Dialectics of Counting and the Mathematics of Vagueness

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Abstract. New concepts of rough natural number systems are introduced in this research paper from both formal and less formal perspectives. These are used to improve most rough set-theoretical measures in general Rough Set theory (RST) and to represent rough semantics. The foundations of the theory also rely upon the axiomatic approach to granularity for all types of general RST recently developed by the present author. The latter theory is expanded upon in this paper. It is also shown that algebraic semantics of classical RST can be obtained from the developed dialectical counting procedures. Fuzzy set theory is also shown to be representable in purely granule-theoretic terms in the general perspective of solving the contamination problem that pervades this research paper. All this constitutes a radically different approach to the mathematics of vague phenomena and suggests new directions for a more realistic extension of the foundations of mathematics of vagueness from both foundational and application points of view. Algebras corresponding to a concept of *rough naturals* are also studied and variants are characterised in the penultimate section.

keywords: Mathematics of Vagueness, Rough Natural Number Systems, Axiomatic Theory of Granules, Granulation, Granular Rough Semantics, Algebraic Semantics, Rough Y-Systems, Cover Based Rough Set Theories, Rough Inclusion Functions, Measures of Knowledge, Contamination Problem.

1 Introduction

Rough and Fuzzy set theories have been the dominant approaches to vagueness and approximate reasoning from a mathematical perspective. Some related references are [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16] . In rough set theory (RST),vague and imprecise information are dealt with through binary relations (for some form of indiscernibility) on a set or covers of a set or through more abstract operators. In classical RST [17], starting from an approximation space consisting of a pair of a set and an equivalence relation over it, approximations

of subsets of the set are constructed out of equivalence partitions of the space (these are crisp or definite) that are also regarded as granules in many senses. Most of the developments in RST have been within the ZFC or ZF set-theoretic framework of mathematics. In such frame works, rough sets can be seen as pairs of sets of the form (A, B), with $A \subseteq B$ or more generally as in the approaches of the present author as collections of "some sense definite elements" of the form

$$\{a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_r\}$$

subject to a_is being 'part of' of some of the b_js (in a Rough Y-system) [18].

Relative RST, fuzzy set theory may be regarded as a complementary approach or as a special case of RST from the membership function perspective [19]. Hybrid rough-fuzzy and fuzzy-rough variants have also been studied. In these a partitioning of the meta levels can be associated for the different types of phenomena, though it can be argued that these are essentially of a rough set theoretic nature. All of these approaches have been confined to ZFC or ZF or the setting of classical mathematics. Exceptions to this trend include the Lesniewski-mereology based approach [20]. Though Rough Y-systems have been introduced by the present author in ZF compatible settings [18], they can be generalised to semi-sets and variants in a natural way. This semi-set theoretical variant is work in progress. Note that the term 'theory' in RST is being used in the loose sense of the literature on the subject. It matters because the paper has strong connections with logical systems and philosophy.

The granular computing paradigm can be traced to the mid nineties and has been used in different fields including fuzzy and rough set theories. An overview is considered in [21]. The dominant interpretation of the paradigm within RST has been that granularity is a context-dependent notion that gets actualized in a quasi inductive way (see [22] for example). A few axiomatic approaches for specific rough set theories are known in the literature [23], while in some others like [24] different types of granules have been used in tolerance approximation spaces. The axiomatic theory [18] developed by the present author is the most general one in the context of general RSTs. [25] considers 'ontology driven formal theory of granules' for application to biological information systems, related specifications and logic programming. The motivations and content are mostly orthogonal to the concerns of the present research paper.

As classical RST is generalised to more general relations and covers, the process of construction and definition of approximations becomes more open ended and draws in different amounts of arbitrariness or hidden principles. The relevant concept of 'granules', the things that generate approximations, may also become opaque in these settings. To address these issues and for semantic compulsions, a new axiomatic theory of granules has been developed over a RYS in [18] and other recent papers by the present author. This theory has been used to formalise different principles, including the local clear discernibility principle in the same paper. In this research, it is extended to various types of general RST and is used to define the concepts of discernibility used in counting procedures, generalised measures and the problems of representation of semantics.

Many types of contexts involving vagueness cannot be handled in elegant way through standard mathematical techniques. Much of RST and FST are not known to be particularly elegant in handling different measures like the degree of membership or inclusion. For example, these measures do not determine the semantics in clear terms or the semantics do not determine the measures in a unique way. But given assumptions about the measures, compatible semantics [26] in different forms are well known. This situation is due to the absence of methods of counting collections of objects including relatively indiscernible ones and methods for performing arithmetical operations on them. However in the various algebraic semantics of classical RST some boundaries on possible operations may be observed.

The process of counting a set of objects, given the restriction that some of these may be indiscernible within themselves, may appear to be a very contextual matter and on deeper analysis may appear to bear no easy relationship with any fine structure concerning the vagueness of a collection of such elements or the rough semantics (algebraic or frame). This is reflected in the absence of related developments on the relationship between the two in the literature and is more due to the lack of developments in the former.

It should be noted that the convenience of choice between concepts of formal logics (in axiomatic or sequent calculus form) preceding algebraic semantics or the converse depend on the context. For many classes of logics, the absence of real distinction is well-known [27,28]. Any intention to deal with models automatically comes with an ontological commitment to the proof-theoretical approach and vice versa. The literature on rough sets shows that the converse works better – for example the rough algebra semantics [29], the double stone algebra semantics [30] and super rough semantics [31] were developed from a 'semantic viewpoint'. The modal perspective originated from both a semantic viewpoint and an axiomatic viewpoint (not proof-theoretic). The more important thing to be noted is that full application contexts of rough sets come with many more kinds of problems (like the reduct problem) that are not represented in rough logics or semantics, since the focus is on reasoning. This means fragments of the full process are abstracted for the formulation of rough logics in proof-theoretical or model-theoretical terms. When I speak of *semantics of RST*, I mean such an abstraction necessarily. More clarifications are provided in the section on semantic domains and the contamination problem.

In this research, theories of *vague numbers* or rather procedures for counting collections of objects including indiscernible ones have been introduced by the present author and have been applied to extend various measures of RST. The extended measures have better information content and also supplement the mereological theory from a quantitative perspective. Proper extension of these to form a basis of RST within ZF/ZFC is also considered in this research paper. Here, by a 'basis of RST', I mean a theory analogous to the theory of numbers from which all mathematics of exact phenomena can be represented. Throughout this paper, the theory may be seen to be restricted to ZF/ZFC set theoretical setting, though a naive or a second order reading will not be problematic. Relax-

ation of the ZF axioms given the dialectical interpretation of semantic domain will be taken up in subsequent papers, but the philosophical motivations for such a paradigm shift will be considered in later sections. From a purely vagueness perspective, the goal is also to enlarge the scope of mathematical perspective of vagueness.

Notation and terminology are fixed in the second section. In the third section, the basic orientation of object and meta levels used, and the relation with concepts is elucidated. The concept of *contamination* of information across meta-levels is introduced and described in the next section. In the fifth section, the reason for using a fragment of mereology as opposed to the Polkowski-Skowron mereology is explained. Some non-standard (with respect to the literature on RST) examples motivating key aspects of the axiomatic approach to granules are presented in the sixth section. In the next section, the entire structure of the proposed program and aspects of the measures used in RST are discussed. In the following section, aspects of counting in domains of vague reasoning are explained in a novel perspective. In the ninth section, the axiomatic theory of granules over rough Y-systems is extended. In the following two sections this is applied to relation-based and cover-based rough set theories. The ninth, tenth and eleventh sections may also be found in a forthcoming paper by the present author and have been included for completeness. Dialectical counting processes are introduced next. These are used to generalise rough inclusion functions, degrees of knowledge dependency and other measures in the following section. In the fourteenth section, possible representation of different types of counts is developed. An application to rough semantics and integration of granularity with a method of counting is considered in the fifteenth section. In the following section, I show how fuzzy set theory can be viewed as a particular form of granularity in the perspective of the contamination problem. The relation with earlier approaches is also indicated. Subsequently I consider the problem of improving the representation of counts in a low-level perspective and develop the algebra of rough naturals in detail. Further directions are mentioned in the eighteenth section.

2 Some Background, Terminology

A *Tolerance Approximation Space* TAS [32] is a pair $S = \langle \underline{S}, T \rangle$, with $\underline{S}$ being a set and T a tolerance relation over it. They are also known as similarity and as tolerance approximation spaces (conflicting the terminology introduced in [33]). For each $x \in S$, the associated set of T-related elements is given by $[x]_T = \{y ; (x, y) \in T\}$. Some references for extension of classical RST to TAS are [33], [34], [35] and [36]. In [24] specific granulations are considered separately in TAS, but many types of duality and connections with logics are not considered. The actual body of work in the field is huge and no attempt to mention all possibly relevant references will be made.

An approach [34] has been to define a new equivalence θ_0 on S via $(x, y) \in \theta_0$ if and only if $dom_T(x) = dom_T(y)$ with $dom_T(z) = \cap\{[x]_T : z \in [x]_T\}$.

This is an unduly cautious 'clear perspective' approach. A generalization of the approximation space semantics using T-related sets (or tolerance sets) can be described from the point of view of generalised covers (see [37]). This includes the approach to defining the lower and upper approximation of a set A as

$$A^l = \bigcup \{[x]_T \,;\, [x]_T \subseteq A\},$$

and

$$A^u = \bigcup \{[x]_T \,;\, [x]_T \cap A \neq \emptyset, \, x \in A\}.$$

A *bited modification* proposed in [38], valid for many definable concepts of granules, consists in defining a *bited upper approximation*. Algebraic semantics of the same has been considered by the present author in [39]. It is also shown that a full representation theorem is not always possible for the semantics.

The approximations

$$A^{l*} = \{x \,;\, (\exists y)\,(x,\,y) \in T,\, [y]_T \subseteq A\},$$

and

$$A^{u*} = \{x \,;\, (\forall y)\,((x,\,y) \in T \longrightarrow [y]_T \cap A \neq \emptyset)\} = (A^c)^{l*c}$$

were considered in [40,35]. It can be shown that, for any subset A,

$$A^l \subseteq A^{l*} \subseteq A \subseteq A^{u*} \subseteq A^u.$$

In the BZ and Quasi-BZ algebraic semantics [41], the lower and upper rough operators are generated by a preclusivity operator and the complementation relation on the power set of the approximation space, or on a collection of sets under suitable constraints in a more abstract setting. Semantically, the BZ-algebra and variants do not capture all the possible ways of arriving at concepts of discernibility over similarity spaces.

Let S be a set and $\mathcal{S} = \{K_i\}_1^n : n < \infty$ be a collection of subsets of it. We will abbreviate subsets of natural numbers of the form $\{1, 2, \ldots, n\}$ by $\mathbf{N}(n)$. For convenience, we will assume that $K_0 = \emptyset$, $K_{n+1} = S$. $\langle S, \mathcal{S} \rangle$ will also be referred to as a *Cover Approximation System* (CAS).

Cover-based RST can be traced to [42], where the approximations A^l and A^u are defined over the cover $\{[x]_T ; x \in S\}$. A 1-neighbourhood [43] $n(x)$ of an element $x \in S$ is simply a subset of S. The collection of all 1-neighbourhoods $\mathcal{N}$ of S will form a cover if and only if $(\forall x)(\exists y) x \in n(y)$ (anti-seriality). So in particular a reflexive relation on S is sufficient to generate a cover on it. Of course, the converse association does not necessarily happen in a unique way.

If $\mathcal{S}$ is a cover of the set S, then the *Neighbourhood* [44] of $x \in S$ is defined via,

$$nbd(x) = \bigcap \{K : x \in K \in \mathcal{S}\}.$$

The sixth type of lower and upper approximations [45,43] of a set X are then defined by

$$X_\$ = \{x : nbd(x) \subseteq X\},$$

and
$$X^\$ = \{x : nbd(x) \cap X \neq \emptyset\}.$$

The minimal description of an element $x \in S$ is defined to be the collection

$$\mathrm{Md}(x) = \{A : x \in A \in \mathcal{S}, \forall B(x \in B \to\sim (A \subset B))\}.$$

The *Indiscernibility* (or friends) of an element $x \in S$ is defined to be

$$Fr(x) = \bigcup\{K : x \in K \in \mathcal{S}\}.$$

The definition was used first in [40], but has been redefined again by many others (see [46]). An element $K \in \mathcal{S}$ will be said to be *Reducible* if and only if

$$(\forall x \in K)K \neq \mathrm{Md}(x).$$

The collection $\{nbd(x) : x \in S\}$ will be denoted by $\mathcal{N}$. The cover obtained by the removal of all reducible elements is called a covering reduct. The terminology is closest to [45] and many variants can be found in the literature (see [46]).

If $X \subseteq S$, then let

(i) $X^{l1} = \bigcup\{K_i : K_i \subseteq X, \ i \in \{0, 1, ..., n\}\}$.

(ii) $X^{l2} = \bigcup\{\cap_{i \in I}(S \setminus K_i) : \cap_{i \in I}(S \setminus K_i) \subseteq X, \ I \subseteq \mathbf{N}(n+1)\}$; the union is over the I's.

(iii) $X^{u1} = \bigcap\{\cup_{i \in I}K_i : X \subseteq \cup_{i \in I} K_i, \ I \subseteq \mathbf{N}(n+1)\}$; the intersection is over the I's.

(iv) $X^{u2} = \bigcap\{S \setminus K_i : X \subseteq S \setminus K_i, \ i \in \{0, ..., n\}\}$.

The pair (X^{l1}, X^{u1}) is called an *AU-rough set* by union, while (X^{l2}, X^{u2}) an *AI-rough set* by intersection (in the present author's notation [47]). In the notation of [37], these are $(\mathcal{F}^{\cup}_*(X), \mathcal{F}^*_{\cup}(X))$ and $(\mathcal{F}^{\cap}_*(X), \mathcal{F}^*_{\cap}(X))$, respectively. I will also refer to the pair $\langle S, \mathcal{K}\rangle$ as an AUAI-*approximation system*.

Theorem 1. *The following hold in AUAI approximation systems:*

(i) $X^{l1} \subseteq X \subseteq X^{u1}; X^{l2} \subseteq X \subseteq X^{u2}; \emptyset^{l1} = \emptyset^{l2} = \emptyset$,

(ii) $(\cup\mathcal{K} = S \longrightarrow S^{u1} = S^{u2} = S) \ ; (\cup\mathcal{K} = S \longrightarrow \emptyset^{u2} = \emptyset, S^{l1} = S)$,

(iii) $(\cap\mathcal{K} = \emptyset \longrightarrow \emptyset^{u1} = \emptyset, S^{l2} = S)$,

(iv) $(X \cap Y)^{l1} \subseteq X^{l1} \cap Y^{l1} \ ; (X \cap Y)^{l2} = X^{l2} \cap Y^{l2}$,

(v) $(X \cup Y)^{u1} = X^{u1} \cup Y^{u1} \ ; X^{u2} \cup Y^{u2} \subseteq (X \cup Y)^{u2}$,

(vi) $(X \subseteq Y \longrightarrow X^{l1} \subseteq Y^{l1}, X^{l2} \subseteq Y^{l2}, X^{u1} \subseteq Y^{u1}, X^{u2} \subseteq Y^{u2})$,

(vii) If $(\forall i \neq j)K_i \cap K_j = \emptyset$ then $(X \cap Y)^{l1} = X^{l1} \cap Y^{l1}, (X \cup Y)^{u2} = X^{u2} \cup Y^{u2}$,

(viii) $X^{l1} \cup Y^{l1} \subseteq (X \cup Y)^{l1} \ ; X^{l2} \cup Y^{l2} \subseteq (X \cup Y)^{l2}$,

(ix) $(X \cap Y)^{u1} \subseteq X^{u1} \cap Y^{u1} \ ; (X \cap Y)^{u2} \subseteq X^{u2} \cap Y^{u2}$,

(x) $(S \setminus X)^{l1} = S \setminus X^{u2} \ ; (S \setminus X)^{l2} = S \setminus X^{u1}$,

(xi) $(S \setminus X)^{u1} = S \setminus X^{l2} \ ; (S \setminus X)^{u2} = S \setminus X^{l1}$,

(xii) $(X^{l1})^{l1} = X^{l1} \ ; (X^{l2})^{l2} = X^{l2} \ ; (X^{u1})^{u1} = X^{u1}$,

(xiii) $(X^{u2})^{u2} = X^{u2} \ ; (X^{l1})^{u1} = X^{l1} \ ; (X^{u2})^{l2} = X^{u2}$,

(xiv) $X^{l2} \subseteq (X^{l2})^{u2}, (X^{u1})^{l1} \subseteq X^{u1}$,

(xv) $(\mathcal{K}_j^{\cap}(X))^{u2} = \mathcal{K}_j^{\cap}(X)$, $j = 1, 2, ..., t_1$; $(\mathcal{K}_j^{\cup}(X))^{l1} = \mathcal{K}_j^{\cup}(X)$, $j = 1, 2, ..., t_2$.

In this, $(\mathcal{K}_j^{\cup}(X))$ is the minimal union of sets of the form K_i that include X (for j being in the indicated range) and $(\mathcal{K}_j^{\cap}(X))$ is the maximal intersection of sets of the form K_i^c that are included in X.

All of the above concepts can be extended to covers with an arbitrary number of elements. The concepts of indiscernibility, neighbourhood and minimum description can be extended to subsets of S. The concept of a *Neighbourhood Operator* has been used in the literature in many different senses. These can be relevant in the context of the sixth type $(l6+, u6+)$ (see the sixth section) approximations for dealing with covers generated by partially reflexive relations [47]. A large number of approximations in the cover-based approximation context have been studied in the literature using a far larger set of notations. An improved nomenclature is also proposed in the eleventh section.

Cover-based RST is more general than relation-based RST and the question of when covers over a set correspond to relations over the set is resolved through duality results. It is well known that partitions correspond to equivalences and normal covers to tolerances. The approach based on neighbourhoods [43] provides many one way results. A more effective way of reducing cover-based RST to relation-based RST is in [48].

3 Semantic Domains, Meta and Object Levels

This section is intended to help with the understanding of the section on the contamination problem, the definition of RYS and clarify the terminology about meta and object levels among other things. In classical RST (see [17]), an approximation space is a pair of the form $\langle S, R \rangle$, with R being an equivalence on the set S. On the power set $\wp(S)$, lower and upper approximation operators, apart from the usual Boolean operations, are definable. The resulting structure constitutes a semantics for RST (though not satisfactory) from a classical perspective. This continues to be true even when R is some other type of binary relation. More generally (see fourth section) it is possible to replace $\wp(S)$ by some set with a parthood relation and some approximation operators defined on it. The associated semantic domain in the sense of a collection of restrictions on possible objects, predicates, constants, functions and low level operations on those will be referred to as the classical semantic domain for general RST. In contrast, the semantics associated with sets of roughly equivalent or relatively indiscernible objects with respect to this domain will be called the rough semantic domain. Actually many other semantic domains, including hybrid semantic domains, can be generated (see [49], [39] [31]) for different types of rough semantics, but these two broad domains will always be - though not necessarily with a nice correspondence between the two. In one of the semantics developed in [39], the reasoning is within the power set of the set of possible order-compatible partitions of the set of roughly equivalent elements. The concept of *semantic domain* is therefore

similar to the sense in which it is used in general abstract model theory [50] (though one can object to formalisation on different philosophical grounds).

Formal versions of these types of semantic domains will be useful for clarifying the relation with categorical approaches to fragments of granular computing [51]. But even without a formal definition, it can be seen that the two approaches are not equivalent. Since the categorical approach requires complete description of fixed type of granulations, it is difficult to apply and especially when granules evolve relative particular semantics or semantic domains. The entire category **ROUGH** of rough sets in [51], assumes a uniform semantic domain as evidenced by the notion of objects and morphisms used therein. A unifying semantic domain may not also be definable for many sets of semantic domains in our approach. This means the categorical approach needs to be extended to provide a possibly comparable setting.

