

Directed Accelerated Growth: Application in Citation Network

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Abstract

In many real world networks, the number of links increases nonlinearly with the number of nodes. Models of such accelerated growth have been considered earlier with deterministic and stochastic number of links. Here we consider stochastic accelerated growth in a network where links are directed. With the number of out-going links following a power law distribution, the results are similar to the undirected case. As the accelerated growth is enhanced, the degree distribution becomes independent of the “initial attractiveness”, a parameter which plays a key role in directed networks. As an example of a directed model with accelerated growth, the citation network is considered, in which the distribution of the number of outgoing link has an exponential tail. The role of accelerated growth is examined here with two different growth laws.

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The number of links shows a nonlinear growth in time in many real-world networks which evolve in time [1,2]. Examples of such network are the Internet [3], World Wide Web (WWW) [4], collaboration [5], word web [6], citation [7,8,9], metabolism [10], gene regulatory network [11,12,13] etc. The number of links may increase in two ways: new nodes may tend to get attached to more nodes as the size of the network increases, and there may be additional links generated between the older nodes in a non-linear fashion. These two factors maybe present either singly or simultaneously resulting in the accelerated growth. In some networks like the citation and the gene regulatory network, new links between older nodes are forbidden and therefore only the first scheme is valid, while in collaboration network or internet, the second factor is dominating.

In general, in a network with accelerated growth, the total number of links l shows a non-linear increase in time, i.e., $l(t) \sim t^\delta$ with $\delta > 1$. Such an acceleration will be possible when the number of links $m(\tau)$ at any time τ

available to the network is not a constant but increases in such a manner that $l(t) = \sum_{\tau=1}^t m(\tau)$ shows a non-linear increase with time.

The case when the new node gets a fixed number of links but older nodes get new links the number of which grows in a non-linear fashion has been considered in both isotropic and directed models of growing networks [5,6,14], showing that it is distinct from networks with a linear growth rule. Networks in which older nodes do not get new links have also been considered recently, with the possibility of a new node to have either deterministic or stochastic number of links [15].

Many real systems like the citation network and WWW are directed networks [9,16]. In the citation network, accelerated growth has been shown to take place [9]. This network also evolves in such a way that new links are forbidden between old nodes. We propose in this paper directed models of accelerated growth with number of outgoing links given by a particular distribution and where older nodes are not allowed to get linked. One of these models mimics the citation network.

Directed accelerated growth with power law distribution

As in [15], here we assume that in the growing network, the number of outgoing links m is determined by a power-law distribution, i.e.,

$$P(m) \sim m^{-\lambda} \tag{1}$$

with $1 \leq m \leq N(\tau)$ at any time τ when the number of nodes in the network is $N(\tau)$. The number of nodes is usually a linear function of time and $N(\tau) = \tau$ when nodes are added one by one to the network. λ is the parameter controlling the measure of accelerated growth. As a new node comes in, it will get preferentially attached to m existing nodes with the probability

$$\Pi \propto A + k_{in} \tag{2}$$

where k_{in} is the in-degree of the existing node and A is the initial attractiveness of the network. A is a parameter which is necessary in all directed networks to ensure that a node even with $k_{in} = 0$ has the chance to get linked with the new node. Without acceleration, when m is fixed, one gets a scale-free network with the the degree distribution $P(k) \sim k^{-\gamma}$ where [18]

$$\gamma = 2 + A/m. \tag{3}$$

With m varying (growing) in time, the nature of the degree distribution $P(k)$ is expected to change. This is obvious if one considers the average value of m which shows the following behaviour:

