001) 27.
[4] B. Ghosh, A. S. Majumdar and N. Nayak, *Phys. Rev. A* **74** (2007) 052315; A. Datta, B. Ghosh, A. S. Majumdar and N. Nayak, *Europhys. Lett.* 67 (2004) 934.

[5] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin and W. K. Wootters, *Phys. Rev. Lett.* **76** (1996) 722.
[6] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, *Phys. Rev. A* **54** (1996) 3824.
[7] K. Chen, S. Albeverio and Shao-Ming Fei, *Phys. Rev. Lett.* **95** (2005) 040504.
[8] F. Mintert, M. Kus and A. Buchleitner, *Phys. Rev. Lett.* **92** (2004) 167902.
[9] G. Gour, *Phys. Rev. A* **72** (2005) 042318.
[10] H. Heydari, *J. Phys. A: Math. Gen.* **38** (2005) 8667.
[11] H. Heydari, *J. Phys. A: Math. Gen.* **38** (2005) 11007.
[12] F. Mintert, M. Kus and A. Buchleitner, *Phys. Rev. Lett.* **95** (2005) 260502.
[13] S. Popescu and D. Rohrlich, *Phys. Rev. A* **56** (1997) R3319.
[14] B.M. Terhal and K.G.H. Vollbrecht, *Phys. Rev. Lett.* **85** (2000) 2625.
[15] K. Chen, S. Albeverio and S. M. Fei, *Phys. Rev. Lett.* **95** (2005) 210501.
[16] P. Rungta and C. M. Caves, *Phys. Rev. A* **67** (2003) 012307.
[17] M. A. Nielsen, *Phys. Rev. Lett.* **83** (1999) 436.
[18] I. Chattopadhyay and D. Sarkar, *Phys. Rev. A* **77**, (2008) 050305(R).
[19] G. Vidal and R. F. Werner, *Phys. Rev. A* **65** (2002) 032314.
[20] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
[21] S. Ishizaka, *Phys. Rev. A* **69** (2004) 020301.

Curve of Concurrence for three different equi−entangled classes