The term *object level* will mean a description that can be used to directly interface with fragments (sufficient for the theories or observations under consideration) of the concrete real world. Meta levels concern fragments of theories that address aspects of dynamics at lower meta levels or the object level. Importantly, we permit meta level aspects to filter down to object levels relative different object levels of specification. So it is always possible to construct more meta levels and expressions carry intentions.

Despite all this, two particular meta levels namely Meta-C (or Meta Classical), Meta-R (or Meta Rough) and an object level will be used for comparing key notions introduced with the more common approaches in the literature. Meta-R is the meta level corresponding to the observer or agent experiencing the vagueness or reasoning in vague terms (but without rough inclusion functions and degrees of inclusion), while Meta-C will be the more usual higher order classical meta level at which the semantics is formulated. It should be noted that rough membership functions and similar measures are defined at Meta-C, but they do not exist at Meta-R. A number of meta levels placed between Meta-R and Meta-C can possibly be defined and some of these will be implicit in the section on rough naturals.

Many logics have been developed with the intent of formalising 'rough sets' as 'well-formed formulae' in a fixed language. They do not have a uniform domain of discourse and even ones with category theoretically equivalent models do not necessarily see the domain in the same way (though most meanings can be mapped in a bijective sense). For example, the regular double stone algebra semantics and complete rough algebra semantics correspond to different logical systems of classical RST (see [52,53]). The super rough algebra semantics in [31] actually adds more to the rough algebra semantics of [29]. It is possible to express the ability of objects to approximate in the former, while this is not possible in the latter. This is the result of a higher order construction used for generating the former.

The relation of some rough semantics and topology mentioned in the previous section is again a statement about the orientation of the semantic domains in the respective subjects formulated in more crude mathematical terms.

3.1 Granules and Concepts

In [54] for example, concepts of human knowledge are taken to consist of an intensional part and an extensional part. The *intension* of a concept is the collection of properties or attributes that hold for the set of objects to which the concept applies. The *extension* is to consist of actual examples of the object. Yao writes, 'This formulation enables us to study concepts in a logic setting in terms of intensions and also in a set-theoretic setting in terms of extensions'. The description of granules characterise concepts from the intensional point of view, while granules themselves characterise concepts from the extensional point of view. *Granulations* are collections of granules that contain every object of the universe in question. In a seemingly similar context, in [55] (or [3]) the authors speak of *extensional granules* and *intensional granules* that are respectively related to objects and properties. In my opinion the semantic domains in use are different and these are not conflicting notions, though it is equally well to call the latter a more strong platonic standpoint. Yao does not take sides on the debate in *what a concept is* and most of it is certainly nonclassical and non empiricist from a philosophical point of view.

In modern western philosophy, intentions and extensions are taken to be possessed by linguistic expressions and not by concepts. Thus, for example, from Frege's point of view, the intension is the concept expressed by the expression, and the extension is the collection of items to which the expression applies. In this perspective, the concept applies to the same collection of items. It also follows that concepts, in this perspective, must be tied to *linguistic expressions* as well.

Concepts are constituents of thinking containing the meaning of words or intended action or response. As such a linguistic expression for such *concepts* may not be supplied by the reasoner. Apparently the Fregean point of view speaks of concepts with associated linguistic expression alone. Even if we use a broad-sense notion of 'linguistic expression', this may fall short of the *concept* mentioned in the former viewpoint. Another key difference is that the former version of *concepts* are bound to be more independent of interpreters (or agents) than the latter. The concept of *granules* actually evolves across the temporal space of the theory and may be essentially *a priori* or *a posteriori* (relative to the theory or semantics) in nature. Because of these reasons, I will not try to map the two *concepts* into each other in this paper at least. In the present paper, *a priori* granules will be required in an essential way.

It is only natural that possible concepts of granules are dependent on the choice of semantic domain in the contexts of RST. But *a priori* granules may even be identified at some stage after the identification of approximations.

4 Contamination Problem

Suppose the problem at hand is to model vague reasoning in a particular context and relative to the agents involved in the context. It is natural for the model

to become *contaminated* with additional inputs from a classical perspective imposed on the context by the person doing the modelling. In other words, meta level aspects can contaminate the perception of object level features. From an algebraic perspective, if the model concerns objects of specific types like 'roughly equivalent objects in some sense', then the situation is relatively better than a model that involves all types of objects. But the operations used in the algebra or algebraic system can still be viewed with suspicion.

By the contamination problem, I mean the problem of minimising or eliminating the influences of the classicist perspective imposed on the context. In other words, the problem is to minimize the contamination of models of meta-R fragments by meta-C aspects. One aspect of the problem is solved partially in [56] by the present author. In the paper, a more realistic conception of rough membership functions and other measures of RST have been advanced from a minimalist perspective avoiding the real-valued or rational-valued rough measures that dominate the rough literature. Most of the rough measures based on cardinalities are of course known to lack mathematical rigour and have the potential to distort the analysis.

In the mathematics of exact phenomena, the natural numbers arise in the way they do precisely because it is assumed that things being counted are well-defined and have exact existence. When a concrete collection of identical balls on a table are being counted, then it is their relative position on the table that helps in the process. But there are situations in real life, where

- such identification may not be feasible,
- the number assigned to one may be forgotten while counting subsequent objects,
- the concept of identification by way of attributes may not be stable,
- the entire process of counting may be 'lazy' in some sense,
- the mereology necessary for counting may be insufficient.

Apart from examples in [56], the most glaring examples for avoiding the measures comes from attempts to apply rough sets to modelling the development of human concepts. The 'same' concept X may depend on ten other concepts in one perspective and nine other concepts in another perspective and concepts of knowing the concept X and gradation does not admit a linear measure in general. Using one in fields like education or education research would only encourage scepticism. The quality of measures like 'impact factor' of journals [57] provide a supportive example.

The underlying assumptions behind rough measures are much less than in a corresponding statistical approach (subject to being possible in the first place in the application context in question) and do not make presumptions of the form -'relative errors should have some regularity'. Still the contamination problem is relevant in other domains of application of RST and more so when the context is developed enough to permit an evaluation of semantic levels.

There may be differences in the semantic approach of proceeding from algebraic models to logics in sequent calculus form in comparison to the approach of directly forming the logic as a sequent calculus, or the approach of forming the

logic in Kripke-like or Frame-related terminology, but one can expect one to feed the other. It should also be noted that this has nothing to do with supervaluationary perspectives [58], where the goal is to reduce vagueness by improving the language. Moreover the primary concerns in the contamination problem are not truth-values or gaps in them. The contamination problem is analogous to the problem of intuitionist philosophers to find a perfect logic free from problematic classicist assumptions. A difficult approach to the latter problem can be found in [59]. The important thing to note in [59] is the suggestion that it is better to reason within a natural deduction system to generate 'pure logic'. In case of the contamination problem, general understanding stems from model theoretic interpretations and so should be more appropriate.

If a model-theoretic perspective is better, then it should be expected to provide justification for the problem. The problems happen most explicitly in attempts to model human reasoning, in conceptual modelling especially (in learning contexts), in attempts to model counting processes in the presence of vagueness and others. In applications to machine intelligence, an expression of contamination would be 'are you blaming machines without reason?'

5 Formalism Compatibility and Mereology

In the literature various types of mereologies or theories of part-whole relationships [60,61] are known. For the axiomatic theory, I used a minimal fragment derived from set-theory compatible mereology in [62]. This fragment may also be argued to be compatible with even Lesniewskian mereology - but such arguments must be founded on scant regard of Lesniewski's nominalism and the distortions of his ideas by later authors. Such distortion is used as the base for generalization in all of 'Lesniewski ontology-based rough mereology'. This is evident for example, from section-2.1 of [63]. Even the perspective of Gregoryck [64] is accepted - 'theorems of ontology are those that are true in every model for atomic Boolean algebras without a null element'. Other papers by the same author, confirm the pragmatic excesses as classical rough set theory is shown to be embedded in the rough mereological generalisation as well [65]. New problems/conflicts of a logical/philosophical nature are bound to crop up if the theory is applied to model reasoning and attempts are made to link it to Lesniewski's approach. In my opinion, it would be better to term the Polkowski-Skowron approach a 'Lesniewski-inspired' mereology rather than a 'Lesniewski-ontology-based' one.

The reader can find a reasonable re-evaluation of formal aspects of the mereology of Lesniewski from a 'platonized perspective' in [66]. Importantly, it highlights difficulties in making the formalism compatible with the more common languages of modern first order or second order logic. The correct translation of expression in the language of ontology to a common language of set theory (modulo simplifying assumptions) requires many axiom schemas and the converse translation (modulo simplifying assumptions) is doubtful (see pp 184-194,[66]). I am stressing this because it also suggests that the foundational aspects of [20] should be investigated in greater detail in relation to:

- the apparent stability of the theory in related application contexts, and
- the exact role of the rough parthood relation and role of degrees of membership and t-norms in diluting logical categories.

I will not go into detailed discussion of the philosophical aspects of the points made above since it would be too much of a deviation for the present paper.

One of the eventual goals of the present approach is also to extend general RST to semi-set theoretical contexts [67,68] (or modifications thereof). Semiset theory has been in development since the 1960s and its original goals have been to capture vagueness and uncertainty, to be clear about what exactly is available for reasoning, to understand the infinite as 'something non-transparent', to impose a more sensible constraint on the relation between objects and properties and to require that any grouping or association is actualized (i.e. available at our disposal). It can be formalised as a conservative extension of ZFC, but irrespective of this aspect, the philosophical framework can be exploited in other directions. However, it is obviously incompatible with the Lesniewski ontology and nominalism. This is another reason for using a fragment of set-theoretically compatible mereology in the definition of a general rough Y-system. In this paper, I will continue to do the theory over ZFC-compatible settings, since most of the present paper will be relevant for all types of rough theorists.

In summary, the differences with the Polkowski-Skowron style mereological approach are:

(i) The mereology is obviously distinct. The present approach is compatible with Godel-Bernays classes.
(ii) No assumptions are made about the degree of inclusion or of 'x being an ingredient of y to a degree r'.
(iii) Concepts of degree are expected to arise from properties of granules and 'natural ways' of counting collections of discernible and indiscernible objects.

6 Motivating Examples for RYS

Motivating examples for the general concept of RYS introduced in [18] are provided in this section. These examples do not explicitly use information or decision tables though all of the information or decision table examples used in various RSTs would be naturally relevant for the general theory developed. Other general approaches like that of rough orders [69] and abstract approximation spaces [70] are not intended for directly intercepting approximations as envisaged in possible applications. They would also be restrictive and problematic from the point of view of the contamination problem. Here the focus is on demonstrating the need for many approximation operators, choice among granules and conflict resolution.

Example-1:

Consider the following statements associable with the description of an apple in a plate on a table:

  (i) Object is apple-shaped; Object has maroon colour,
 (ii) Object has vitreous skin; Object has shiny skin,
(iii) Object has vitreous, smooth and shiny skin,
(iv) Green apples are not sweet to taste,
 (v) Object does not have coarse skin as some apples do,
(vi) Apple is of variety A; Apple is of variety X.

Some of the individual statements like those about shape, colour and nature of skin may be 'atomic' in the sense that a more subtle characterization may not be available. It is also obvious that only some subsets of these statements may consistently apply to the apple on the table. This leads to the question of selecting some atomic statements over others. But this operation is determined by consistency of all the choices made. Therefore, from a RST perspective, the atomic statements may be seen as granules and then it would also seem that choice among sets of granules is natural. More generally 'consistency' may be replaced by more involved criteria that are determined by the goals. A nice way to see this would be to look at the problem of discerning the object in different contexts - discernibility of apples on trees require different kind of subsets of granules.

Example-2:

In the literature on educational research [71] it is known that even pre-school going children have access to powerful mathematical ideas in a form. A clear description of such ideas is however not known and researchers tend to approximate them through subjective considerations. For example, consider the following example from [71]:

Four-year-old Jessica is standing at the bottom of a small rise in the preschool yard when she is asked by another four-year-old on the top of the rise to come up to her.

- No, you climb down here. Its much shorter for you.

The authors claim that "Jessica has adopted a developing concept of comparison of length to solve at least for her - the physical dilemma of having to walk up the rise". But a number of other concepts like 'awareness of the effects of gravitational field', 'climbing up is hard', 'climbing up is harder than climbing down', 'climbing down is easier', 'climbing up is harder', 'others will find distances shorter', 'make others do the hard work' may or may not be supplemented by linguistic hedges like *developing* or *developed* and assigned to Jessica. The well known concept of concept maps cannot be used to visualise these considerations, because the concept under consideration is not well defined. Of these concepts some will be assuming more and some less than the actual concept used and some will be closer than others to the actual concept used. Some of the proposals may be conflicting, and that can be a problem with most approaches of

RST and fuzzy variants. The question of one concept being closer than another may also be dependent on external observers. For example, how do 'climbing up is harder' and 'climbing up is harder than climbing down' compare?

The point is that it makes sense to:

(i) accommodate multiple concepts of approximation,

(ii) assume that subsets of granules may be associated with each of these approximations,

(iii) assume that disputes on 'admissible approximations' can be resolved by admitting more approximations.

It is these considerations and the actual reality of different RST that motivates the definition of Rough Y-systems.

7 Objectivity of Measures and General RST

In RST, different types of rough membership and inclusion functions are defined using cardinality of associated sets. When fuzzy aspects are involved then these types of functions become more controversial as they tend to depend on the judgement of the user in quantifying linguistic hedges and other aspects. These types of functions are also dependent on the way in which the evolution of semantics is viewed. But even without regard to this evolution, the computation of rough inclusion functions invariably requires one to look at things from a higher meta level - from where all objects appear exact. In other words an estimate of a perfect granular world is also necessary for such computations and reasoning.

Eventually, this leads to mix up (contamination) of information obtained from perceiving things at different levels of granularity. I find all this objectionable from the point of view of rigour and specific applications too. To be fair such a mix up seems to work fine without much problems in many applications. But that is due to the scope of the applications and the fact that oversimplifications through the use of cardinality permits a second level of 'intuitive approximation'.

In applications of RST that start from information or decision tables including information about values associated with attributes (and possibly decisions) for different objects, the evolution of the theory adheres to the following dependency schemas:

```
              ┌─────────────────┐
              │    Granules     │
              │ and Granulation │
              └─────────────────┘
```
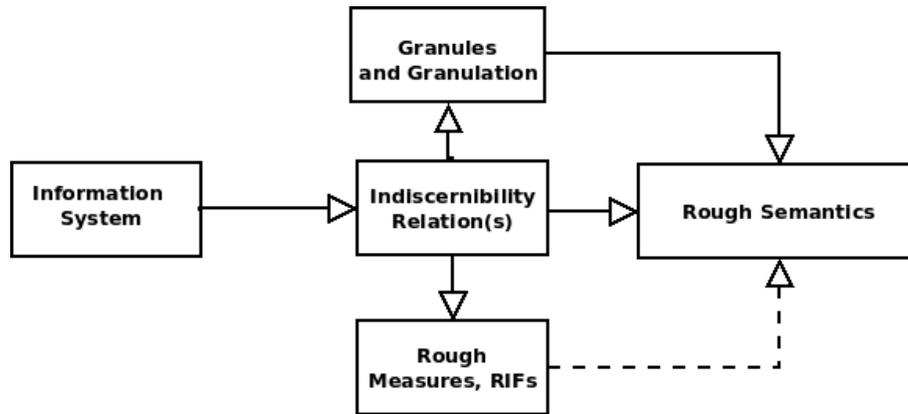
Type-1: Dependencies in Evolution of Components
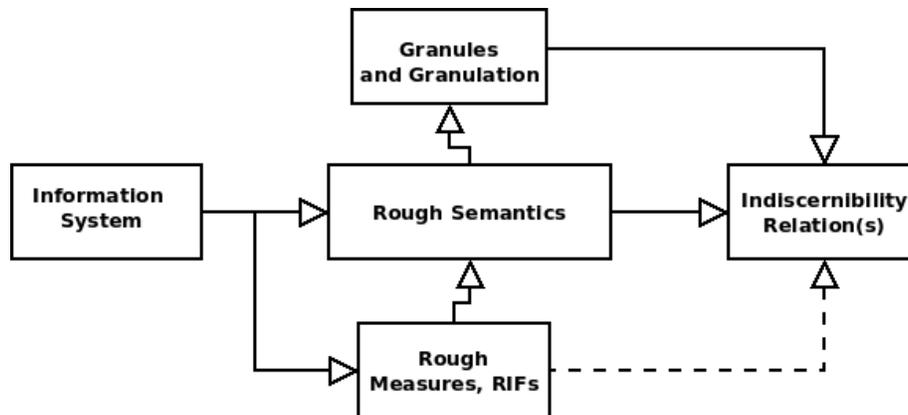
Type-2: Dependencies in Evolution of Components

In the above two figures, 'rough semantics' can be understood to be in algebraic or in proof-theoretic sense. The intended reading is - 'components at arrow heads depend on those at the tail' and multiple directed paths suggest that 'components in alternate paths may be seen in weaker forms relatively'. These figures do not show the modified information system that invariably results due to the computation of reducts of different kinds, as the entire picture merely gets refreshed to the refined scenario. The Lesniewski-style ontology-based mereological approach of [20,26] fits into type-1 schemas. Rule discovery approaches would fall within type-2 schemas.

The approach of the present paper is aimed at using measures that are more natural in the rough setting and to use fewer assumptions about their evolution at the meta level. Eventually this is likely to result in changes on methods of reduct computation in many cases. The theory is also aimed at tackling the so-called *inverse problems* of [31] and later papers, which is essentially 'Given a collection of definite objects and objects of relatively less definite objects with some concepts of approximations (that is result of vague and possibly biased considerations), find a description of the context from a general rough perspective'. From a semantic perspective these may reduce to abstract representation problems. The following dependency schema shows how the different parts fit in.
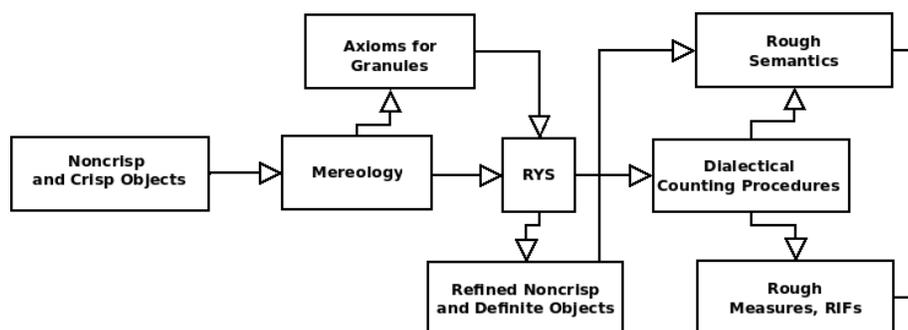
Dependencies Between Components of the Theory (in this Paper)

The link between 'Rough Semantics' and 'Rough Measures' should be read as 'possibly related'.

8 Numbers and their Generalization

The problems with using natural numbers for counting collections of objects including indiscernibles have been mentioned in the fourth section. It was pointed out that there are situations in real life, where

(i) the discernibility required for normal counting may not be feasible,
(ii) the number assigned to one may be forgotten while counting subsequent objects,
(iii) the concept of identification by way of attributes may not be stable,
(iv) the entire process of counting may be 'lazy' in some sense,
(v) the mereology necessary for counting may be insufficient.

Some specific examples of such contexts are:

1. Direct counting of fishes in a lake is difficult and the sampling techniques used to estimate the population of fishes do not go beyond counting samples and

pre-sampling procedures. For example some fishes may be caught, marked and then put back into the lake. Random samples may be drawn from the mixed population to estimate the whole population using proportion related statistics. The whole procedure however does not involve any actual counting of the population.

2. In crowd management procedures, it is not necessary to have exact information about the actual counts of the people in crowds.

3. In many counting procedures, the outcome of the last count (that is the total number of things) may alone be of interest. This way of counting is known to be sufficient for many apparently exact mathematical applications.

4. Suppose large baskets containing many varieties of a fruit are mixed together and suppose an observer with less-than-sufficient skills in classifying the fruits tries to count the number of fruits of a variety. The problem of the observer can be interpreted in mereological terms.