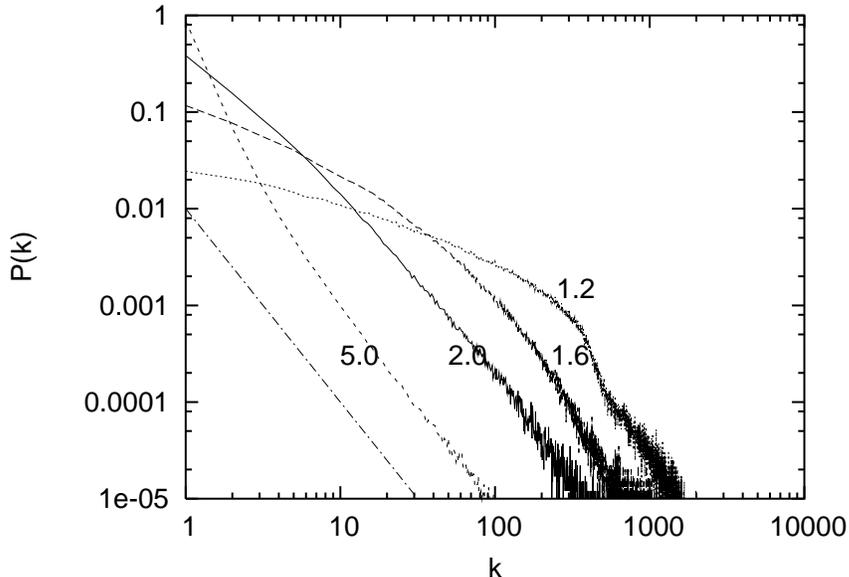


Fig. 1. The total degree distribution $P(k)$ for different values of λ (shown as labels of the curves) are plotted against k . All the data are for networks evolved upto 4000 nodes in this and in the other figures. The value of $A = 0.1$. The straightline has slope 2.0.

$$\begin{aligned}
 \langle m \rangle &= \text{const for } \lambda > 2 \\
 &\propto N^{2-\lambda} \text{ for } 1 < \lambda < 2 \\
 &\propto N \text{ for } \lambda < 1.
 \end{aligned}$$

For large values of λ , one should expect that the the distribution retains the features of a non-accelerated growing network. This is verified in the simulation. In Fig. 1, the data for $P(k)$ for different λ values are shown. With $A = 0.1$, γ should be close to 2 if equation (3) is still valid. The data indicates that the exponent is indeed close to 2 for $\lambda \geq 2$. However, for smaller values of λ , $P(k)$ is different from that of the non-accelerated case and not a power law at all. $P(k)$ is a stretched exponential function for $2 \geq \lambda \geq 1$. At $\lambda = 1$, $P(k)$ decays weakly with k and in fact shows a weak growth when λ is further decreased (curves (c) and (d) in Fig. 2). These results are very similar to the undirected model [15]. The fact that for $\lambda < 2$, $P(k)$ does not have a power-law decay shows that the randomness in m is relevant here.

From the simulations, one can directly obtain the in-degree distribution, which is the distribution of the number of incoming links to a node [17]. In Fig. 2, we have plotted the in-degree distribution alongwith the total degree distribu-

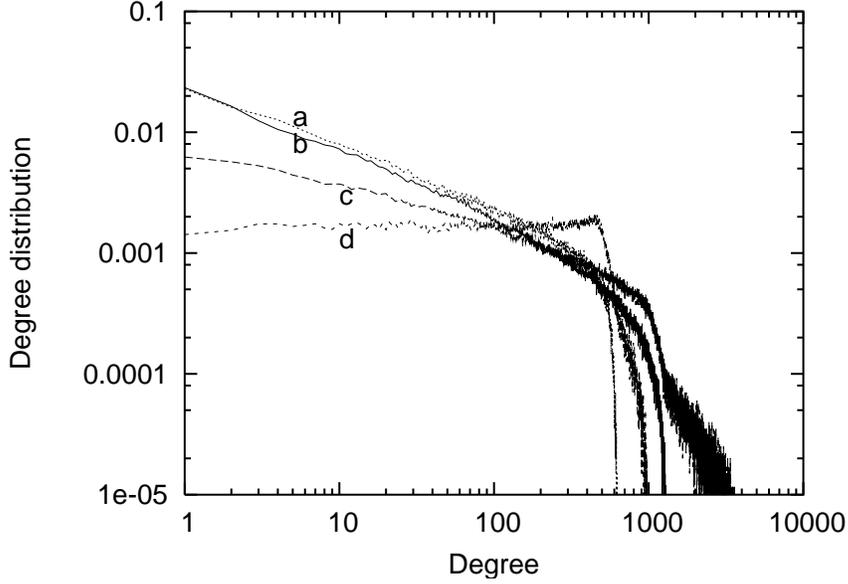


Fig. 2. The in-degree distributions are shown for (a) $\lambda = 0.6$ and (b) $\lambda = 1.0$. The total degree distributions are markedly different: (c) is for $\lambda = 1$ and (d) for $\lambda = 0.6$. $A = 0.1$ in this figure.

tion as they show noticeable difference when λ is less than 2. To explain the above feature, we note that if P_{in} and P_{out} are the in-degree and out-degree distributions respectively then the total degree distribution can be expressed as

$$P(k) = \sum_{k_1=1}^{k_1^{max}} P_{out}(k_1) P_{in}(k - k_1) \quad (4)$$

where k_1^{max} is the maximum value of the out-degree k_1 . Evidently, if $k_1^{max} \ll k$, the in-degree and total degree distributions have similar behaviour with k . As λ becomes smaller, k_1^{max} increases and this inequality does not hold good anymore and therefore the in-degree and total degree distributions become different.