5. Partial algebras are very natural in a wide variety of mathematics. For example, in semigroup theory the set of idempotents can be endowed with a powerful partial algebraic structure. Many partial operations may be seen to originate from failure of standard counting procedures in specific contexts.

Various generalizations of the concept of natural numbers, integers and real numbers are known in mathematics. These generalizations are known to arise from algebraic, topological or mixed considerations. For example, a vast amount of ring and semigroup theory arises from properties of the integers. These include Euclidean Rings, UFD, Integral Domains, Positively totally ordered Semigroups [72], and Totally Ordered Commutative Semigroups. Partial Well Orders and Variants thereof [73] and Difference orders can also be seen in a similar perspective. In all these cases none of the above mentioned aspects can be captured in any obvious way and neither have they been the motivation for their evolution. Their actual motivations can be traced to concrete examples of subsets of real numbers and higher order structures derived from real numbers having properties defining the structures. Further structures like these often possess properties quite atypical of integers.

In counting collections of objects including relatively exact and indiscernible objects, the situation is far more complex - the first thing to be modified would be the relative orientation of the object and different meta levels as counting in any sense would be from a higher meta level. Subsequently the concept of counting (as something to be realised through injective maps into N) can be modified in a suitable way. The eventual goal of such procedures should be the attainment of order-independent representations.

Though not usually presented in the form, studies of group actions, finite and infinite permutation groups and related automorphisms and endomorphisms can throw light on lower level counting. In the mathematics of exact phenomena, these aspects would seem superfluous because cardinality is not dependent on the order of counting. But in the context of the present generalization somewhat related procedures are seen to be usable for improving representation. A more direct algebra of Meta-R counts is also developed in the penultimate section.

They can be regarded as a natural generalization of the ordered integral domain associated with integers and was not considered in [56] by the present author. The former approach does have feasibility issues associated. For one thing a string of relatively discernible and indiscernible things may not be countable in all possible ways in actual practice. The latter approach takes a more holistic view and so the two approaches can be expected to complement each other.

9   Granules: An Axiomatic Approach

Different formal definitions of granules have been used in the literature on rough sets and in granular computing. An improved version of the axiomatic theory of granules introduced in [18] is presented here. The axiomatic theory is capable of handling most contexts and is intended to permit relaxation of set-theoretic axioms at a later stage. The axioms are considered in the framework of Rough Y-Systems mentioned earlier. RYS maybe seen as a generalised form of *rough orders* [69], *abstract approximation spaces* [70] and approximation framework [74]. It includes relation-based RST, cover-based RST and more. These structures are provided with enough structure so that a classical semantic domain and at least one rough semantic domain of roughly equivalent objects along with admissible operations and predicates are associable.

Within the domain of naive set theory or ZFC or second order ZFC, the approximation framework of [74] is not general enough because:

 (i) It assumes too much of order structure.
 (ii) Assumes the existence of a De Morgan negation.
(iii) It may not be compatible with the formulations aimed at inverse problems. That holds even if a flexible notion of equality is provided in the language.

An application to the context of example-2 in the previous section will clearly show that there is no direct way of getting to a lattice structure or the negation from information of the type in conjunction with knowledge base about concepts represented in suitable way unless the context is very special.

As opposed to a lattice order in a *rough order*, I use a parthood relation that is reflexive and antisymmetric. It may be non transitive. The justification for using such a relation can be traced to various situations in which restrictive criteria operating on inclusion of attributes happen. In many cases, these may be dealt with using fuzzy methodologies. Contexts using the so-called rough-fuzzy-rough approximations and extensions thereof [75] can be dealt with in a purely rough way through such relations. The unary operations used in the definitions of the structures are intended as approximation operators. More than two approximation operators are common in cover-based RST [45], dynamic spaces [76], Esoteric RST [47], multiple approximation spaces [77], in dialectical rough set theory [39] and elsewhere. The requirement of equal number of upper and lower approximation operators is actually a triviality and studies with non matching number of operators can be traced to considerations on rough bottom and top equalities [78]. Concrete examples are provided after definitions.

The intended concept of a *rough set* in a rough Y-system is as a collection of some sense definite elements of the form $\{a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_r\}$ subject to a_is being 'part of' of some of the b_js.

Both RYS+ and RYS can be seen as the generalization of the algebra formed on the power set of the approximation space in classical RST. $\mathbf{P}xy$ can be read as 'x is a part of y' and is intended to generalise inclusion in the classical case. The elements in S may be seen as the collection of approximable and exact objects - this interpretation is compatible with S being a set. The description operator of FOPL ι is used in the sense: $\iota(x)\Phi(x)$ means 'the x such that $\Phi(x)$'. It helps in improving expression and given this the metalogical $','$ can be understood as $\wedge$ (which is part of the language). The description operator actually extends FOPL by including more of the metalanguage and from the meaning point of view is different, though most logicians will see nothing different. For details, the reader may refer to [79].

For those understanding ',' as being part of the metalanguage, statements of the form $a + b = \iota(x)\Phi(x)$ can be safely read as $a + b = z$ if and only if $\Phi(z)$. It is of course admissible to read ',' as being in the metalanguage with ι being part of the language - the resulting expression would be more readable.

Definition 1. *A* Rough Y System *(RYS+) will be a tuple of the form*

$$\langle S, W, \mathbf{P}, (l_i)_1^n, (u_i)_1^n, +, \cdot, \sim, 1\rangle$$

satisfying all of the following ($\mathbf{P}$ is intended as a binary relation on S and $W \subset S$, n being a finite positive integer. ι is the description operator of FOPL: $\iota(x)\Phi(x)$ means 'the x such that $\Phi(x)$ '. W is actually superfluous and can be omitted):

1. $(\forall x)\mathbf{P}xx$; $(\forall x, y)(\mathbf{P}xy, \mathbf{P}yx \longrightarrow x = y)$,
2. *For each i, j, l_i, u_j are surjective functions : $S \longmapsto W$,*
3. *For each i, $(\forall x, y)(\mathbf{P}xy \longrightarrow \mathbf{P}(l_ix)(l_iy), \mathbf{P}(u_ix)(u_iy))$,*
4. *For each i, $(\forall x)\mathbf{P}(l_ix)x, \mathbf{P}(x)(u_ix))$,*
5. *For each i, $(\forall x)(\mathbf{P}(u_ix)(l_ix) \longrightarrow x = l_ix = u_ix)$.*

The operations $+$, $\cdot$ and the derived operations $\mathbf{O}$, $\mathbb{P}$, $\mathbf{U}$, $\mathbb{X}$, $\mathbb{O}$ will be assumed to be defined uniquely as follows:

Overlap: $\mathbf{O}xy$ iff $(\exists z)\,\mathbf{P}zx \wedge \mathbf{P}zy$,
Underlap: $\mathbf{U}xy$ iff $(\exists z)\,\mathbf{P}xz \wedge \mathbf{P}yz$,
Proper Part: $\mathbb{P}xy$ iff $\mathbf{P}xy \wedge \neg\mathbf{P}yx$,
Overcross: $\mathbb{X}xy$ iff $\mathbf{O}xy \wedge \neg\mathbf{P}xy$,
Proper Overlap: $\mathbb{O}xy$ iff $\mathbb{X}xy \wedge \mathbb{X}yx$,
Sum: $x + y = \iota z(\forall w)(\mathbf{O}wz \leftrightarrow (\mathbf{O}wx \vee \mathbf{O}wy))$,
Product: $x \cdot y = \iota z(\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \mathbf{P}wy))$,
Difference: $x - y = \iota z(\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \neg\mathbf{O}wy))$,
Associativity: *It will be assumed that $+$, $\cdot$ are associative operations and so the corresponding operations on finite sets will be denoted by $\oplus$, $\odot$ respectively.*

Remark:

W can be dropped from the above definition and it can be required that the range of the operations u_i, l_j are all equal for all values of i, j.

Definition 2. *In the above definition, if we relax the surjectivity of l_i, u_i, require partiality of the operations $+$ and $\cdot$, weak associativity instead of associativity and weak commutativity instead of commutativity, then the structure will be called a* General Rough Y-System *(RYS). In formal terms,*

Sum1: $x + y = \iota z (\forall w)(\mathbf{O}wz \leftrightarrow (\mathbf{O}wx \vee \mathbf{O}wy))$ *if defined*
Product1: $x \cdot y = \iota z (\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \mathbf{P}wy))$ *if defined*
wAssociativity $x \oplus (y \oplus z) \stackrel{\omega*}{=} (x \oplus y) \oplus z$ *and similarly for product.* $\stackrel{\omega*}{=}$',
  essentially means if either side is defined, then the other is and the two terms are equal.
wCommutativity $x \oplus y \stackrel{\omega*}{=} y \oplus x;\ x \cdot y \stackrel{\omega*}{=} y \cdot x$

Both RYS and a RYS+ are intended to capture a minimal common fragment of different RSTs. The former is more efficient due to its generality. Note that the parthood relation $\mathbf{P}$, taken as a general form of rough inclusion (in a suitable semantic domain), is not necessarily transitive. Transitivity of $\mathbf{P}$ is a sign of fair choice of attributes (at that level of classification), but non transitivity may result by the addition of attributes and so the property by itself says little about the quality of classification.

The meaning of the equality symbol in the definition of RYS *depends on the application domain. It may become necessary to add additional stronger equalities to the language or as a predicate in some situations. In this way, cases where any of conditions $1, 3, 4, 5$ appear to be violated can be accommodated. All weaker equalities are expected to be definable in terms of other equalities.*

For example, using the variable precision RST procedures [80,81], it is possible to produce lower approximations that are not included in a given set and upper approximations of a set that may not include the set. In [47], methods for transforming the VPRS case are demonstrated. But nothing really needs to be done for accommodating the VPRS case - the axioms can be assumed. The predicate $\mathbf{P}$ would become complicated as a consequence, though we can have $(\forall x, y)(x \subseteq y \longrightarrow \mathbf{P}xy)$. A stronger equality should be added to the language if required.

Vague predicates may be involved in the generation of RYS and RYS+. Suppose crowds assembling at different places are to be comparatively studied in relation to a database of information on 'relevant' people and others through audiovisual information collected by surveillance cameras. Typically automatic surveillance equipment will not be able to identify much, but information about various subsets of particular crowds and even 'specific people' can be collected through surveillance cameras. Processing information (so called off-line processing) using the whole database with classical rough methods will not work because of scalability issues and may be just irrelevant. Suppose that data about different gatherings have been collected at different times. The collection of the observed

subsets of these crowds can be taken as S. The operations l_i, u_i can originate on the basis of the capabilities of the surveillance equipment like cameras. If one camera can see better in infra-red light, another may see better in daylight, cameras do not have uniform abilities to move and finally placement of the individual camera in question will affect its sight. Definite abstract collections of people may be also taken as approximations of subsets of the crowds based on information from the database and the set of these may form the W of RYS+ or this may be a RYS. These can be used to define predicates like 'vaguely similar' between subsets of the crowd. Because of difficulties in scalability of the computation process of identification, the collections $\underline{S}$ of possible subsets of the crowd should be considered with a non-transitive parthood relation-based on a criteria derived from inclusion of 'relevant' people and others (possibly number of people with some gross characteristics), instead of set inclusion. The latter would naturally lead to aggravation of errors and so should not be used. As of now automated surveillance is imperfect at best and so the example is very real. RYS+ can also be used to model classical rough set theory and some others close to it, but not esoteric RST [47] as approximations of 'definite objects' may not necessarily be the same 'definite objects'. RYS on the other hand can handle almost all types of RST.

In the above two definitions, the parthood relation is assumed to be reflexive and antisymmetric, while the approximation operators are required to preserve parthood. Further any of the lower approximations of an object is required to be part of it and the object is required to be part of any of its upper approximations. The fifth condition is a very weak form of transitivity. The Venn diagram way of picturing things will not work with respect to the mereology, but some intuitive representations may be constructed by modification. Two objects *overlap* if there is a third object that is part of both the objects. Two objects *underlap* if both are part of a third object. In general such an object may not exist, while in ZF all sets taken in pairs will underlap. However if the considerations are restricted to a collection of sets not closed under union, then underlap will fail to hold for at least one pair of sets. *Overcross* is basically the idea of a third object being part of two objects, with the first being not a part of the second. In the above example a set of 'relevant people' may be part of two subsets of the crowd (at different times), but the first crowd may contain other people with blue coloured hair. So the first crowd is not part of the second. If the second crowd contains other people with their hair adorned with roses while such people are not to be located in the first crowd then the two crowds have *proper overlap*.

From the purely mereological point of view a RYS+ is a very complicated object. The sum, product and difference operations are assumed to be defined. They do not follow in general from the conditions on $\mathbf{P}$ in the above. But do so with the assumptions of closed extensional mereology or in other words of the first five of the following axioms. They can also follow from the sixth (Top) and part of the other axioms.

Transitivity $(\mathbf{P}xy, \mathbf{P}yz \longrightarrow \mathbf{P}xz)$,
Supplementation $(\neg\mathbf{P}xy \longrightarrow \exists z(\mathbf{P}zx \wedge \neg\mathbb{O}zy))$,

P5 $\mathbf{U}xy \rightarrow (\exists z)(\forall w)(\mathbf{O}wz \leftrightarrow (\mathbf{O}wz \vee \mathbf{O}wy))$,

P6 $\mathbf{O}xy \rightarrow (\exists z)(\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wz \wedge \mathbf{P}wy))$,

P7 $(\exists z)(\mathbf{P}zx \wedge \neg\mathbf{O}zy) \rightarrow (\exists z)(\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \neg\mathbf{O}wy))$,

Top $(\exists z)(\forall x)\mathbf{P}xz$.

In classical RST, 'supplementation' does not hold, while the weaker version $(\neg\mathbf{P}xy \longrightarrow \exists z(\mathbf{P}zx \wedge \neg\mathbf{O}zy))$ is trivially satisfied due to the existence of the empty object ($\emptyset$). Proper selection of semantic domains is essential for avoiding possible problems of ontological innocence [82], wherein the 'sum' operation may result in non existent objects relative the domain. A similar operation results in 'plural reference' in [65,26], and related papers. The Lesniewski ontology inspired approach originating in [20] assumes the availability of measures beforehand for the definition of the parthood predicate and is not always compatible with and distinct from the present approach.

Examples of Non-Transitivity:

Example-1:

In the classical *handle-door-house* example, parthood is understood in terms of attributes and a level of being part of. The latter being understood in terms of attributes again. The example remains very suggestive in the context of applications of RST and specifically in the context of a RYS. The basic structure of the example has the form:

- Handle is part of a Door,
- Door is part of a House,
- If 'part of' is understood in the sense of 'substantial part of' (defined in terms of attributes), then the handle may not be part of the house.

From the application point of view all the concepts of 'Handle', 'Door' and 'House' can be expected to be defined by sets of relatively atomic sensor (for machines) or sense data. Additionally a large set of approximation related data (based on not necessarily known heuristics) can also be expected. But if we are dealing with sensor data, then it can be reasonable to assume that the data is the result of some rough evolution. Finding one is an inverse problem.

Example-2:

- Let Information Set-A be the processed data from a grey scale version of a colour image.
- Let 'Information-B' be the processed data about distribution of similar colours (at some level).
- In this situation, when 'Information set A' is processed in the light of 'Information set B', then transitivity of the parthood relations about colour related attributes can be expected to be violated.

This example is based on the main context of [75], but is being viewed from a pure rough perspective. $\square$

Example-3:

In processing information from videos in off-line or real time mode, it can be sensible to vary the partitions on the colour space (used in analytics) across key frames. $\square$

Definition 3. *In the above, two approximation operators u_i and l_i will be said to be* S5-dual *to each other if and only if*

$$(\forall A \subset S)\, A^{u_i l_i} = A^{u_i} \;;\; A^{l_i u_i} = A^{l_i}.$$

Throughout this paper it will not be assumed that the operators u_i are S5-dual or dual to the operators l_i in the classical sense in general. It is violated in the study of the lower, upper and bitten upper approximation operators in a tolerance space [39] as RYS. There it will also be required to take the identity operator or repeat the lower approximation operator as a lower approximation operator (as the number of lower and upper approximation are required to match - a trivial requirement).

In almost all applications, the collection of all granules $\mathcal{G}$ forms a subset of the RYS S. But a more general setting can be of some interest especially in a semi-set theoretical setting. This aspect will be considered separately.

Definition 4. *When elements of $\mathcal{G}$ are all elements of S, it makes sense to identify these elements with the help of a unary predicate γ defined via, γx if and only if $x \in \mathcal{G}$. A RYS or a RYS+ enhanced with such a unary predicate will be referred to as a* Inner *RYS (or γRYS for short) or a* Inner *RYS+ (γRYS+ for short) respectively. γRYS will be written as ordered pairs of the form (S, γ) to make the connection with γ clearer. (S, γ) should be treated as an abbreviation for the algebraic system (partial or total)*

$$\langle S,\, \mathbf{P},\, \gamma,\, (l_i)_1^n,\, (u_i)_1^n,\, +,\, \cdot,\, \sim,\, 1 \rangle .$$

Some important classes of properties possibly satisfiable by granules fall under the broad categories of *representability*, *crispness*, *stability*, *mereological atomicity* and *underlap*. If the actual representations are taken into account then the most involved axioms will fall under the category of representability. Otherwise the possible defining properties of a set of granules in a RYS include the following (t_i, s_i are term functions formed with $+$, $\cdot$, $\sim$, while p, r are finite positive integers. $\forall i$, $\exists i$ are meta level abbreviations.) Not all of these properties have been considered in [18]:

Representability, RA $\forall i$, $(\forall x)(\exists y_1, \ldots y_r \in \mathcal{G})\, y_1 + y_2 + \ldots + y_r = x^{l_i}$ and $(\forall x)(\exists y_1, \ldots y_p \in \mathcal{G})\, y_1 + y_2 + \ldots + y_p = x^{u_i}$,

Weak RA, WRA $\forall i$, $(\forall x \exists y_1, \ldots y_r \in \mathcal{G})\, t_i(y_1, y_2, \ldots y_r) = x^{l_i}$ and $(\forall x)(\exists y_1, \ldots y_r \in \mathcal{G})\, t_i(y_1, y_2, \ldots y_p) = x^{u_i}$,

Sub RA $\exists i$, $(\forall x)(\exists y_1, \ldots y_r \in \mathcal{G})\, y_1 + y_2 + \ldots + y_r = x^{l_i}$ and $(\forall x)(\exists y_1, \ldots y_p \in \mathcal{G})\, y_1 + y_2 + \ldots + y_p = x^{u_i}$,

Sub TRA, STRA $\forall i,$ $(\forall x \exists y_1, \dots y_r \in \mathcal{G})$ $t_i(y_1, y_2, \dots y_r) = x^{l_i}$
  and $(\forall x)(\exists y_1, \dots y_r \in \mathcal{G})$ $t_i(y_1, y_2, \dots y_p) = x^{u_i},$
Lower RA, LRA $\forall i,$ $(\forall x)(\exists y_1, \dots y_r \in \mathcal{G})$ $y_1 + y_2 + \dots + y_r = x^{l_i},$
Upper RA, URA $\forall i,$ $(\forall x)(\exists y_1, \dots y_p \in \mathcal{G})$ $y_1 + y_2 + \dots + y_p = x^{u_i},$
Lower SRA, LSRA $\exists i,$ $(\forall x)(\exists y_1, \dots y_r \in \mathcal{G})$ $y_1 + y_2 + \dots + y_r = x^{l_i},$
Upper SRA, USRA $\exists i,$ $(\forall x)(\exists y_1, \dots y_p \in \mathcal{G})$ $y_1 + y_2 + \dots + y_p = x^{u_i},$
Absolute Crispness, ACG For each $i,$ $(\forall y \in \mathcal{G})$ $y^{l_i} = y^{u_i} = y,$
Sub Crispness, SCG $\exists i,$ $(\forall y \in \mathcal{G})y^{l_i} = y^{u_i} = y$ (In [18], this was termed
  'weak crispness'),
Crispness Variants LACG, UACG, LSCG, USCG will be defined as for representability,
Mereological Atomicity,MER $\forall i,$ $(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy, x^{l_i} = x^{u_i} = x \longrightarrow x = y),$
Sub MER,SMER $\exists i,$ $(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy, x^{l_i} = x^{u_i} = x \longrightarrow x = y)$ (In
  [18], this was termed 'weak MER'),
Inward MER, IMER

$$(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy, \bigwedge_i (x^{l_i} = x^{u_i} = x) \longrightarrow x = y),$$

Lower MER, LMER $\forall i,$ $(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy, x^{l_i} = x \longrightarrow x = y),$
Inward LMER, ILMER $(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy, \bigwedge_i (x^{l_i} = x) \longrightarrow x = y),$
MER Variants UMER, LSMER, USMER, IUMER will be defined as for representability,
Lower Stability, LS $\forall i,$ $(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}yx \longrightarrow \mathbf{P}(y)(x^{l_i})),$
Upper Stability, US $\forall i,$ $(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{O}yx \longrightarrow \mathbf{P}(y)(x^{u_i})),$
Stability, ST Shall be the same as the satisfaction of LS and US,
Sub LS, LSS $\exists i,$ $(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}yx \longrightarrow \mathbf{P}(y)(x^{l_i}))$ (In [18], this was
  termed 'LS'),
Sub US, USS $\exists i,$ $(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{O}yx \longrightarrow \mathbf{P}(y)(x^{u_i}))$ (In [18], this was
  termed 'US'),
Sub ST, SST Shall be the same as the satisfaction of LSS and USS,
No Overlap, NO $(\forall x, y \in \mathcal{G})\neg \mathbb{O}xy,$
Full Underlap, FU $\forall i,$ $(\forall x, y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}yz, z^{l_i} = z^{u_i} = z,$
Lower FU, LFU $\forall i,$ $(\forall x, y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}yz, z^{l_i} = z,$
Sub FU, SFU $\exists i,$ $(\forall x, y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}yz, z^{l_i} = z^{u_i} = z,$
Sub LFU, LSFU $\exists i,$ $(\forall x, y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}yz, z^{l_i} = z,$
Unique Underlap, UU For at least one $i,$
  $(\forall x, y \in \mathcal{G})(\mathbb{P}xz, \mathbb{P}yz, z^{l_i} = z^{u_i} = z, \mathbb{P}xb, \mathbb{P}yb, b^{l_i} = b^{u_i} = b \longrightarrow z = b),$
Pre-similarity, PS $(\forall x, y \in \mathcal{G})(\exists z \in \mathcal{G})\mathbf{P}(x \cdot y)(z),$
Lower Idempotence, LI $\forall i,$ $(\forall x \in \mathcal{G})x^{l_i} = x^{l_i l_i},$
Upper Idempotence, UI $\forall i,$ $(\forall x \in \mathcal{G})x^{u_i} = x^{u_i u_i},$
Idempotence, I $\forall i,$ $(\forall x \in \mathcal{G})x^{u_i} = x^{u_i u_i}, x^{l_i} = x^{l_i l_i}.$

All of the above axioms can be written without any quantification over $\mathcal{G}$ in an inner RYS or an inner RYS+. The letter 'I' for 'Inner' will be appended

to the axiom abbreviation in this situation. For example, I will rewrite **LS** as **LSI**:

$$\textbf{LSI}: \ \exists \mathbf{i}, (\forall x, y \in S)\, (\gamma x, \, \mathbf{P}yx \ \longrightarrow \ \mathbf{P}(y)(x^{l_i})).$$

Further, statements of the form $(S, \gamma) \models \textbf{RAI} \rightarrow (S, \gamma) \models \textbf{WRAI}$ ($\models$ being the model satisfaction relation in FOPL) will be abbreviated by '$\textbf{RAI} \twoheadrightarrow \textbf{WRAI}$'.

Proposition 1. *The following holds:*

1. ***RAI*** $\twoheadrightarrow$ ***WRAI****,*
2. ***ACGI*** $\twoheadrightarrow$ ***SCGI****,*
3. ***MERI*** $\twoheadrightarrow$ ***SMERI****,*
4. ***MERI*** $\twoheadrightarrow$ ***IMERI****,*
5. ***FUI*** $\twoheadrightarrow$ ***LUI****.*

The axioms RA, WRA are actually very inadequate for capturing representability in the present author's opinion. Ideally the predicate relating the set in question to the granules should be representable in a nice way. A hierarchy of axioms have been developed for a good classification by the present author and will appear separately in a more complete study. But I will not digress to this issue in the present paper.

In any RST, at least some of these axioms can be expected to hold for a given choice of granules. In the following sections various theorems are proved on the state of affairs.

9.1 Concepts Of Discernibility

In 1-neighbourhood systems, cover-based RST, relation-based RST and more generally in a RYS various types of indiscernibility relations can be defined. In most cases, indiscernibility relations that are definable by way of conditions using term functions involving approximation operators are of interest. Some examples of such conditions, in a RYS of the form specified in the third section, are:

(i) $x \approx_i y$ if and only if $x^{l_i} = y^{l_i}$ and $x^{u_i} = y^{u_i}$ for a specific i,

(ii) $x \approx_a y$ if and only if $x^{l_i} = y^{l_i}$ and $x^{u_i} = y^{u_i}$ for any i,

(iii) $x \approx_b y$ if and only if $x^{l_i} = y^{l_i}$ and $x^{u_i} = y^{u_i}$ for all i,

(iv) $a \approx_c y$ if and only if $(\forall g \in \mathcal{G})(\mathbf{P}gx^\alpha \leftrightarrow \mathbf{P}gy^\alpha)$ with $\alpha \in \{l_i, u_i\}$ for a specific i,

(v) $a \approx_e y$ if and only if $(\forall g \in \mathcal{G})(\mathbf{P}gx^\alpha \leftrightarrow \mathbf{P}gy^\alpha)$ with $\alpha \in \{l_i, u_i\}$ for a specific i,

(vi) $a \approx_f y$ if and only if $(\forall g \in \mathcal{G})(\mathbf{P}gx^\alpha \leftrightarrow \mathbf{P}gy^\alpha)$ with $\alpha \in \{l_i, u_i\}$ for any i,

(vii) $a \approx_h y$ if and only if $(\forall g \in \mathcal{G})(\mathbf{P}gx^\alpha \leftrightarrow \mathbf{P}gy^\alpha)$ with $\alpha \in \{l_i, u_i\}$ for all specific i.

Note that the subscript of $\approx$ has been chosen arbitrarily and is used to distinguish between the different generalised equalities. Weaker varieties of such indiscernibility relations can also be of interest.

9.2 Relative- and Multi-Granulation

Concepts of relativised granulation have been studied in a few recent papers [83,84], under the name 'Multi-Granulation'. These are actually granulations in one application context considered relative the granulation of another application context. For example if two equivalences are used to generate approximations using their usual respective granulations, then 'multi-granulations' have been used according to authors. The relative part is not mentioned explicitly but that is the intended meaning. In our perspective all these are seen as granulations. Multiple approximation spaces, for example use 'multi-granulations'.

The relation between the two contexts has not been transparently formulated in the mentioned papers, but it can be seen that there is a way of transforming the present application context into multiple instances of the other context in at least one perspective. In general it does not happen that approximations in one perspective are representable in terms of the approximation in another perspective (see [85]) and is certainly not a requirement in the definition of multi-granulation. Such results would be relevant for the representation problem of granules mentioned earlier.

10 Relation-Based Rough Set Theory

Theorem 2. *In classical* RST, *if* $\mathcal{G}$ *is the set of partitions, then all of* RA, ACG, MER, AS, FU, NO, PS *hold.* UU *does not hold in general*

Proof. The granules are equivalence classes and RA, NO, ACG, PS follow from the definitions of the approximations and properties of $\mathcal{G}$. MER holds because both approximations are unions of classes and no crisp element can be properly included in a single class. If a class overlaps with another subset of the universe, then the upper approximation of the subset will certainly contain the class by the definition of the latter.

In esoteric RST [47], partial equivalence relations (symmetric, transitive and partially reflexive relations) are used to generate approximations instead of equivalence relations. In the simplest case, the upper and lower approximations of a subset A of a partial approximation space $\langle S, R \rangle$ are defined via ($[x] = \{y;\, (x,y) \in R\}$ being the pseudo-class generated by x)

$$A^l = \bigcup \{[x];\, [x] \subseteq A\}; \ A^u = \bigcup \{[x];\, [x] \cap A \neq \emptyset\}.$$

Theorem 3. *In case of esoteric* RST *[47], with the collection of all pseudo-classes being the granules, all of* RA, MER, NO, UU, US *hold, but* ACG *may not.*

Proof. RA, NO follow from the definition. It is possible that $[x] \subset [x]^u$, so ACG may not hold. US holds as if a granule overlaps another subset, then the upper approximation of the set would surely include the granule.

If we consider a reflexive relation R on a set S and define, $[x] = \{y : (x, y) \in R\}$ -the set of x-relateds and define the lower and upper approximation of a subset $A \subseteq S$ via

$$A^l = \cup\{[x] : [x] \subseteq A,\ x \in A\} \text{ and}$$

$$A^u = \cup\{[x] : [x] \cap A \neq \emptyset\ x \in A\},$$

($A^l \subseteq A \subseteq A^u$ for a binary relation R is equivalent to its reflexivity [43,86]) then we can obtain the following about granulations of the form $\{[x] : x \in S\}$:

Theorem 4. *RA, LFU holds, but none of MER, ACG, LI, UI, NO, FU holds in general.*

Proof. RA holds by definition, LFU holds as the lower approximation of the union of two granules is the same as the union. It is easy to define an artificial counter example to support the rest of the statement.

Let $\langle S, (R_i)_i \in K \rangle$ be a multiple approximation space [77], then the strong lower, weak lower, strong upper and weak upper approximations of a set $X \subseteq S$ shall be defined as follows (modified terminology):

1. $X^{ls} = \bigcap_i X^{li}$,
2. $X^{us} = \bigcup_i X^{ui}$,
3. $X^{lw} = \bigcup_i X^{li}$,
4. $X^{uw} = \bigcap_i X^{ui}$.

Theorem 5. *In a multiple approximation of the above form, taking the set of granules to be the collection of all equivalence classes of the R_is, LSRA, USRA, LSS, USS holds, but all variants of rest do not hold always.*

Proof. X^{lw}, X^{us} are obviously unions of equivalence classes. The LSS, USS part for these two approximations respectively can also be checked directly. Counterexamples can be found in [77]. But it is possible to check these by building on actual possibilities. If there are only two distinct equivalences, then at least two classes of the first must differ from two classes of the second. The ls approximation of these classes will strictly be a subset of the corresponding classes, so CG will certainly fail for the (ls, us) pair. Continuing the argument, it will follow that SCG, ACG cannot hold in general. The argument can be extended to other situations.

Since multiple approximations spaces are essentially equivalent to special types of tolerance spaces equipped with the largest equivalence contained in the tolerance, the above could as well have been included in the following subsection.

10.1 Tolerance Spaces

In TAS of the form $\langle S, T \rangle$, all of the following types of granules with corresponding approximations have been used in the literature:

1. The collection of all subsets of the form $[x] = \{y : (x, y) \in T\}$ will be denoted by $\mathcal{T}$.
2. The collection of all blocks, the maximal subsets of S contained in T, will be denoted by $\mathcal{B}$. Blocks have been used as granules in [35,31,87,24] and others.
3. The collection of all finite intersections of blocks will be denoted by $\mathcal{A}$.
4. The collection of all intersections of blocks will be denoted by $\mathcal{A}_\sigma$ [24].
5. Arbitrary collections of granules with choice functions operating on them [18].
6. The collection of all sets formed by intersection of sets in $\mathcal{T}$ will be denoted by $\mathcal{TI}$.

For convenience $H_0 = \emptyset$, $H_{n+1} = S$ will be assumed whenever the collection of granules $\mathcal{G}$ is finite and $\mathcal{G} = \{H_1, \ldots H_n\}$.

In a TAS $\langle S, T \rangle$, for a given collection of granules $\mathcal{G}$ definable approximations of a set $A \subseteq S$ include:

(i) $A^{l\mathcal{G}} = \bigcup \{H : H \subseteq A, H \in \mathcal{G}\}$,
(ii) $A^{u\mathcal{G}} = \bigcup \{H : H \cap A \neq \emptyset, H \in \mathcal{G}\}$,
(iii) $A^{l2\mathcal{G}} = \bigcup \{\cap_{i \in I} H_i^c : \cap_{i \in I} H_i^c \subseteq A, H \in \mathcal{G}\ I \subseteq \mathbf{N}(n+1)\}$,
(iv) $A^{u1\mathcal{G}} = \bigcap \{\cup_{i \in I} H_i : A \subseteq \cup_{i \in I} H_i, I \subseteq \mathbf{N}(n+1)\}$,
(v) $A^{u2\mathcal{G}} = \bigcap \{H_i^c : A \subseteq H_i^c, I \in \{0, 1, \ldots, n\}\}$.

But not all approximations fit it into these schemas in an obvious way. These include:

(i) $A^{l+} = \{y : \exists x (x, y) \in T, [x] \subseteq A\}$ [35],
(ii) $A^{u+} = \{x ; (\forall y)((x, y) \in T \longrightarrow [y]_T \cap A \neq \emptyset)\}$,
(iii) Generalised bitten upper approximation : $A^{ubg} = A^{ug} \setminus A^{clg}$ - this is a direct generalisation of the bitten approximation in [39,38].

Theorem 6. *In the TAS context, with $\mathcal{T}$ being the set of granules and restricting to the approximations $l\mathcal{T}, u\mathcal{T}$, all of RA, MER, ST and weakenings thereof hold, but others do not hold in general.*

Proof. RA follows from definition. For MER, if $A \subseteq [x]$ and $A^{l\mathcal{T}} = A^{u\mathcal{T}} = A$, then as $[x] \cap A \neq \emptyset$, so $[x] \subseteq A^{u\mathcal{T}} = A$. So $A = [x]$. Crispness fails because it is possible that $[x] \cap [y] \neq \emptyset$ for distinct x, y.

Theorem 7. *If $\langle \underline{S}, T \rangle$ is a tolerance approximation space with granules $\mathcal{T}$ and the approximations $l\mathcal{T}, l+, u\mathcal{T}, u+$, then RA, NO, ACG do not hold, but SRA, SMER, SST, IMER, MER, US holds*

Proof. RA does not hold due to $l+, u+$, ACG fails by the previous theorem. 'Sub' properties have been verified in the previous theorem, while the rest can be checked directly.

Theorem 8. *In Bitten* RST *[39,38], (taking* $\mathcal{G}$ *to be the set of T-relateds and restricting ourselves to the lower, upper and bitten upper approximations alone),* SRA *does not hold for the bitten upper approximation if '+, ·' are interpreted as set union and intersection respectively.* MER, NO *do not hold in general, but* IMER, SCG, LS, LU, SRA *hold.*

Proof. The proof can be found in [39]. If unions and intersections alone are used to construct terms operations, then the bited upper approximation of a set of the form $[x]_T$ (x being an element of the tolerance approximation space) may not be representable as it is the difference of the upper approximation and the lower approximation of the complement of $[x]_T$. But if $\sim$, for set complements is also permitted, then there are no problems with the representability in the WRA sense.

In [24], a semantics of tolerance approximation spaces for the following theorem context are considered, but the properties of granules are not mentioned.

Theorem 9. *Taking* $\mathcal{G}$ *to be* $\mathcal{A}_\sigma$ *and restricting ourselves to lower and bitten upper approximations alone* RA, ACG, NO *do not hold while* LRA, MER, LACG, LMER, UMER, ST *do hold.*

Proof. If H is a granule, then it is an intersection of blocks. It can be deduced that the lower approximation of H coincides with itself, while the bitten upper approximation is the union of all blocks including H. LRA is obvious, but URA need not hold due to the bitten operation. If a definite set is included in a granule, then it has to be a block that intersects no other block and so the granule should coincide with it. So MER holds.

11   Cover-Based Rough Set Theory

The notation for approximations in cover-based RST, is not particularly minimalistic. This is rectified for easy comprehension below. I follow superscript notation strictly. 'l, u' stand for lower and upper approximations and anything else following those signify a type. If $X \subseteq S$, then let

(i) $X^{u1+} = X^{l1} \cup \bigcup \{\mathrm{Md}(x) : x \in X\}$,
(ii) $X^{u2+} = \bigcup \{K : K \in \mathcal{S}, K \cap X \neq \emptyset\}$,
(iii) $X^{u3+} = \bigcup \{\mathrm{Md}(x) : x \in X\}$,
(iv) $X^{u4+} = X^{l1} \cup \{K : K \cap (X \setminus X^{l1}) \neq \emptyset\}$,
(v) $X^{u5+} = X^{l1} \cup \bigcup \{nbd(x) : x \in X \setminus X^{l1}\}$,
(vi) $X^{u6+} = \{x : nbd(x) \cap X \neq \emptyset\}$,
(vii) $X^{l6+} = \{x : nbd(x) \subseteq X\}$.

The approximation operators $u1+, \ldots, u5+$ (corresponding to first, ..., fifth approximation operators used in [45] and references therein) are considered with the lower approximation operator $l1$ in general. Some references for cover-based

RST include [42,40,46,88,43,45,89,37,48]. The relation between cover-based RST and relation-based RST are considered in [48,45]. For a cover to correspond to a tolerance, it is necessary and sufficient that the cover be normal - a more general version of this result can be found in [90]. When such reductions are possible, then good semantics in Meta-R perspective are possible. The main results of [48] provide a more complicated correspondence between covers and sets equipped with multiple relations or a relation with additional operations. The full scope of the results are still under investigation. So, in general, cover-based RST is more general than relation-based RST. From the point of view of expression, equivalent statements formulated in the former would be simpler than in the latter.

It can be shown that:

Proposition 2. *In the above context,*

- $\mathrm{Md}(x)$ *is invariant under removal of reducible elements,*
- $(nbd(x))^{l6+} = nbd(x),$
- $nbd(x) \subseteq (nbd(x))^{6+}.$

The following pairs of approximation operators have also been considered in the literature (the notation of [46] has been streamlined; $lp1$, $lm1$ corresponds to $\underline{P}_1$, $\underline{C}_1$ respectively and so on).