The variation of the degree distribution with a fixed value of λ and different A values is also worth studying. In Fig. 3 we show this variation. The data shows that the dependence on A gradually vanishes when the accelerated growth is increased. This can be explained by noting that when λ is small, each incoming node has to get linked to a large number of existing nodes and therefore the initial attractiveness factor in the attachment probability becomes unimportant. However, it has to be different from zero for reasons mentioned earlier.

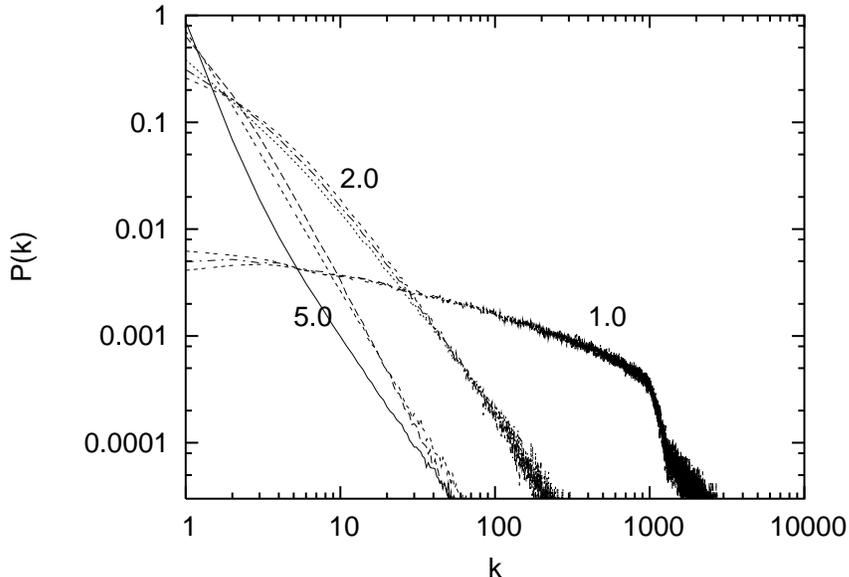


Fig. 3. The variation of $P(k)$ against k with $A = 0.1, 0.5$ and 1 are shown for 3 fixed values of λ : $\lambda = 5$, $\lambda = 2$ and $\lambda = 1$. The dependence on A is seen to disappear as λ becomes smaller.

Accelerated growth model for citation network

The citation network is a directed network in which the nodes are published papers. A new paper establishes a directed link with an old paper by citing it. Obviously new links cannot be formed between old nodes.

In a citation network, one is interested to find out the in-degree distribution $P(k)$, the number of papers which have been cited k times. The citation data $P(k)$ may be studied in various ways: (journal-wise, time-wise [19]) etc. A citation network is obviously a growing network but while the in-degree distribution maybe scale-free, the out-degree distribution is not, as the number of papers cited is usually limited. However, the average number of citations does show an increase with the number of published papers, indicating that it has an accelerated growth [9]. In the present model, we find out the distribution $P(k)$ irrespective of all details - $P(k)$ is obtained for the set of all published papers and all citations made to them after their publication to date. As such it is difficult to compare with the available data for citation, and instead of attempting to do so, we investigate the role of accelerated growth in a citation network.

In [9], it was shown that the average number of citations (out-degree) $\langle c \rangle$ made by a paper shows a growth with the number of published paper M as

$$\langle c \rangle = \langle c_0 \rangle + b \ln(1 + M/M_0), \quad (5)$$

or

$$\langle c \rangle = \langle c_0 \rangle + bM/M_0, \quad (6)$$

where M_0 is the number of papers published before a reference year and $\langle c_0 \rangle$ is the average number of citation of that year. The out-degree distribution of papers, $P_c(c)$ showed that it had a peak and an exponential decay. While simulating the citation network this is in fact the distribution one must use instead of a power-law distribution discussed earlier.

In order to qualitatively match with the observed out-degree distribution in [9] we have used a distribution

$$P_c(c) \sim c^\alpha \exp(-\beta c) \quad (7)$$

which has an exponential decay and a peak at $c_p = \alpha/\beta$. Here c can be rescaled to obtain

$$P_c(x) \sim x^\alpha \exp(-\alpha x) \quad (8)$$

where $x = c/c_p$. The distribution is then precisely in the form as in [9] where the observed value of α is $O(1)$. The average number of citations is $\langle c \rangle = (\alpha + 1)/\beta$ and in our simulation we keep α constant and vary $\langle c \rangle$ according to eq (5) or (6) such that effectively β is varied as time progresses. Taking $\alpha = 1$, we generate a network of 4000 papers and choose $\langle c_0 \rangle = 5$ when M_0 number of papers have been published (this may be considered small, but as the network has accelerated growth, the average outdegree increases; we have also checked that the results do not change for other values of $\langle c_0 \rangle$).

The model works in the following way: at any time we have a new paper with c outbound links (c is chosen according to the probability distribution (eq. 7)) which are attached to the existing nodes using a preferential attachment scheme (eq. 2) with $A = 1$.

For the logarithmic growth (eq. 5) we find that the degree distribution is scale-free with an exponent close to 2 indicating that the acceleration does not play an important role here. We have used different values of b (2 and 3) and M_0 (100 and 1000)(comparable to the observed data) but the results do not depend seriously on these choices.

For the linear increase of citations (eq (6)): we find that the behaviour of $P(k)$ does become different from that of the logarithmic case, and the distribution has a power law tail with an exponent ~ 1.6 .

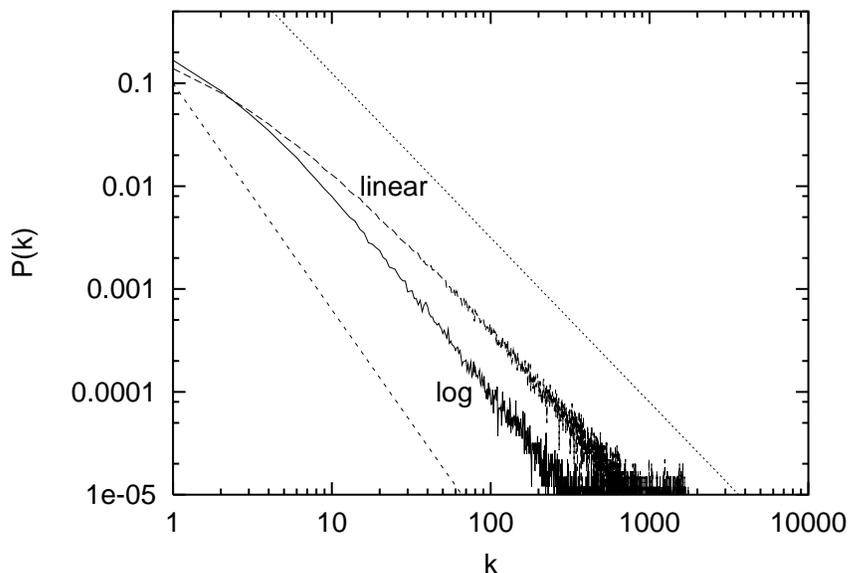


Fig. 4. In-degree distribution $P(k)$ of citation network for linear and logarithmically accelerated citation models. The dotted lines have slope = 2.2 (lower, corresponding to the non-accelerated model) and 1.6 (upper).

We have also we studied the non-accelerated network ($b = 0$). We again find that $P(k)$ has a power law decay with an exponent close to 2. Hence the accelerated growth does not play a very significant role as far as the logarithmic growth is concerned which is not very surprising. The linear growth also does not indicate a serious departure from the non-accelerated case possibly because of the low value of the factor b/M_0 which controls the effect of the growth term.

In the non-accelerated model, the outdegree is a variable but its average is fixed and therefore it is still expected to be described by the general model of [18] (this is what happens in the case of the BA model [20] as observed in [21]). According to [18] the exponent of $P(k)$ should be $2 + A/\langle c \rangle = 2.2$. We obtained a slightly smaller value of the exponent; the disagreement may be due to the finite size of the system. In fact, the exponents for the non-accelerated and the logarithmic growth data are found to be very close.

As mentioned earlier, we have not attempted to compare the results with the available citation data. Actually to do so, one should also consider other factors (e.g., aging [22], the fact that number of papers in each year increases, increased probability of citing a paper if the contents are closer etc.) in the model. In [9], a different algorithm to grow the network had been adopted (the form of the outdegree distribution was not assumed but generated here) which predicted a power law decay of $P(k)$ with exponent 2.

In summary, we have studied two directed networks in which accelerated growth takes place. The number of links is determined by a power-law distribution in the first case and it is more of an academic interest to study it. In

the second, we examine the role of accelerated growth in a real network where the distribution of the outgoing link number is chosen according to observed data. The results of the directed model, where λ appears as a tuning parameter, are qualitatively similar to the undirected case with the behaviour of the degree distributions showing marked difference at different intervals of the value of λ . The new feature in the directed accelerated model is that the degree distributions become independent of the value of the initial attractiveness parameter A ($A \neq 0$) as acceleration is enhanced.

It appears that appreciable difference from the non-accelerated case arises when the out-degree is power-law distributed. However, such a distribution of out-degree is rare, in real world networks one usually gets an exponential distribution for the out-degree. Also, in the citation network the acceleration term happens to be less significant due to its small magnitude such that the effect of acceleration becomes marginal.

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