(i) $X^{lp1} = \{x : Fr(x) \subseteq X\},$
(ii) $X^{up1} = \bigcup\{K : K \in \mathcal{K}, K \cap X \neq \emptyset\},$
(iii) $X^{lp2} = \bigcup\{Fr(x); Fr(x) \subseteq X\},$
(iv) $X^{up2} = \{z : (\forall y)(z \in Fr(y) \rightarrow Fr(y) \cap X \neq \emptyset)\},$
(v) $X^{lp3} = X^{l1},$
(vi) $X^{up3} = \{y : \forall K \in \mathcal{K}(y \in K \rightarrow K \cap X \neq \emptyset)\},$
(vii) X^{lp4}, X^{up4} are the same as the classical approximations with respect to $\pi(\mathcal{K})$ - the partition generated by the cover $\mathcal{K}$.
(viii) $X^{lm1} = X^{l1} = X^{lp3},$
(ix) $X^{um1} = X^{u2},$
(x) $X^{lm2} = X^{l6+},$
(xi) $X^{um2} = X^{u6+},$
(xii) $X^{lm3} = \{x; (\exists u)u \in nbd(x), nbd(u) \subseteq X\},$
(xiii) $X^{um3} = \{x; (\forall u)(u \in nbd(x) \rightarrow nbd(u) \cap X \neq \emptyset)\},$
(xiv) $X^{lm4} = \{x; (\forall x)(x \in nbd(u) \rightarrow nbd(u) \subseteq X)\},$
(xv) $X^{um4} = X^{u6+} = X^{um2},$
(xvi) $X^{lm5} = \{x; (\forall u)(x \in nbd(u) \rightarrow u \in X)\},$
(xvii) $X^{um5} = \bigcup\{nbd(x); x \in X\}.$

Example-1:

Let $S = \{a, b, c, e, f, g, h, i, j\},$

$$\mathcal{K} = \{K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9\},$$

$$K_1 = \{a, b\}, \ K_2 = \{a, c, e\}, \ K_3 = \{b, f\}, \ K_4 = \{j\}, \ K_5 = \{f, g, h\},$$

$$K_6 = \{i\}, \ K_7 = \{f, g, j, a\}, \ K_8 = \{f, g\}, \ K_9 = \{a, j\}.$$

The following table lists some of the other popular granules:

Element: x	$Fr(x)$	$Md(x)$	$nbd(x)$
a	$S \setminus \{h, i\}$	$\{K_1, K_2, K_3\}$	$\{a\}$
b	$\{a, b, f\}$	$\{K_3\}$	$\{b\}$
c	$\{a, c, e\}$	$\{K_2\}$	$\{a, c, e\}$
e	$\{a, c, e\}$	$\{K_2\}$	$\{a, c, e\}$
f	$S \setminus \{c, e, i\}$	$\{K_3, K_8\}$	$\{f\}$
g	$\{a, f, g, h, j\}$	$\{K_8\}$	$\{f, g\}$
h	$\{f, g, h\}$	$\{K_5\}$	$\{f, g, h\}$
i	$\{i\}$	$\{K_6\}$	$\{i\}$
j	$\{a, f, g, j\}$	$\{K_4\}$	$\{j\}$

The best approach to granulation in these types of approximations is to associate an initial set of granules and then refine them subsequently. Usually a refinement can be expected to be generable from the initial set through relatively simple set theoretic operations. In more abstract situations, the main problem would be of representation and the results in these situations would be the basis of possible abstract representation theorems. The theorems proved below throw light on the fine structure of granularity in the cover-based situations:

Theorem 10. *In AUAI RST [37,47], with the collection of granules being $\mathcal{K}$ and the approximation operators being (l_1, l_2, u_1 and u_2), WRA, LS, SCG, LU, IMER holds, but ACG, RA, SRA, MER do not hold in general.*

Proof. WRA holds if the complement, union and intersection operations are used in the construction of terms in the WRA condition. ACG does not hold as the elements of $\mathcal{K}$ need not be crisp with respect to $l2$. Crispness holds with respect to $l1, u1$, so SCG holds. MER need not hold as it is violated when a granule is properly included in another. IMER holds as the pathology negates the premise. It can also be checked by a direct argument.

From the definition of the approximations in AUAI and context, it should be clear that all pairings of lower and upper approximations are sensible generalisations of the classical context. In the following two of the four are considered. These pairs do not behave well with respect to duality, but are more similar with respect to representation in terms of granularity.

Theorem 11. *In AUAI RST, with the collection of granules being $\mathcal{K}$ and the approximation operators being (l_1, u_1), WRA, ACG, ST, LU holds, but MER, NO, FU, RA do not hold in general.*

Theorem 12. *In AUAI RST, with the collection of granules being $\mathcal{K}$ and the two approximation operators being (l_2, u_2),* WRA, ST *holds, but* ACG, MER, RA, NO *do not hold in general.*

Proof. If $K \in \mathcal{K}$, $K \subseteq X$ (X being any subset of S) and $y \in K^{l2}$, then y must be in at least one intersection of the sets of the form $S \setminus K_i$ (for $i \in I_0$, say) and so it should be in each of these $S \setminus K_i \subseteq K \subseteq X$. This will ensure $y \in X^{l2}$. So lower stability holds. Upper stability can be checked in a similar way.

Theorem 13. *When the approximations are $(lp1, up1)$ and with the collection of granules being $\{Fr(x)\}$, all of* MER, URA, UMER *hold, but* ACG, NO, LS *do not hold necessarily*

Proof. For an arbitrary subset X, $X \subseteq Fr(x)$ for some $x \in S$ and $X^{lp1} = X^{up1} = X$ would mean that $Fr(x) = X$ as $Fr(x) \subseteq X^{up1}$ would be essential. So UMER and the weaker condition MER holds. URA is obvious from the definitions.

Theorem 14. *When the approximations are $(lp2, up2)$ and with the collection of granules being $\{Fr(x)\}$, all of* MER, LMER, RA, LCG *hold, but* ACG, NO, LS *do not hold necessarily*

Proof. From the definition, it is clear that RA, LCG hold. If for an arbitrary subset X, $X \subseteq Fr(x)$ for some $x \in S$ and $X^{lp2} = X = X^{up2}$, then X is a union of some granules of the form $Fr(y)$. If $x \in X$, then it is obvious that $X = Fr(x)$.

If $x \in Fr(x) \setminus X$ and $F(x)$ is an element of the underlying cover $\mathcal{S}$, then again it would follow that $X = Fr(x)$. Finally if $x \in Fr(x) \setminus X$ and $Fr(x)$ is a union of elements of the cover intersecting X, then we would have a contradiction. So MER follows.

If for an arbitrary subset X, $X \subseteq Fr(x)$ for some $x \in S$ and $X^{lp2} = X$ and $x \in Fr(x) \setminus X$, then we will have a contradiction. So LMER holds.

Theorem 15. *When the approximations are $(lp3, up3)$ and with the collection of granules being $\mathcal{K}$, all of* MER, RA, ST, LCG, LU *hold, but* ACG, NO *do not hold necessarily.*

Proof. Both the lower and upper approximations of any subset of S is eventually a union of elements $\mathcal{K}$, so RA holds. Other properties follow from the definitions. Counter examples are easy.

Theorem 16. *When the approximations are $(lp4, up4)$ and with the collection of granules being $\pi(\mathcal{K})$, all of* RA, ACG, MER, AS, FU, NO, PS *hold.*

Proof. With the given choice of granules, this is like the classical case.

Theorem 17. *When the approximations are $(lp4, up4)$ and with the collection of granules being $\mathcal{K}$, all of* WRA, ACG, AS *hold, while the rest may not.*

Proof. WRA holds because elements of $\pi(\mathcal{K})$ can be represented set theoretically in terms of elements of $\mathcal{K}$. Lower and upper approximations of elements of $\mathcal{K}$ are simply unions of partitions of the elements induced by $\pi(\mathcal{K})$.

Theorem 18. *When the approximations are* $(lm1, um1)$ *and with the collection of granules being* $\mathcal{K}$*, all of* WRA, LS, LCG *hold, but* RA, ST, LMER *do not hold necessarily. For* WRA*, complementation is necessary.*

Proof. If $\mathcal{K}$ has an element properly included in another, then LMER will fail. If complementation is also permitted, then WRA will hold. Obviously RA does not hold. Note the contrast with the pair $(lp3, up3)$ with the same granulation. ∎

Theorem 19. *When the approximations are* $(lm2, um2)$ *and with the collection of granules being* $\mathcal{N}$*, all of* LCG, LRA, ST, MER *holds, but* RA, ACG, LMER, NO *do not hold necessarily.*

Proof. If $y \in nbd(x)$ for any two $x, y \in S$, then $nbd(y) \subseteq nbd(x)$ and it is possible that $x \notin nbd(y)$, but it is necessary that $x \in nbd(x)$. So $(nbd(x))^{lm2} = nbd(x)$, but $(nbd(x))^{um2}$ need not equal $nbd(x)$. LRA will hold as the lower approximation will be a union of neighbourhoods, but this need happen in case of the upper approximation. NO is obviously false. The upper approximation of a neighbourhood can be a larger neighbourhood of a different point. So ACG will not hold in general. MER can be easily checked. ∎

Theorem 20. *When the approximations are* $(l6+, u6+)$ *and with the collection of granules being* $\mathcal{N}$*, all of* LCG, LRA, ST, MER *holds, but* RA, ACG, LMER, NO *do not hold necessarily.*

Proof. Same as above. ∎

Theorem 21. *When the approximations are* $(l1, u1+)$ *and with the collection of granules being* $\mathcal{K}$*, all of* ACG, RA, FU, LS *holds, but* MER, LMER, NO *do not hold necessarily.*

Proof. RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^l = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u1+} = A$, but $A \neq B$. So $MER, LMER$ cannot hold in general. ∎

Theorem 22. *When the approximations are* $(l1, u2+)$ *and with the collection of granules being* $\mathcal{K}$*, all of* ACG, RA, FU, ST *holds, but* MER, LMER, NO *do not hold necessarily.*

Proof. RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^l = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u1+} = A$, but $A \neq B$. So MER, LMER cannot hold in general. If a granule K is included in a subset X of S, then it will be included in the latter's lower approximation. If K intersects another subset, then the upper approximation of the set will include K. So ST holds. ∎

Theorem 23. *When the approximations are* $(l1, u3+)$ *and with the collection of granules being* $\mathcal{K}$*, all of* ACG, RA, FU, LS *holds, but* MER, LMER, NO *do not hold necessarily.*

Proof. RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^l = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u3+} = A$, but $A \neq B$. So MER, LMER cannot hold in general. The union of any two granules is a definite element, so FU holds.

Theorem 24. *When the approximations are $(l1, u4+)$ and with the collection of granules being $\mathcal{K}$, all of ACG, RA, FU, LS holds, but MER, LMER, NO do not hold necessarily*

Proof. RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^l = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u3+} = A$, but $A \neq B$. So MER, LMER cannot hold in general. The union of any two granules is a definite element, so FU holds.

Theorem 25. *When the approximations are $(l1, u5+)$ and with the collection of granules being $\mathcal{K}$, all of ACG, RA, FU, LS holds, but MER, LMER, NO do not hold necessarily.*

Proof. RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^l = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u3+} = A$, but $A \neq B$. So MER, LMER cannot hold in general. The union of any two granules is a definite element, so FU holds.

Apparently the three axioms WRA, LS, LU hold in most of the known theories and with most choices of granules. This was the main motivation for the following definition of admissibility of a set to be regarded as a set of granules.

Definition 5. *A subset $\mathcal{G}$ of S in a RYS will be said to be an* admissible set of granules *provided the properties WRA, LS and LU are satisfied by it.*

In cover-based RSTs, different approximations are defined with the help of a determinate collection of subsets. These subsets satisfy the properties WRA, LS and FU and are therefore admissible granules. But they do not in general have many of the nicer properties of granules in relation to the approximations. However, at a later stage it may be possible to refine these and construct a better set of granules (see [48], for example) for the same approximations. Similar process of refinement can be used in other types of RSTs as well. For these reasons, the former will be referred to as *initial granules* and the latter as relatively *refined granules*. It may happen that more closely related approximations may as well be formed by such process.

11.1 Classification of Rough Set Theory

From the point of view of logic or rough reasoning associated, RST can be classified according to:

1. General context and definition of approximations.
2. Modal logic perspective from Meta-C (see [52]).

3. Frechet Space perspective from Meta-C [88].
4. Global properties of approximations viewed as operators at Meta-C (see for example [43]).
5. Rough equality based semantic perspective at Meta-R (see for example [47]).
6. Granularity Based Perspective (this paper).
7. Algebraic perspectives at Meta-C.
8. Algebraic perspectives at Meta-R.
9. Others.

In general the meta-C classification is not coherent with meta-R features. The problems are most severe in handling quotients induced by rough equalities. In the algebraic perspective, the operations at meta-C level are not usually preserved by quotients.

For algebraic approaches at Meta-C, the classification goes exactly as far as duality (for formulation of logic) permits. Modal approaches can mostly be accommodated within the algebraic. But the gross classification into relation-based, cover-based and more abstract forms of RST remains coherent with desired duality. Of course, the easier cases that fit into the scheme of [88] can be explored in more ways from the Meta-C perspective. The common algebraic approaches to modal logics further justifies such a classification as:

- The representation problem of algebraic semantics corresponds most naturally to the corresponding problems in duality theory of partially or lattice-ordered algebras or partial algebras. Some of the basic duality results for relation-based RST are considered in [91]. [92] is a brief survey of applications of topology free duality. For studies proceeding from a sequent calculus or modal axiomatic system point of view, the classification also corresponds to the difficulty of the algebraization problem in any of the senses [3,52,93,94].
- The duality mentioned in the first point often needs additional topology in explicit terms. The actual connection of the operator approach in RST with point set topology is : Start from a collection with topological or pre-topological operators on it, form possibly incompatible quotients and study a collection of some of these types of objects with respect to a new topology (or generalisations thereof) on it. This again supports the importance of the classification or the mathematical uniqueness of the structures being studied.

The present axiomatic approach to granules does provide a level of classification at Meta-C. But the way in which approximations are generated by granules across the different cases is not uniform and so comparisons will lack the depth to classify Meta-R dynamics, though the situation is definitely better than in the other mentioned approaches. One way out can be through the representation problem. It is precisely for this reason that the classification based on the present axiomatic approach is not stressed.

For those who do not see the point of the contamination problem, the axiomatic theory developed provides a supportive classification for general RST.

12   Dialectical Counting, Measures

To count a collection of objects in the usual sense it is necessary that they be distinct and well defined. So a collection of well defined distinct objects and indiscernible objects can be counted in the usual sense from a higher meta level of perception. Relative this meta level, the collection must appear as a set. In the counting procedures developed, the use of this meta level is minimal and certainly far lesser than in other approaches. It is dialectical as two different interpretations are used simultaneously to complete the process. These two interpretations cannot be merged as they have contradictory information about relative discernibility. Though the classical interpretation is at a relatively higher meta level, it is still being used in the counting process and the formulated counting is not completely independent of the classical interpretation.

A strictly formal approach to these aspects will be part of a forthcoming paper.

Counting of a set of objects of an approximation space and that of its power set are very different as they have very different kinds of indiscernibility inherent in them. The latter possess a complete evolution for all of the indiscernibility present in it while the former does not. Counting of elements of a RYS is essentially a generalisation of the latter. In general any lower level structure like an approximation space corresponding to a 1-neighbourhood system [43] or a cover need not exist in any unique sense. The axiomatic theory of granules developed in the previous sections provides a formal exactification of these aspects.

Let S be a RYS, with R being a derived binary relation (interpreted as a weak indiscernibility relation) on it. As the other structure on S will not be explicitly used in the following, it suffices to take S to be an at most countable set of elements in ZF, so that it can be written as a sequence of the form: $\{x_1, x_2, \ldots, x_k, \ldots, \}$. Taking $(a, b) \in R$ to mean 'a is weakly indiscernible from b' concepts of *primitive counting* regulated by different types of meta level assumptions are defined below. The adjective *primitive* is intended to express the minimal use of granularity and related axioms.

Indiscernible Predecessor Based Primitive Counting (IPC)

In this form of 'counting', the relation with the immediate preceding step of counting matters crucially.

1. Assign $f(x_1) = 1_1 = s^0(1_1)$.
2. If $f(x_i) = s^r(1_j)$ and $(x_i, x_{i+1}) \in R$, then assign $f(x_{i+1}) = 1_{j+1}$.
3. If $f(x_i) = s^r(1_j)$ and $(x_i, x_{i+1}) \notin R$, then assign $f(x_{i+1}) = s^{r+1}(1_j)$.

The 2-type of the expression $s^{r+1}(1_j)$ will be j. Successors will be denoted by the natural numbers indexed by 2-types.

History Based Primitive Counting (HPC)

In HPC, the relation with all preceding steps of counting will be taken into account.

1. Assign $f(x_1) = 1_1 = s^0(1_1)$.
2. If $f(x_i) = s^r(1_j)$ and $(x_i, x_{i+1}) \in R$, then assign $f(x_{i+1}) = 1_{j+1}$.
3. If $f(x_i) = s^r(1_j)$ and $\bigwedge_{k<i+1}(x_k, x_{i+1}) \notin R$, then assign $f(x_{i+1}) = s^{r+1}(1_j)$.

In any form of counting in this section, if $f(x) = \alpha$, then $\tau(\alpha)$ will denote the least element that is related to x, while $\epsilon(\alpha)$ will be the greatest element preceding α, which is related to α.

History Based Perceptive Partial Counting (HPPC)

In HPPC, the valuation set shall be $N \cup \{*\} = N^*$ with $*$ being an abbreviation for 'undefined'.

1. Assign $f(x_1) = 1 = s^0(1)$.
2. If $(x_i, x_{i+1}) \in R$, then assign $f(x_{i+1}) = *$.
3. If $Max_{k<i}f(x_k) = s^r(1)$ and $\bigwedge_{k<i}(x_k, x_i) \notin R$, then assign $f(x_i) = s^{r+1}(1)$.

Clearly, HPPC depends on the order in which the elements are counted.

Indiscernible Predecessor Based Partial Counting (IPPC)

This form of counting is similar to HPPC, but differs in using the IPC methodology.

1. Assign $f(x_1) = 1 = s^0(1)$.
2. If $(x_i, x_{i+1}) \in R$, then assign $f(x_{i+1}) = *$.
3. If $Max_{k<i}f(x_k) = s^r(1)$ and $(x_{i-1}, x_i) \notin R$, then assign $f(x_i) = s^{r+1}(1)$.

Definition 6. *A generalised approximation space $\langle S, R \rangle$ will be said to be* IPP *Countable (resp HPP Countable) if and only if there exists a total order on S, under which all the elements can be assigned a unique natural number under the IPPC (resp HPPC) procedure . The ratio of number of such orders to the total number of total orders will be said to be the* IPPC *Index (resp HPPC Index).*

These types of countability are related to measures, applicability of variable precision rough methods and combinatorial properties. The counting procedures as such are strongly influenced by the meaning of associated contexts and are not easy to compare. In classical RST, HPC can be used to effectively formulate a semantics, while the IPC method yields a weaker version of the semantics in general. This is virtually proved below:

Theorem 26. *In the HPC procedure applied to classical RST, a set of the form $\{l_p, 1_q, \ldots 1_t\}$ is a granule if and only if it is the greatest set of the form with every element being related to every other and for any element α in it $\tau(\alpha) = f^\dashv(l_p)$ and $\epsilon(\alpha) = f^\dashv(1_t)$ (for at least two element sets). Singleton granules should be of the form $\{l_p\}$ with this element being unrelated to all other elements in S.*

Theorem 27. *All of the following are decidable in a finite number of steps in a finite AS by the application of the HPC counting method :*

1. *Whether a given subset A is a granule or otherwise.*
2. *Whether a given subset A is the lower approximation of a set B.*
3. *Whether a given subset A is the upper approximation of a set B.*

Proof. Granules in the classical RST context are equivalence classes and elements of such classes relate to other elements within the class alone.

Theorem 28. *The above two theorems do not hold in the IPC context. The criteria of the first theorem defines a new type of granules in totally ordered approximation spaces and relative this the second theorem may not hold.*

Proof. By the IPC way of counting, equivalence classes can be split up into many parts and the numeric value may be quite deceptive. Looking at the variation of the 2-types, the criteria of the first theorem above will define parts of granules, that will yield lower and upper approximations distinct from the classical ones respectively.

Extension of these counting processes to TAS for the abstract extraction of information about granules, becomes too involved even for the simplest kind of granules. However if the IPC way of counting is combined with specific parthood orders, then identification of granules can be quite easy. The other way is to consider entire collections of countings according to the IPC scheme.

Theorem 29. *Among the collection of all IPC counts of a RYS S, any one with maximum number of 1_is occurring in succession determines all granules. If S is the power set of an approximation space, then all the definite elements can also be identified from the distribution of sequences of successive 1_is.*

Proof. • Form the collection $\mathcal{I}(S)$ of all IPC counts of S.
- For $\alpha, \beta \in \mathcal{I}(S)$, let $\alpha \preceq \beta$ if and only if β has longer strings of 1_i s towards the left and more number of strings of 1_is of length greater than 1, than α.
- The maximal elements in this order suffice for this theorem.

Example

Let $S = \{f, b, c, a, k, i, n, h, e, l, g, m\}$ and let R, Q be the reflexive and transitive closure of the relation

$$\{(a, b),\ (b, c),\ (e, f),\ (i, k), (l, m),\ (m, n),\ (g, h)\}$$

and

$$\{(a, b),\ (e, f),\ (i, k),\ (l, m),\ (m, n)\}$$

respectively. Then $\langle S,\ R \rangle$ and $\langle S,\ Q \rangle$ are approximation spaces. This set can be counted (relative R) in the presented order as follows:

IPC $\{1_1, 2_1, 1_2, 1_3, 2_3, 1_4, 2_4, 3_4, 1_5, 2_5, 1_6, 2_6\}$,

HPC $\{1_1, 2_1, 1_2, 1_3, 2_3, 1_4, 1_5, 2_5, 1_6, 1_7, 1_8, 1_9\}$,

HPPC $\{1_1, 2_1, *, *, 3_1, *, 4_1, 5_1, *, *, *, *\}$,

HPC Relative Q: $\{1_1, 2_1, 3_1, 1_2, 2_2, 1_3, 2_3, 3_3, 1_4, 1_5, 2_5, 1_6\}$.

Now $S|Q = \{\{a, b\}, \{c\}, \{e, f\}, \{i, k\}, \{l, m, n\}, \{g\}, \{h\}\}$ and the positive region of Q relative R is

$$POS_R(Q) = \bigcup_{X \in S|Q} X^l = \{e, f, l, m, n\} = \{f, n, e, l, m\}.$$

The induced HPC counts of this set are respectively $\{1_1, 2_5, 1_6, 1_7, 1_9\}$ and $\{1_1, 2_3, 1_4, 1_5, 1_6\}$.

13 Generalized Measures

According to Pawlak's approach [17] to theories of knowledge from a classical rough perspective, if $\underline{S}$ is a set of attributes and R an indiscernibility relation on it, then sets of the form A^l and A^u represent clear and definite concepts. If Q is another stronger equivalence ($Q \subseteq R$) on $\underline{S}$, then the state of the knowledge encoded by $\langle \underline{S}, Q \rangle$ is a *refinement* of that of $S = \langle \underline{S}, R \rangle$. The R-positive region of Q is defined to be

$$POS_R(Q) = \bigcup_{X \in S|Q} X^{l_R}; \quad X^{l_R} = \bigcup \{[y]_R; [y]_R \subseteq X\}.$$

The degree of dependence of knowledge Q on R $\delta(Q, R)$ is defined by

$$\delta(Q, R) = \frac{Card(POS_R(Q))}{Card(S)}.$$

Definition 7. *The granular dependence degree of knowledge Q on R, $gk(Q, R)$ will be the tuple $(k_1, \ldots, k_r)$, with the k_i's being the ratio of the number of granules of type i included in $POS_R(Q)$ to card(S).*

Note that the order on S, induces a natural order on the granules (classes) generated by R, Q respectively. This vector cannot be extracted from a single HPC count in general (the example in the last section should be suggestive). However if the granulation is taken into account, then much more information (apart from the measure) can be extracted.

Proposition 3. *If $gk(Q, R) = (k_1, k_2, \ldots, k_r)$, then $\delta(Q, R) = \sum k_i$.*

The concepts of consistency degrees of multiple models of knowledge introduced in [95] can also be improved by a similar strategy:

If $\delta(Q, R) = a$ and $\delta(R, Q) = b$, then the consistency degree of Q and R, $Cons(Q, R)$ is defined by

$$Cons(Q, R) = \frac{a + b + nab}{n + 2},$$

where n is the consistency constant. With larger values of n, the value of the consistency degree becomes smaller.

Definition 8. *If $gk(Q, R) = (k_1, k_2, \ldots, k_r)$ and $gk(R, Q) = (l_1, l_2, \ldots, l_p)$ then the* granular consistency degree *$gCons(Q, R)$ of the knowledge represented by Q, R, will be*

$$gCons(Q, R) = (k_1^*, \ldots, k_r^*, l_1^*, \ldots, l_p^*, nk_1^* l_1, \ldots nk_r^* l_p),$$

where $k_i^* = \frac{k_i}{n+2}$ for $i = 1, \ldots, r$ and $l_j^* = \frac{l_j}{n+2}$ for $j = 1, \ldots, p$.

The knowledge interpretation can be extended in a natural way to other general RST (including TAS) and also to choice inclusive rough semantics [18]. Construction of similar measures is however work in progress. With respect to the counting procedures defined, these two general measures are relatively constructive provided granules can be extracted. This is possible in many of the cases and not always. They can be replaced by a technique of defining atomic sub-measures based on counts and subsequently combining them.

Algebraic semantics for these and related measures have been developed in [6] by the present author (after the writing of this paper). The reader is referred to the same for a fuller treatment of this section.

13.1 Rough Inclusion Functions

Various rough inclusion and membership functions with related concepts of degrees are known in the literature (see [96] and references therein). If these are not representable in terms of granules through term operations formed from the basic ones [18], then they are not truly functions/degrees of the rough domain. To emphasize this aspect, I will refer to such measures as being *non-compliant for the rough context.* I seek to replace such non-compliant measures by tuples satisfying the criteria. Based on this heuristic, I would replace the rough inclusion function

$$k(X, Y) = \begin{cases} \dfrac{\#(X \cap Y)}{\#(X)}, & \text{if } X \neq \emptyset, \\ 1, & \text{else,} \end{cases}$$

with

$$k^*(X, Y) = \begin{cases} (y_1, y_2, \ldots, y_r) & \text{if } X^l \neq \emptyset, \\ \left(\dfrac{1}{r}, \ldots, \dfrac{1}{r}\right), & \text{else} \end{cases}$$

where

$$\frac{\#(G_i) \cdot \chi_i(X \cap Y)}{\#(X^l) = y_i}, \quad i = 1, 2, \ldots r.$$

Here it is assumed that $\{G_1, \ldots, G_r\} = \mathcal{G}$ (the collection of granules) and that the function χ_i is being defined via,

$$\chi_i(X) = \begin{cases} 1, & \text{if } G_i \subseteq X, \\ 0, & \text{else,} \end{cases}$$

Similarly,

$$k_1(X, Y) = \begin{cases} \dfrac{\#(Y)}{\#(X \cup Y)}, & \text{if } X \cup Y \neq \emptyset, \\ 1, & \text{else}, \end{cases}$$

can be replaced by

$$k_1^*(X, Y) = \begin{cases} (h_1, h_2, \ldots, h_r) & \text{if } X^l \neq \emptyset, \\ \left(\frac{1}{r}, \ldots, \frac{1}{r}\right), & \text{else} \end{cases}$$

where

$$\frac{\#(G_i) \cdot \chi_i(Y)}{\#((X \cup Y)^l)} = h_i \, ; i = 1, 2, ..., r,$$

and

$$k_2(X, Y) = \frac{\#(X^c \cup Y)}{\#(S)},$$

can be replaced by

$$k_2^*(X, Y) = (q_1, q_2, \ldots, q_r),$$

where

$$\frac{\#(G_1) \cdot \chi_1(X^c \cup Y)}{\#(S)} = q_i, \quad i = 1, 2, \ldots, r.$$

This strategy can be extended to every other non-compliant inclusion function. Addition, multiplication and their partial inverses for natural numbers can be properly generalised to the new types of numbers with special regard to meaning of the operations on elements of dissimilar type. This paves the way for the representation of these general measures in the new number systems (given the granulation).

14   On Representation of Counts

In this section, ways of extending the representation of the different types of counts considered in the previous section are touched upon. The basic aim is to endow these with more algebraic structure through higher order constructions. The generalised counts introduced have been given a representation that depend on the order of arrangement of relatively discernible and indiscernible things from a Meta-C perspective. A representation that accommodates all possible arrangements is of natural interest from both concrete and abstract perspectives. The global structure that corresponds to Z or N is not this and may be said to correspond to rather superfluous implicit interpretations of counting exact objects by natural numbers. From a different perspective a distinction is made between the names of objects and the 'generalised numbers' associated in all this.

Consider the following set of statements:

A Set S has finite cardinality n.

B Set S can be counted in $n!$ ways.

C The set of counts $\mathcal{C}(S)$ of the set S has cardinality $n!$.

E The set of counts $\mathcal{C}(S)$ of the set S bijectively corresponds to the permutation group S_n.

F The set of counts $\mathcal{C}(S)$ can be endowed with a group operation so that it becomes isomorphic to S_n.

Statements C, E and F are rarely explicitly stated or used in mathematical practice in the stated form. To prove 'F', it suffices to fix an initial count. From the point of view of information content it is obvious that $A \subset B \subset C \subset E \subset F$. It is also possible to replace 'set' in the above statements by 'collection' and then from the axiomatic point of statements C, E, and F will require stronger axioms (This aspect will not be taken up in the present paper). I will show that partial algebraic variants of statement F correspond to IPC.

An important aspect that will not carry over under the generalisation is:

Proposition 4. *Case F is fully determined by any of the two element generating sets (for finite n).*

14.1 Representation Of IPC

The group S_n can be associated with all the usual counting of a collection of n elements. The composition operation can be understood as the action of one counting on another. This group can be associated with the RYS being counted and 'similarly counted pairs' in the latter can be used to generate a partial algebra. Formally, the structure will be deduced using knowledge of S_n and then abstracted:

- Form the group S_n with operation $*$ based on the interpretation of the collection as a finite set of n elements at a higher meta level,
- Using the information about similar pairs, associate a second interpretation $s(x)$ based on the IPC procedure with each element $x \in S_n$,
- Define $(x, y) \in \rho$ if and only if $s(x) = s(y)$.

On the quotient $S_n|\rho$, let

$$a \circ b = \begin{cases} c & \text{if } \{z : x * y = z, \, x \in a, \, y \in b\}|\rho \in S_n|\rho, \\ \infty & \text{else.} \end{cases}$$

The following proposition follows from the form of the definition.

Proposition 5. *The partial operation $\circ$ is well defined.*

Definition 9. *A partial algebra of the form $\langle S_n|\rho, \circ \rangle$ defined above will be called a* Concrete IPC-partial Algebra *(CIPCA)*

A partially ordered set $\langle F, \leq \rangle$ is a *lower semi-linearly ordered set* if and only if:

- For each x, $x \downarrow$ is linearly ordered,
- $(\forall x, y)(\exists z) z \leq x \wedge z \leq y$.

It is easy to see that IPC counts form lower semi-linearly ordered sets. Structures of these types and properties of their automorphisms are all of interest and will be considered in a separate paper.

15   Semantics from Dialectical Counting

An algorithmic method for deducing the algebraic semantics of classical RST using the IPC method of counting is demonstrated in this section. Derivation of semantics for other types of RST will be taken up in a separate paper.

Let $S = \langle \underline{S}, R \rangle$ be an approximation space, then $\left\langle \wp(S), \cup, \cap, ^l, ^u, ^c, 0, 1 \right\rangle$ can be regarded as a RYS. 0, 1, corresponding to the empty set and S, can be regarded as distinguished elements or defined through additional axioms. The parthood operation corresponds to set inclusion and is definable using $\cap$, $\cup$. Indiscernibility of any $x, y \in \wp(S)$ will be assumed to be defined by the rough equality;
$$x \approx y \text{ iff } x^l = y^l \wedge x^u = y^u.$$

If $\mathcal{I}(\wp(S))$ is the set of all IPC counts of the RYS, then by the last theorem of section on dialectical counting it is clear that the granules and definite elements can be identified. It remains to define the other operations on the identified objects along the lines of [29]. The corresponding logics in modal perspective can be found in [97]. So,

Theorem 30. *The rough algebra semantics of classical RST can be deduced from any one of the maximal elements of $\mathcal{I}(\wp(S))$ in the $\preceq$ order.*

15.1   Rough Entanglement: Granular HPC

By the term *Entangled*, I mean to grasp the sharp increase in semantic content that happens when counting processes are made to take aspects of the distribution of granules into account. For the following version of counting, it is assumed that the set S is associated with a RYS $\mathcal{S}$, generated by it according to some process and that a collection of granules $\mathcal{G}$ is included in the RYS $\mathcal{S}$. It is also assumed that a total order on S is available relative the lower level classical semantic domain. These assumptions can be relaxed. The following definitions will also be used:

- For $a, b \in S$ and $G \in \mathcal{G}$, $ind_G(a, b)$ if and only if $a, b \in G$.
- For $a, b \in S$ $disc(a, b)$ if and only if $(\exists G_1, G_2 \in \mathcal{G})$ $a \in G_1, b \in G_2, a \notin G_2, b \notin G_1$ (or the pair guarantees $disc(a, b)$).
- For $a, b \in S$ $pdisc(a, b)$ if and only if $(\exists G_1, G_2 \in \mathcal{G})$ $a, b \in G_1, b \in G_2, a \notin G_2$.

The granular part of the granular extension of HPC can be done as follows:

1. Assume a total order on $\mathcal{G}$ and form $\mathcal{G}^2 \setminus \Delta$,
2. Form words with elements (alphabets) from $\mathcal{G}^2 \setminus \Delta$ and order them lexicographically,
3. A word that includes all instances of pairs that guarantee $disc(a, b)$ will be said to be *complete* for (a, b),
4. A word will be said to be *reduced* with respect to a pair (a, b) if and only if no letter x in the word guarantees $pdisc(a, b)$.
5. The existence of unique complete least reduced words (least with respect to the lexicographic order, provided $\mathcal{G}$ is finite) for pairs of elements in S permits simple valuation as a tuple and adds another dimension to HPC.
6. It suffices to adjoin the tuple corresponding to the encoded word between the element and its predecessor in the counting procedure of HPC.

Theorem 31. *Any two distinct elements a, $b \in S$ generate a least unique complete reduced word from $\mathcal{G}$.*

Proof. I will need to use transfinite induction in the absence of the finiteness assumption. Otherwise, forming the set of complete words, and then extracting the reduced words and finally ordering the rest by the lexicographic graphic will yield the least element. A contradiction argument can be used to check the uniqueness.

16   Connections Between Fuzzy and Rough Sets

In this section, I will use granularity-related features to establish a new transformation between fuzzy sets and classical rough sets. The transformation can be the basis for translations or transformations between different types of logics associated with the corresponding semantics. In the literature of the different attempts at connections between fuzzy and rough sets, the non granular approach via the Brouwer Zadeh MV algebras in [41] is somewhat related. The exact connections with the present derivations, however, requires further investigation especially when rough membership functions in the classical sense are not permitted into the discourse. The relation of this result to the connections based on membership functions as developed in [98] and earlier papers are mentioned at the end of this section. The result is very important in the light of the contamination problem and the generalised number systems of [56]. It shows that both concepts of rough measures and 'fuzzy sets as maps' are not needed to speak of rough sets, fuzzy sets and possible connections between them.

A non-controversial definition of fuzzy sets, with the purpose of removing the problems with the 'membership function formalism', was proposed in [99]. I show that the definition can be used to establish interesting links between fuzzy set theory and rough sets. The connection is essentially of a mathematical nature. In my view, the results should be read as *in a certain perspective, the granularity of particular rough contexts originate from fuzzy contexts and vice versa*. The existence of any such perspective and its possible simplicity provides another classification of general rough set theories.

Definition 10. *A* fuzzy subset *(or* fuzzy set*)* $\mathbb{A}$ *of a set S is a collection of subsets $\{A_i\}_{i \in [0,1]}$ satisfying the following conditions:*

- $A_0 = S$,
- $(0 \leq a < b \leq 1 \rightarrow A_b \subseteq A_a)$,
- $A_b = \bigcap_{0 \leq a < b} A_a$.

A fuzzy membership function $\mu_\mathbb{A} : X \mapsto [0,1]$ is defined via $\mu_\mathbb{A}(x) = \mathrm{Sup}\{a : a \in [0,1], x \in A_a\}$ for each x. The *core* of $\mathbb{A}$ is defined by $\mathrm{Core}(A) = \{x \in S : \mu_\mathbb{A}(x) = 1\}$. $\mathbb{A}$ is *normalized* if and only if it has non-empty core. The *support* of $\mathbb{A}$ is defined as the closure of $\{x \in S; \mu_\mathbb{A}(x) > 0\}$. The *height* of $\mathbb{A}$ is $H(\mathbb{A}) = \mathrm{Sup}\{\mu_\mathbb{A}(x); x \in S\}$. The *upper level set* is defined via $U(\mu, a) = \{x \in S : \mu_\mathbb{A}(x) \geq a\}$. The class of all fuzzy subsets of S will be denoted by $\mathcal{F}(S)$. The standard practice is to refer to 'fuzzy subsets of a set' as simply a 'fuzzy set'.

Proposition 6. *Every fuzzy subset $\mathbb{A}$ of a set S is a granulation for S which is a descending chain with respect to inclusion and with its first element being S.*

The cardinality of the indexing set and the second condition in the definition of fuzzy sets is not a problem for use as granulations in RST, but almost all types of upper approximations of any set will end up as S. From the results proved in the previous sections it should also be clear that many of the nice properties of granulations will not be satisfied modulo any kind of approximations. I will show that simple set theoretic transformations can result in better granulations. Granulations of the type described in the proposition will be called *phi-granulations*.

Construction-1:

1. Let $P = \{0, p_1, \ldots p_{n-1}, 1\}$ be a finite set of rationals in the interval [0,1] in increasing order.
2. From $\mathbb{A}$ extract the collection $\mathbb{B}$ corresponding to the index P.
3. Let $B_0 \setminus B_{p_1} = C_1$, $B_{p_1} \setminus B_{p_2} = C_2$ and so on.
4. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$.
5. This construction can be extended to countable and uncountably infinite P in a natural way.

Theorem 32. *The collection $\mathcal{C}$ formed in the fourth step of the above construction is a partition of S. The reverse transform is possible, provided P has been selected in an appropriate way.*

It has been shown that fuzzy sets can be corresponded to classical rough sets in at least one way and conversely by way of stipulating granules and selecting a suitable transform. But a full semantic comprehension of these transforms cannot be done without imposing a proper set of restrictions on admissibility of transformation and is context dependent. The developed axiomatic theory makes these connections clearer.

The result also means that rough membership functions are not necessary to establish a semantics of fuzzy sets within the rough semantic domain as considered in [98]. Further as noted in [98], the semantics of fuzzy sets within rough sets is quite restricted and form a special class. The core and support of a fuzzy set is realized as lower and upper approximations. Here this need not happen, but it has been shown that any fuzzy set defined as in the above is essentially equivalent to a granulation that can be transformed into different granulations for RST. A more detailed analysis of the connections will appear separately.

17   Operations on Low-Level Rough Naturals

The operations that should be defined are those that have meaning and correspond to semantic actions. An important aspect of the meaning relates to strings associated with counts. In case of natural numbers, the number 2 may be associated with two distinct objects, but an IPC count will have a Meta-C sequence of objects which may be Meta-R distinct or indiscernible from their predecessors and different kinds of operations can be performed on these. Addition for example needs to be defined over strings and then corresponded to new IPC counts.

As mentioned earlier, two different types of structures relative the classical meta level can be associated with possible concepts of 'rough naturals'. With respect to IPC for example, the raw IPC count may be seen as a rough natural or the set of IPC counts of a collection of things may be seen as a rough natural. I will refer to the former as the *low-level* concept of rough natural and the latter as a *high-level* concept. Operations on the former can be expected to influence the structure of the latter. In this section operations on collections of low-level rough naturals will be considered.

of representing other possible operations in a easy way):

If x is an IPC count, then the string of relatively discernible and indiscernible objects (abstract) associated with the count will be denoted by $\overline{x}$. When I speak of 'the string corresponding to x', it can be understood as

- Any of the strings that have the same kind of distribution of discernible and indiscernible objects with respect to their immediate predecessors, or
- the abstract classes associated, or
- a representative 'arbitrary' object (I am not going into the philosophical viewpoints on the issue in this paper).

Obviously this makes the appropriate concept of $\overline{x}$ for a method of counting to depend on the method. Specifically, the IPC concept of $\overline{x}$ is not suitable for getting nice structure in HPC contexts. The rough equality (indiscernibility or similarity or whatever) relation will be denoted by $\approx$. The basic string operations will then be defined as follows:

If $I_k(x)$ is the Meta-R count of the Meta-C string formed by interchanging $\overline{x}_k$ and $\overline{x}_{k+1}$ in $\overline{x}$.

$$\iota_k(x) = \begin{cases} x, & \text{if } (\overline{x}_k, \overline{x}_{k+1}) \in \approx, \\ I_k(x) & \text{otherwise.} \end{cases}$$

If y is the Meta-R count of the result of the Meta-C removal of $\overline{x}_k$ from $\overline{x}$, ($x \neq 0$)

$$\varrho_k(x) = \begin{cases} y, \text{if } \neg(\overline{x}_k, \overline{x}_{k+1}) \in \approx \text{ or } \neg(\overline{x}_{k-1}, \overline{x}_k) \in \approx, \\ x \text{ otherwise.} \end{cases}$$

If z is the IPC count of the result of insertion of the string $\overline{y}$ between $\overline{x}_k$ and $\overline{x}_{k+1}$, then for non-zero x, y,

$$\eta_k(x,\ y) = \begin{cases} z, \text{if } (\overline{x}_k, \overline{x}_{k+1}) \in \approx, \\ x \text{ otherwise.} \end{cases}$$

Definition 11. *For any of the rough counts x, $\nu(x)$ will represent the length of (cardinal associated with) $\overline{x}$ in Meta-C.*

Note that $\eta_k(x,\ y)$ does not include $\oplus$, but is very similar. In fact the two operations are not distinct from the point of view of interpretation at Meta-C.

Definition 12. *Basic arithmetical operations can be generalised to rough naturals as follows:*

(i) *$x \oplus y$: The count (at meta-R) of the 'object level string' (relative meta-C) defined as the $\overline{x}$ items followed by the $\overline{y}$ items.*

(ii) *$x \odot y$: The count (at meta-R) of the 'object level string' defined as $\nu(y)$ copies of $\overline{x}$ placed at each of the atomic $\overline{y}$ places.*

(iii) *$\mu(x, y)$: Form $\nu(y)$ copies of x at meta-C.*

(iv) *x^c: Meta-R Count of $\overline{x}$ in reverse order.*

(v) *$x \otimes y$: The meta-R count of the string formed by the conditional substitution of $\overline{x}$ at each of the $\overline{y}$ places subject to the rules:*

    (a) *Replace $\overline{y}_1$ with $\overline{x}$.*

    (b) *Replace $\overline{y}_{k+1}$ with $\overline{x}$ if $(\overline{y}_k, \overline{y}_{k+1}) \notin \approx$.*

    (c) *If $(\overline{y}_k, \overline{y}_{k+1}) \in \approx$, then $\overline{y}_{k+1}$ should be dropped.*

(vi) *$x \ominus y$: The meta-R count of the string formed by the deletion of $\overline{y}$ from $\overline{x}$ from the right if defined. This definition can always be improved through the ϱ_k operation.*

Other more interesting definitions of subtraction can be obtained by imposing conditions relating to discernibility on the *target $\overline{x}$* and/or *source $\overline{y}$*.

(i) The condition 'a sub-string will be removable from the target from the right if and only if each letter in the sub-string is discernible from its successor or predecessor' will be corresponded to $x \ominus_{1 \vee 2} y$.

(ii) The condition 'a sub-string will be removable from the target from the right if and only if each letter in the sub-string is discernible from its successor and predecessor' will be corresponded to $x \ominus_{12} y$.

(iii) The condition 'a sub-string will be removable from the target from the right if and only if each letter in the sub-string is discernible from its successor' will be corresponded to $x \ominus_1 y$.

(iv) The condition 'a sub-string will be removable from the target from the right if and only if each letter in the sub-string is discernible from its predecessor' will be corresponded to $x \ominus_2 y$.

Definition 13. *For any x, y, $x \preceq y$ if and only if x is obtainable from y in Meta-R by a finite number of recursive applications of ϱ_k and ι_k operations for different values of k.*

Definition 14. *For any x, y,*

$$x \leq_{\oplus} y \text{ iff } (\exists z)\,(x \oplus z = y) \vee (z \oplus x = y).$$

Definition 15. *For any x, y,*

$$x \leq_{\odot} y \text{ iff } (\exists z)\,(x \odot z = y).$$

Definition 16. *For any x, y,*

$$x \leq_{\otimes} y \text{ iff } (\exists z)\,(x \otimes z = y).$$

Definition 17. *For any x, y, $x \leq_p y$ in Meta-R if and only if $\zeta(x), \zeta(y)$ and $\nu(x) \leq \nu(y)$ in Meta-C ($\leq$ being the usual total order on integers).*

Definition 18. *For any x, y, $x \trianglelefteq y$ if and only if $\nu(x) \leq \nu(y)$ in Meta-C ($\leq$ being the usual total order on integers).*

Definition 19. *For any x, y, $x \sqsubseteq y$ in Meta-R if and only if x is obtainable from y in Meta-R by a finite number of recursive applications of ϱ_k for different values of k.*

Using some of the different operations, many new partial algebras modelling the essence of rough naturals at meta-R and meta-C are defined next. It should be noted that the operations are well defined because IPC counts have an injective correspondence with distributions of relatively discernible and indiscernible objects.

Definition 20. *By a* Rough IPC-Natural Algebra *(RIPCNA) will be meant a partial algebraic system of the form*

$$S = \langle \underline{S}, \zeta, \oplus, \odot, \ominus_{1 \vee 2}, 0, 1, (1, 2, 2, 2, 0, 0) \rangle,$$

with the set $\underline{S}$ being the set of rough naturals formed according to the IPC schema and with the above defined operations according to the IPC schema. 1 can be treated as an abbreviation of 1_1, numbers of the form $1_1 2_1...k_1$ can also be abbreviated by k. 0 will be understood as the IPC count of the empty string. ζ is a one place predicate for indicating the usual integers:

$$\zeta(x) \text{ iff } (\exists k)\, x = 1_1 2_1 ... k_1.$$

Theorem 33. *In a RIPCNA of the form $S = \langle \underline{S}, \oplus, \odot, \ominus_{1 \vee 2}, 0, 1 \rangle$ of type $(2, 2, 2, 0, 0)$ all of the following hold:*

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
2. $x \oplus 0 = x = 0 \oplus x$,
3. $x \oplus x = x \odot 2$,
4. $(\forall x, y, z)(x \oplus y = x \oplus z \longrightarrow y = z)$,
5. $(\forall x, y, z)(x \oplus z = y \oplus z \longrightarrow x = y)$,
6. $x \odot (y \odot z) = (x \odot y) \odot z$,
7. $(\forall x, y, z)(x \odot y = z \odot y \longrightarrow x = z)$,
8. $x \odot 1 = x; \ x \ominus x = 0$,
9. $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$,
10. $(x \odot x = x \longrightarrow (x = 0) \vee (x = 1))$,
11. $(\forall x, y, z)(x = y \longrightarrow x \odot z = x \odot z)$,
12. $(\forall x, y, z)(x \oplus y = z \longrightarrow z \ominus y = x)$,
13. $(\forall x, y)(\zeta x, \zeta y \longrightarrow x \oplus y = y \oplus x, \ x \odot y = y \odot x)$,
14. $(\forall x, y, z)(\zeta x, \zeta y, \zeta z \longrightarrow (x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$,
15. $(\forall x, y, z)(\zeta x, \zeta y \longrightarrow (x \oplus y) \odot z = (x \odot z) \oplus (y \odot z))$.

Proof.   1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ holds because either side is the IPC count of the string formed by placing $\bar{x}$, $\bar{y}$ and $\bar{z}$ in succession.
2. 0 by definition is the count of the empty string, so $x \oplus 0 = x = 0 \oplus x$.
3. 2 is the same as $1_1 2_1$ and $x \odot 2$ would be the IPC count of the string formed by placing two copies of x in 2 places. So $x \oplus x = x \odot 2$
4. If $x \oplus y = x \oplus z$ holds then either side will be the IPC count of $\overline{x \oplus y}$ and the count of the initial string $\bar{x}$ will be common to both. If $y \neq z$, then $x \oplus y$ cannot be equal to $x \oplus z$, so $(\forall x, y, z)(x \oplus y = x \oplus z \longrightarrow y = z)$
5. Proof of $(\forall x, y, z)(x \oplus z = y \oplus z \longrightarrow x = y)$ is similar to the above.
6. The length of the string $\overline{y \odot z}$ will be the same as the length of $\bar{y}$ multiplied by the length of $\bar{z}$. So $\bar{x}$ will be equally replicated during the IPC counting operation corresponding to either side of $x \odot (y \odot z) = (x \odot y) \odot z$.
7. $(\forall x, y, z)(x \odot y = z \odot y \longrightarrow x = z)$ can be verified from the strings associated and a contradiction argument.
8. $x \odot 1 = x$ is obvious. If $\bar{x}$ is removed from $\bar{x}$, then the result would be an empty string. So $x \ominus x = 0$
9. The length of $\overline{y \oplus z}$ is the same as the length of $\bar{y}$ plus that of $\bar{z}$. So $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.
10. $(x \odot x = x \leftrightarrow (x = 0) \vee (x = 1))$ is easy to verify.
11. $(\forall x, y, z)(x = y \longrightarrow x \odot z = x \odot z)$ is easy to prove, but note that $x \oplus z = y \oplus z$ need not follow from $x = y$ in general.
12. $(\forall x, y, z)(x \oplus y = z \longrightarrow z \ominus y = x)$ follows from the definition of $\ominus$.
13. $(\forall x, y)(\zeta x, \zeta y \longrightarrow x \oplus y = y \oplus x, \ x \odot y = y \odot x)$ follows from the properties of integers.
14. $(\forall x, y, z)(\zeta x, \zeta y, \zeta z \longrightarrow (x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ follows from the properties of integers.
15. $(\forall x, y, z)(\zeta x, \zeta y \longrightarrow (x \oplus y) \odot z = (x \odot z) \oplus (y \odot z))$ is again a statement about integers.

Theorem 34. $\{x : \zeta(x), x \in S\}$ *with the operations* $\oplus, \odot, \ominus$ *is an integral domain of characteristic 0 that is also a unique factorization domain. In fact it is isomorphic to* Z.

Proof. $\{x : \zeta(x), x \in S\}$ is in bijective correspondence with Z by definition. The restrictions of the operations $\oplus, \odot, \ominus$ to the domain coincide with usual arithmetic operations on Z.

As $\odot$ uses a certain level of knowledge of the natural numbers that is correctly at Meta-C, the question of admissibility of such in some contexts can be a matter of dispute. Relative this understanding $\otimes$ would be a better suited operation that is at the same time a proper generalisation of usual multiplication and $\odot$.

Definition 21. *By a* Rough IPC Algebra *(RIPCA) will be meant a partial algebraic system of the form*

$$S = \langle \underline{S}, \zeta, \oplus, \otimes, \ominus_{1\vee 2}, 0, 1, (1, 2, 2, 2, 0, 0) \rangle,$$

with the set $\underline{S}$ being the set of rough naturals formed according to the IPC schema and with the above defined operations according to the IPC schema. 1 can be treated as an abbreviation of 1_1, numbers of the form $1_1 2_1 ... k_1$ will also be abbreviated by k. 0 will be understood as the IPC count of the empty string. ζ is a one place predicate for indicating the usual integers:

$$\zeta(x) \text{ iff } (\exists k) \, x = 1_1 2_1 ... k_1.$$

Of the orders $\sqsubseteq, \preceq, \leq_\odot, \leq_\otimes, \leq_\oplus$ and $\trianglelefteq$, the last is basically a Meta-C order corresponding to the usual order on integers. Among the other five, $\preceq$ is the strongest with respect to inclusion, but not much can be said about comparisons of their naturality in Meta-R. So it seems best to define ordered versions of the partial algebras as follows:

Definition 22. *By a* R-Ordered Rough IPC Algebra *(RORIPCA) will be meant a partial algebraic system of the form*

$$S = \langle \underline{S}, \zeta, \sqsubseteq, \preceq, \leq_\otimes, \leq_\oplus, \oplus, \otimes, \ominus_{1\vee 2}, 0, 1, (1, 2, 2, 2, 2, 2, 2, 2, 0, 0) \rangle,$$

with the set $\underline{S}$ being the set of rough naturals formed according to the IPC schema and with

$$S = \langle \underline{S}, \zeta, \oplus, \otimes, \ominus_{1\vee 2}, 0, 1, (1, 2, 2, 2, 0, 0) \rangle,$$

being a RIPCA.

Definition 23. *By a* C-Ordered Rough IPC Algebra *(CORIPCA) will be meant a partial algebraic system of the form*

$$S = \langle \underline{S}, \zeta, \trianglelefteq, \oplus, \otimes, \ominus_{1\vee 2}, 0, 1, (1, 2, 2, 2, 2, 0, 0) \rangle,$$

with the set $\underline{S}$ being the set of rough naturals formed according to the IPC schema and with

$$S = \langle \underline{S}, \zeta, \oplus, \otimes, \ominus_{1\vee 2}, 0, 1, (1, 2, 2, 2, 0, 0) \rangle,$$

being a RIPCA.

Definition 24. *By a* Full Ordered Rough IPC Algebra *(FORIPCA) will be meant a partial algebraic system of the form*

$$S \,=\, \langle \underline{S}, \zeta, \trianglelefteq, \sqsubseteq, \preceq, \leq_{\otimes}, \leq_{\odot} \leq_{\oplus}, \leq_p, \oplus, \otimes, \odot, \ominus_{1\vee 2}, \ominus, 0, 1 \rangle \,,$$

of type $(1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0)$ *with the set* $\underline{S}$ *being the set of rough naturals formed according to the IPC schema and with*

$$S \,=\, \langle \underline{S}, \zeta, \oplus, \otimes, \ominus_{1\vee 2}, 0, 1, (1, 2, 2, 2, 0, 0) \rangle \,,$$

being a RIPCA.

Analogously, the concepts of FORIPCNA, CORIPCNA and RORIPCNA can be defined.

Proposition 7. *In a FORIPCA S,* $\trianglelefteq$ *is a quasi order that satisfies*

$$\trianglelefteq \cap \, \zeta^2 \, = \leq .$$

Proof. It is easy to find elements x, y that violate possible antisymmetry of $\trianglelefteq$. The restriction to ζ^2 is the same as the intersection with Z^2. So on Z, the usual order $\leq$ coincides with $\trianglelefteq$. Strictly speaking this is a category theoretic result involving S and Z.

Proposition 8. *In a FORIPCA S,* $\preceq$ *is a quasi order. The class of any element in* $\{x \,;\, \zeta(x)\, x \neq 0\}$ *with respect to the equivalence induced by* $\preceq$ *includes* $\{x \,;\, \zeta(x)\, x \neq 0\}$.

Proof. Transitivity holds as if x is obtainable from y by the ϱ_k, eta_k operations for different k and y is obtainable from z similarly, then x would be obtainable from z.

If $\zeta(x), \zeta(y)$, then $\overline{x}$ and $\overline{y}$ would be strings of discernibles and so x, y would be transformable into each other by the ϱ_k, eta_k operations. Defining $x \sim y$ if and only if $x \preceq y$ and $y \preceq x$, it can be seen that $\sim$ is an equivalence and through this all of the elements of $\{x \,;\, \zeta(x)\, x \neq 0\}$ can be identified.
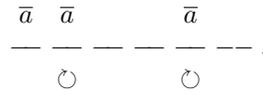
Theorem 35. *In a FORIPCA S, the following implications between the different orders hold:*

1. $(\forall x, y)(x \leq_{\odot} y \longrightarrow x \leq_{\oplus} y)$,
2. $(\forall x, y)(x \leq_{\oplus} y \longrightarrow x \sqsubseteq y)$,
3. $(\forall x, y)(\zeta x, \zeta y \longrightarrow x \trianglelefteq y \vee y \trianglelefteq x \vee x = y)$,
4. $(\forall x, y)(x \leq_p y \longrightarrow x \trianglelefteq y)$,
5. $(\forall x, y)(x \otimes y \leftrightarrow x \odot y)$,
6. *For any x, y, $x \leq_{\oplus} y$ does not not imply that $x \preceq y$ and neither does the converse implication hold.*

Proof. Most of the statements can be proved from the corresponding definitions. $\leq_p$ is a partial order that is coincides with the usual order when restricted to Z. Statement 5 is ensured by the existence of elements corresponding to all possible relative discernibility-indiscernibility patterns.

If $\bar{z}$ is a string of elements, some of which are indiscernible from $\bar{x}$ and $y = x \oplus z$, then x may not be obtainable from y. Concrete patterns can be constructed at different levels of complexity. The easiest case corresponds to all elements of $\bar{z}$ being indiscernible from those of $\bar{x}$.

A useful visual representation of the computing process for $a \otimes b$ is illustrated below through a specific example. In the first line blank spaces are drawn for the alphabets in $\bar{b}$. The curved arrows are used to indicate discernibility. The second figure represents a gross view of the strings in $a \otimes b$.

$$\overline{a} \quad \overline{a} \qquad \overline{a}$$
$$\underline{\quad} \ \underline{\quad} \ \underline{\quad} \ \underline{\quad} \ \underline{\quad} \ \underline{\quad \quad} \ .$$
$$\circlearrowleft \qquad \circlearrowleft$$

From the above, it can be deduced that $a \otimes b$ is equivalent to the count of

$$\overline{a} \quad \overline{a} \quad \overline{a}$$
$$\underline{\quad} \ \underline{\quad} \ \underline{\quad \quad} \ .$$
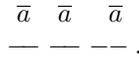
Key properties of the relatively difficult operation $\otimes$ are considered in the following theorems:

Theorem 36. *In a FORIPCA S, all of the following hold:*

1. *$(\forall a, b, c)(a \otimes b) \otimes c \trianglelefteq a \otimes (b \otimes c)$,*
2. *$(\forall a, b, c)(\zeta(b) \longrightarrow (a \otimes b) \otimes c = a \otimes (b \otimes c))$,*
3. *$(\forall a, b)(a \neq 0, \ a \otimes b = a \odot b \leftrightarrow b \ominus_{12} b = 0)$.*

Proof. 1. The length of $a \otimes b$ is determined by the distribution of indiscernible pairs in b. The proof can be done by considering the different cases of the $\otimes$ multiplication in the left and right side of the inequality corresponding to the relation of the first and last element of the strings corresponding to a, b and $a \otimes b$.

2. If $\zeta(b)$, then the string corresponding to it will have mutually discernible elements of the length of b. In $(a \otimes b) \otimes c$, $a \otimes b$ will be repeated the same number of times as b in $b \otimes c$. So the equality will hold.

3. If $a \otimes b = a \odot b$, then as a is not 0, it is necessary that each object in $\bar{b}$ be discernible from its immediate predecessor (that is for every admissible k, $(\bar{b}_k, \bar{b}_{k+1}) \notin \approx$). So $b \ominus_{12} b = 0$ must hold (irrespective of the length of b). Note that for an arbitrary element x, it need not happen that $x \ominus_{12} x = 0$ in general.

For the converse, note that if $b \ominus_{12} b = 0$, then it will be necessary that for every admissible k, $(\bar{b}_k, \bar{b}_{k+1}) \notin \approx$. Both the multiplications of a with such a b will be equal.

In the next theorem, the compatibility of the different orders defined are considered:

Theorem 37. *In a FORIPCA S, the following hold:*

1. $(\forall a, b, c, e)(a \trianglelefteq b,\ c \trianglelefteq e\ \longrightarrow\ (c \oplus a) \trianglelefteq (e \oplus b),\ (a \odot c) \trianglelefteq (b \odot e))$,
2. $(\forall a, b, c, e)(a \leq_{\oplus} b,\ c \leq_{\oplus} e\ \longrightarrow\ a \oplus c \trianglelefteq b \oplus e)$,
3. $(\forall a, b, c)(a \leq_{\oplus} b\ \longrightarrow\ c \oplus a \leq_{\oplus} c \oplus b,\ \text{or}\ a \oplus c \leq_{\oplus} b \oplus c)$,
4. $(\forall a, b)(a \leq_{\oplus} b, a \sqsubseteq b\ \longrightarrow\ b \ominus_{1 \vee 2} a \sqsubseteq b)$,
5. $(\forall a, b, c, e)(a \leq_{\odot} b, c \leq_{\odot} e\ \longrightarrow\ a \odot c \trianglelefteq b \odot e,\ a \odot c \sqsubseteq b \odot e)$,
6. $(\forall a, b, c, e)(a \sqsubseteq b, c \subseteq e\ \longrightarrow\ a \oplus c \sqsubseteq b \oplus e,\ a \odot c \sqsubseteq b \odot e,)$.

Proof.   1. $\trianglelefteq$ corresponds to the meta-C interpretation by the length of associated. So the $\odot$ and $\oplus$ part of the implication should be obvious. However as $c \trianglelefteq e$ does not imply that the number of objects in c that are discernible from their predecessor are less than the corresponding number in e. So $\otimes$ will not preserve $\trianglelefteq$.

2. In general if $a \leq_{\oplus} b$, $c \leq_{\oplus} e$, then it need not happen that $a \oplus c \leq_{\oplus} b \oplus e$, because of the different possible relations between the objects at the terminal and initial position of $\oplus$. But the gross length of $\overline{a \oplus c}$ will be $\nu(a) + \nu(c)$. So the implication holds.

3. Follows from the definition of $\leq_{\oplus}$.

4. If $a \leq_{\oplus} b$, then there exists a c such that $a \oplus c = b$ or $c \oplus a = b$. In either case, if $\overline{a}$ is obtainable from $\overline{b}$ by a finite number of recursive applications of ρ_k, then it is necessary that these operations must have been applied at one end of $\overline{b}$. This causes $b \ominus_{1 \vee 2} a \sqsubseteq b$.

5. In $\overline{a \odot c}$ and $\overline{b \odot e}$, $\overline{a}$ is repeated $\nu(c)$ times and b is repeated νe times. So the length of $\overline{a \odot c}$ will be less than that of $\overline{b \odot e}$. Further as $a \leq_{\odot} b$ and $c \leq_{\odot} e$, it will be possible to obtain $\overline{a \odot c}$ from $\overline{b \odot e}$ through a finite number of applications of ρ_k for different values of k. So the result holds.

6. This follows from definition.

Interestingly it is not possible to define a unary negation operator from any of the subtraction-like $\ominus$ operations and $\oplus$ in a consistent way. So the concept of negative elements does not generalise well to the present contexts. The extent to which a consistent definition is possible will be of natural interest.

18   Further Directions: Conclusion

The broad classes of problems that will be part of future work fall under:

(i) Improvement of the representation of different classes of counts. One class of questions also relate to using morphisms or automorphisms in a more streamlined way.

(ii) Extension of the algebraic approach (and the concept of rough natural numbers) of the last section to non IPC cases.

(iii) The contamination problem.

(iv) Description of other semantics of general RST in terms of counts. This is the basic program of representing all types of general rough semantics in terms of counts.

(v) Can the division operation be eliminated in the general measures proposed? In other words, a more natural extensions of fractions (rough rationals) would be of much interest. Obviously this is part of the representation problem for counts.

(vi) How do algorithms for reduct computation get affected by the generalised measures?

Apart from these a wide variety of combinatorial questions would be of natural interest.

In this research paper, the axiomatic theory of granules and granulation developed by the present author has been extended to cover most types of general RST and new methods of counting collections of well defined and indiscernible objects have been integrated with it. These new methods of counting have been shown to be applicable to the extension of fundamental measures of RST, rough inclusion measures and consistency degrees of knowledge. The redefined measures possess more information than the original measures and have more realistic orientation with respect to counting. The connections with semantic domains, that are often never explicitly formulated, has been brought into sharp focus through the approach.

The concept of rough naturals in the IPC perspective has also been developed in this paper. The new program centred around the contamination problem proposed in this paper can also be found in another forthcoming paper by the present author on axiomatic theory of granules for RST. Here the direction is made far more clearer through the integration with rough naturals.

By the mathematics of vagueness, I do not mean a blind transfer of the results of the mathematics of exact contexts to inexact contexts. It is intended to incorporate vagueness in more natural and amenable ways in the light of the contamination problem. An essential part of this is achieved in this research paper.

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References

1. Skowron, A., Peters, J.: Rough-granular computing. In Pedrycz, W., Skowron, A., Kreinovich, V., eds.: Granular Computing. John Wiley & Sons, Ltd., Chichester, UK (2008) 225–328
2. Pawlak, Z.: An inquiry into vagueness and uncertainty. PAS Institute of Computer Science **29/94** (1994)

3. Pagliani, P., Chakraborty, M.: A Geometry of Approximation: Rough Set Theory: Logic, Algebra and Topology of Conceptual Patterns. Springer, Berlin (2008)
4. Bazan, J., Skowron, A., Swiniarski, R.: Rough sets and vague concept approximation: From sample approximation to adaptive learning. Trans. on Rough Sets **V, LNCS 4100** (2006) 39–62
5. Skowron, A.: Rough sets and vague concepts. Fund. Inform. **64** (2005) 417–431
6. Mani, A.: Towards logics of some rough perspectives of knowledge. In Suraj, Z., Skowron, A., eds.: Intelligent Systems Reference Library dedicated to the memory of Prof. Pawlak,. In Press (2011) 25pp
7. Banerjee, M., Chakraborty, M.K.: Foundations of vagueness: A categorytheoretic approach. Electronic Notes in Theoretical Computer Science **82** (2003) 10–19
8. Weiner, J.: Science and semantics: The case of vagueness and supervaluation. Pacific Philosophical Quarterly **88** (2007) 355–374
9. Shapiro, S.: Vagueness in Context. Oxford University Press, Oxford, UK (2006)
10. Keefe, R.: Theories of Vagueness. Cambridge University Press, Cambridge, UK (2000)
11. Keefe, R., Smith, P.: Vagueness: A Reader. MIT Press, Cambridge, MA (1999)
12. Russell, B.: Vagueness. The Australasian Journal of Psychology and Philosophy **1** (1923) 84–92
13. Pawlak, Z.: Vagueness - a rough set view. In Mycielski, J., Rozenberg, G., Salomaa, A., eds.: Structures in Logic and Computer Science, LNCS 1261. Springer, Berlin (1997) 106–117
14. Bonikowski, Z., Wybraniec-Skardowska, U.: Vagueness and roughness. Trans. on Rough Sets **IX, LNCS 5390** (2008) 1–13
15. Dubois, D., Prade, H.: Fuzzy Sets and Systems. Academic Press, New York (1980)
16. Polkowski, L.: Rough Sets: Mathematical Foundations. Physica-Verlag, Heidelberg (2002)
17. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning About Data. Kluwer Academic Publishers, Dodrecht (1991)
18. Mani, A.: Choice inclusive general rough semantics. Information Sciences **181** (2011) 1097–1115
19. Pawlak, Z.: Some issues in rough sets. In Skowron, A., Peters, J.F., eds.: Transactions on Rough Sets- I. Volume 3100. Springer Verlag (2004) 1–58
20. Polkowski, L., Skowron, A.: Rough mereology: A new paradigm for approximate reasoning. Internat. J. Appr. Reasoning **15** (1996) 333–365
21. Lin, T.Y.: Granular computing -1: The concept of granulation and its formal model. Int. J. Granular Computing, Rough Sets and Int Systems **1** (2009) 21–42
22. Yao, Y.: The art of granular computing. In Kryszkiewicz, M., et al., eds.: RSEISP'2007, LNAI 4585, Springer Verlag (2007) 101–112
23. Zhang, M.W.X., Wu, W.Z., Li, T.J.: Granulation. In An, A., Yao, J.T., et al., eds.: RSFDGrC 2007, LNAI 4481. Springer-Verlag (2007) 93–100
24. Wasilewski, P., Slezak, D.: Foundations of rough sets from vagueness perspective. In Hassanien, A., et al., eds.: Rough Computing: Theories, Technologies and Applications. Information Science Reference. IGI, Global (2008) 1–37
25. Keet, C.M.: A Formal Theory of Granules - Phd Thesis. PhD thesis, Fac of Comp.Sci., Free University of Bozen (2008)
26. Polkowski, L.: Rough mereology as a link between rough and fuzzy set theories. In Peters, J.F., Skowron, A., eds.: Transactions on Rough Sets. Volume LNCS 3135. Springer Verlag (2004) 253–277
27. Takeuti, G.: Proof Theory. 2nd edn. North Holland, Amsterdam (1987)

28. Font, J.M., Jansana, R.: A General Algebraic Semantics for Sentential Logics. Volume 7. Association of Symbolic Logic (2009)
29. Banerjee, M., Chakraborty, M.K.: Rough sets through algebraic logic. Fundamenta Informaticae **28** (1996) 211–221
30. Duntsch, I.: Rough sets and algebras of relations. In Orlowska, E., ed.: Incomplete Information and Rough Set Analysis, Physica, Heidelberg (1998) 109–119
31. Mani, A.: Super rough semantics. Fundamenta Informaticae **65** (2005) 249–261
32. Pawlak, Z.: Rough sets. Internat. J. of Computing and Information Sciences **18** (1982) 341–356
33. Skowron, A., Stepaniuk, O.: Tolerance approximation spaces. Fundamenta Informaticae **27** (1996) 245–253
34. Komorowski, J., Pawlak, Z., Polkowski, L., Skowron, A.: Rough sets – a tutorial. In Pal, S.K., Skowron, A., eds.: Rough Fuzzy Hybridization. Springer Verlag (1999) 3–98
35. Cattaneo, G.: Abstract approximation spaces for rough set theory. In Polkowski, L., Skowron, A., eds.: Rough Sets in Knowledge Discovery 2. Physica Heidelberg (1998) 59–98
36. Konikowska, B.: A logic for reasoning with similarity. In Orlowska, E., ed.: Incomplete Information and Rough Set Analysis, Physica Verlag, Heidelberg (1998) 461–490
37. Inuiguchi, M.: Generalisation of rough sets and rule extraction. In Peters, J.F., Skowron, A., eds.: Transactions on Rough Sets-1. Volume LNCS-3100. Springer Verlag (2004) 96–116
38. Slezak, D., Wasilewski, P.: Granular sets - foundations and case study of tolerance spaces. In An, A., Stefanowski, J., Ramanna, S., Butz, C.J., Pedrycz, W., Wang, G., eds.: RSFDGrC 2007, LNCS. Volume 4482. Springer (2007) 435–442
39. Mani, A.: Algebraic semantics of similarity-based bitten rough set theory. Fundamenta Informaticae **97** (2009) 177–197
40. Pomykala, J.A.: Approximation, similarity and rough constructions. Technical Report CT-93-07, ILLC, Univ of Amsterdam (1993)
41. Cattaneo, G., Ciucci, D.: Algebras for rough sets and fuzzy logics. In Skowron, A., Peters, J.F., eds.: Transactions on Rough Sets II. Volume 2. Springer Verlag (2004) 208–252
42. Zakowski, W.: Approximation in the space (u, π). Demonstration Math **XVI** (1983) 761–769
43. Yao, Y.: Relational interpretation of neighbourhood operators and rough set approximation operators. Information Sciences (1998) 239–259
44. Li, T.J.: Rough approximation operators in covering approximation spaces. In Greco, S., et al., eds.: RSCTC 2006. LNCS (LNAI) 4259, Springer Verlag (2006) 174–182
45. Zhu, W.: Relationship between general rough set based on binary relation and covering. Information Sciences **179** (2009) 210–225
46. Samanta, P., Chakraborty, M.K.: Covering based approaches to rough sets and implication lattices. In Sakai, H., et al., eds.: RSFDGrC '2009, LNAI 5908, Springer (2009) 127–134
47. Mani, A.: Esoteric rough set theory-algebraic semantics of a generalized vprs and vprfs. In Skowron, A., Peters, J.F., eds.: Transactions on Rough Sets VIII. Volume LNCS 5084. Springer Verlag (2008) 182–231
48. Mani, A.: Towards an algebraic approach for cover based rough semantics and combinations of approximation spaces. In Sakai, H., et al., eds.: RSFDGrC 2009. Volume LNAI 5908., Springer-Verlag (2009) 77–84

49. Mani, A.: Integrated dialectical logics for relativised general rough set theory. In: Internat. Conf. on Rough Sets, Fuzzy Sets and Soft Computing, Agartala, India, http://arxiv.org/abs/0909.4876 (2009) 6pp (Refereed)
50. Mundici, D.: Generalization of abstract model theory. Fundamenta Math **124** (1984) 1–25
51. Banerjee, M., Yao, Y.: Categorical basis for granular computing. In Stefanowski, J., Ramanna, S., Butz, C.J., Pedrycz, W., Wang, G., eds.: RSFDGrC 2007, LNAI. Springer (2007) 427–434
52. Banerjee, M., Khan, M.A.: Propositional logics for rough set theory. In: Transactions on Rough Sets VI, LNCS 4374, Springer Verlag (2007) 1–25
53. Banerjee, M., Chakraborty, M.K.: Algebras from rough sets – an overview. In Pal, S.K., et. al, eds.: Rough-Neural Computing. Springer Verlag (2004) 157–184
54. Yao, Y.: A partition model for granular computing. In Skowron, A., Peters, J.F., eds.: Transactions on Rough Sets-1, LNCS 3100. Springer (2004) 232–253
55. Pagliani, P., Chakraborty, M.K.: Formal topology and information systems. In: Transactions on Rough Sets VI, Springer (2007) 253–297
56. Mani, A.: Dialectics of counting and measures of rough set theory. In: IEEE Proceedings of NCESCT'2011, Pune, Feb-1-3, Arxiv:1102.2558 (2011) 17pp
57. Arnold, D., Fowler, K.: Nefarious numbers. Notices Amer. Math. Soc. **58** (2011)
58. Fine, K.: Vagueness, truth and logic. Synthese **30** (1975) 265–300
59. Dummet, M.: The Logical Basis of Metaphysics. Harvard University Press (1991)
60. Simon, P.: Parts - A Study in Ontology. Oxford Univ. Press (1987)
61. Lewis, D.K.: Parts of Classes. Oxford; Basil Blackwell (1991)
62. Varzi, A.: Parts, wholes and part-whole relations: The prospects of mereotopology. Data and Knowledge Engineering **20** (1996) 259–286
63. Polkowski, L. In: Rough Neural Computation Model Based on Rough Mereology. Springer Verlag (2004) 85–108
64. Grzegorczyk, A.: The systems of lesniewski in relation to contemporary logical research. Studia Logica **3** (1955) 77–95
65. Polkowski, L., Polkowska, S.M.: Reasoning about concepts by rough mereological logics. In Wang, G., et al., eds.: RSKT 2008, LNAI 5009, Springer (2008) 197–204
66. Urbaniak, R.: Lesniewski's Systems of Logic and Mereology;History and Re-evaluation. PhD thesis, Department of Philosophy, Univ of Calgary (2008)
67. Vopenka, P.: Mathematics in the Alternative Set Theory. Teubner-Verlag (1979)
68. Vopenka, P., Hajek, P.: The Theory of Semisets. Academia, Prague (1972)
69. Iwinski, T.B.: Rough orders and rough concepts. Bull. Pol. Acad. Sci (Math) **(3–4)** (1988) 187–192
70. Cattaneo, G., Ciucci, D.: Lattices with interior and closure operators and abstract approximation spaces. In Peters, J.F., et al., eds.: Transactions on Rough Sets X, LNCS 5656. Springer (2009) 67–116
71. Perry, B., Dockett, S.: Young children's access to powerful mathematical ideas. In English, L., et al., eds.: Handbook of International Research in Mathematics Education. LEA (2002) 81–112
72. Satyanarayana, M.: Positively Ordered Semigroups. Volume 42 of Lecture Notes in Pure and Applied Mathematics. Marcel Dekker Inc. (1979)
73. Mani, A.: V-perspectives, pseudo-natural number systems and partial orders. Glasnik Math. **37** (2002) 245–257
74. Ciucci, D.: Approximation algebra and framework. Fundamenta Informaticae **94** (2009) 147–161
75. Sen, D., Pal, S.K.: Generalized rough sets, entropy and image ambiguity measures. IEEE Trans. -PartB: Cybernetics **39** (2008) 117–128

76. Pagliani, P.: Pretopologies and dynamic spaces. Fundamenta Informaticae **59** (2004) 221–239

77. Khan, M.A., Banerjee, M.: Formal reasoning with rough sets in multiple-source approximation spaces. Internat. J. Approximate Reasoning **49** (2008) 466–477

78. Novotny, M., Pawlak, Z.: Characterization of rough top and bottom equalities. Bull. Pol. Acad. Sci (Math) **33** (1985) 91–97

79. Hodges, W.: Elementary predicate logic. In Gabbay, D., Guenthner, F., eds.: Handbook of Philosophical Logic. Academic, Kluwer (2004) 1–130

80. Ziarko, W.: Variable precision rough set model. J. of Computer and System Sciences **46** (1993) 39–59

81. Rolka, A.M., Rolka, L.: Variable precision fuzzy rough sets. In Peters, J.F., Skowron, A., eds.: Transactions in Rough Sets-1. Volume LNCS 3100. Springer Verlag (2004) 144–160

82. Carrara, M., Martino, E.: On the ontological commitment of mereology. Rev. Symb. Logic **2** (2009) 164–174

83. Qian, Y.H., Liang, J.Y.: Rough set method based on multi- granulations. In: Proc. 5th IEEE Conf. Cognitive Informatics. Volume 1. (2006) 297–304

84. Qian, Y., Liang, J.Y., Yao, Y.Y., Dang, C.Y.: Mgrs: A multi granulation rough set. Information Science **180** (2010) 949–970

85. Qian, Y.H., Liang, J.Y., Dang, C.Y.: Incomplete mutigranulation rough set. IEEE Transactions on Systems, Man and Cybernetics, Part A **20** (2010) 420–430

86. Slowinski, R., Vanderpooten, D.: A generalized definition of rough approximations based on similarity. IEEE Trans on Knowledge and Data Engg. **12** (2000) 331–336

87. Mani, A.: Meaning, choice and similarity based rough set theory. Internat. Conf. Logic and Appl., Jan'2009 Chennai;(Refereed), http://arxiv.org/abs/0905.1352 (2009) 1–12

88. Lin, T.Y.: Neighbourhood systems- applications to qualitative fuzzy and rough sets. In Wang, P.P., et al., eds.: Advances in Machine Intelligence and Soft Computing' Duke University, Durham'1997. (1997) 132–155

89. Zhu, W.: Topological approaches to covering rough sets. Information Sciences **177** (2007) 1499–1508

90. Chajda, I., Niederle, J., Zelinka, B.: On existence conditions for compatible tolerances. Czech. Math. J **26** (1976) 304–311

91. Jarvinen, J.: Lattice theory for rough sets. In Peters, J.F., et al., eds.: Transactions on Rough Sets VI. Volume LNCS 4374. Springer Verlag (2007) 400–498

92. Orlowska, E., Rewitz, I.: Discrete duality and its application to reasoning with incomplete information. In Kryszkiewicz, M., et al., eds.: RSEISP'2007, LNAI 4585, Springer Verlag (2007) 51–56

93. Orlowska, E.: Kripke semantics for knowledge representation logics. Studia Logica **XLIX** (1990) 255–272

94. Hartonas, C.: Duality for lattice ordered algebras and for normal algebraizable logics. Studia Logica **58** (1997) 403–450

95. Chakraborty, M.K., Samanta, P.: Consistency degree between knowledges. In Kryszkiewicz, M., et al., eds.: RSEISP'2007. Volume LNAI 4583., Springer Verlag (2007) 133–141

96. Gomolinska, A.: On certain rough inclusion functions. In Peters, J.F., et al., eds.: Transactions on Rough Sets IX, LNCS 5390. Springer Verlag (2008) 35–55

97. Banerjee, M., Chakraborty, M.K., Bunder, M.: Some rough consequence logics and their interrelations. In Skowron, A., Peters, J.F., eds.: Transactions on Rough Sets VIII. Volume LNCS 5084., Springer Verlag (2008) 1–20

98. Yao, Y.: Semantics of fuzzy sets in rough set theory. In Skowron, A., Peters, J.F., eds.: Transactions on Rough Sets II. Number LNCS 3135. Springer Verlag (2005) 297–318

99. Ramik, J., Vlach, M.: A non-controversial definition of fuzzy sets. In Skowron, A., Peters, J.F., et al., eds.: Transactions on Rough Sets II. Volume II., Springer (2004) 201–207