

***CPT*–conserving Hamiltonians and their
nonlinear supersymmetrization
using differential charge-operators \mathcal{C}**

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Abstract

A brief overview is given of recent developments and fresh ideas at the intersection of \mathcal{PT} - and/or \mathcal{CPT} -symmetric quantum mechanics with supersymmetric quantum mechanics (SUSY QM). Within the framework of the resulting supersymmetric version of \mathcal{CPT} -symmetric quantum mechanics we study the consequences of the assumption that the “charge” operator \mathcal{C} is represented in a differential-operator form of the second or higher order. Besides the freedom allowed by the Hermiticity constraint for the operator \mathcal{CP} , encouraging results are obtained in the second-order case. In particular, the integrability of intertwining relations proves to match the closure of our nonlinear (*viz.*, polynomial) SUSY algebra. In a particular illustration, our form of \mathcal{CPT} -symmetric SUSY QM leads to a new class of non-Hermitian polynomial oscillators with real spectrum which turn out to be \mathcal{PT} -asymmetric.

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\mathcal{PT} -symmetric Hamiltonians; \mathcal{CPT} -symmetric quantum mechanics; supersymmetric quantum mechanics; nonlinear SUSY algebra; intertwining relations; \mathcal{PT} -asymmetric potentials

1 Introduction

The recent growth of interest in the possibility of working with non-Hermitian observables in quantum theory (cf. the concise review papers [1]) is mainly due to the influential Bender's and Boettcher's letter [2] where its authors observed that the spectrum of certain Hamiltonians $H \neq H^\dagger$ seems real and discrete and bounded below.

They conjectured that such an observation may find a firmer mathematical background and explanation in the symmetry of their models with respect to the combined action of the parity \mathcal{P} and time-reversal (i.e., complex conjugation) \mathcal{T} . This inspiring idea has been further developed and re-formulated as proposals of the so called \mathcal{PT} -symmetric quantum mechanics [3], pseudo-Hermitian quantum mechanics [4] and \mathcal{CPT} -symmetric quantum mechanics [5]. They all deal with more or less the same class of the specific non-Hermitian models characterized, in the language of the latter reference, by another symmetry operator \mathcal{C} which is very conveniently called "charge".

There exists an extensive literature on \mathcal{PT} -symmetric quantum mechanics [6, 7]. In particular, in a number of papers [8, 9, 10, 11], unexpected consequences of the non-Hermiticity of Hamiltonians have been noticed to emerge after its supersymmetrization a la Witten [12]. In terms of local models $H = p^2 + V(x)$ on the real line ($x \in \mathbb{R}$) where $V(x) = V^*(-x)$, these Hamiltonians satisfy the intertwining relation

$$H^\dagger \mathcal{P} = \mathcal{P} H. \tag{1}$$

Such \mathcal{PT} -symmetric Hamiltonians may have either complex, or real spectra. When the \mathcal{PT} symmetry remains spontaneously unbroken and all the spectrum is real [2], one has elaborated the concept of quasi-Hermiticity of the Hamiltonian [13, 14]. This means that the intertwining relation (1) holds also with \mathcal{P} replaced by a positive-definite operator $\Theta = \Theta^\dagger > 0$ which plays the role of a metric operator. The physical interpretation of such models is standard [15]. When the spectrum is complex, relation (1) can still be written with \mathcal{P} replaced by a pseudo-metric [16, 17, 18].

We shall now generalize the previous considerations to a new type of symmetry. The framework of our constructions proposed in our recent letter [19] will incorporate Hamiltonians with both real and complex spectra. Correspondingly, we shall also deal, in general, with non-positive metric (i.e., pseudo-metric). As for the case of \mathcal{PT} symmetry, the interpretation of this type of quantum mechanics can be disputable [14] and might require some innovation. However, we stress that, in our framework, we can find models which have real spectra, where, in particular, a non-Hermitian Hamiltonian is related by similarity transformations not only to a Hermitian operator, but, more specifically, to a Hermitian Schrödinger operator. Thus, for these

cases, we recover the standard quantum mechanics, after similarity. Therefore, these cases are not disputable in their interpretation. From a conservative point of view, one might restrict the interest of our supersymmetric approach insofar as one takes it instrumentally as a strategy to find complex Hamiltonians with real spectrum that do not satisfy \mathcal{PT} invariance (see Section 4 below).

1.1 SUSY intertwining relations

In the same spirit as in Ref. [19], we shall study the intertwining relations

$$\mathcal{F}H^\dagger = H\mathcal{F} \quad (2)$$

mediated by the Hermitian operator

$$\mathcal{F} = \mathcal{C}\mathcal{P} \quad (\mathcal{F} = \mathcal{F}^\dagger), \quad (3)$$

where \mathcal{P} is the parity operator, and \mathcal{C} a generalized "charge" operator, assumed to be a polynomial in the differential operator d/dx . For any Hamiltonian H , Eq. (2) is equivalent to \mathcal{CPT} conservation, with \mathcal{T} the time reversal operator,

$$\mathcal{CPT}H = H\mathcal{CPT}. \quad (4)$$

In this paper we shall not discuss in detail the metric interpretation for \mathcal{F} , but only stress the fact that, if \mathcal{F} and H satisfy Eq. (2), then also \mathcal{F}^{-1} (if it exists) and H meet an intertwining

$$H^\dagger\mathcal{F}^{-1} = \mathcal{F}^{-1}H, \quad (5)$$

which means that H is pseudo-Hermitian with respect to \mathcal{F}^{-1} , i.e., \mathcal{F}^{-1} -pseudo-Hermitian [4].

This observation may be useful for implementing the metric based on \mathcal{F}^{-1} , when \mathcal{F}^{-1} has a better behavior than \mathcal{F} , *e.g.* with respect to boundedness. Nevertheless, in our text we also use for a \mathcal{F} satisfying Eqs. (3), (2), the word "metric" operator. In fact, Eq. (2) implies that $H\mathcal{F}$ is Hermitian. As a consequence of (5), if $|\phi\rangle$ and $|\psi\rangle$ are two arbitrary vectors of the Hilbert space $L^2(\mathbf{R})$, we have

$$\int \phi^*(x) (\mathcal{F}^{-1}H\psi)(x) dx = \int \psi^*(x) (\mathcal{F}^{-1}H\phi)(x) dx.$$

This can be interpreted as a Hermiticity condition for H provided the scalar product is defined as

$$\begin{aligned} \langle \phi | \psi \rangle_{\mathcal{F}^{-1}} &= \int \phi^*(x) (\mathcal{F}^{-1}\psi)(x) dx; \\ \langle \psi | \phi \rangle_{\mathcal{F}^{-1}} &= \int \psi^*(x) (\mathcal{F}^{-1}\phi)(x) dx. \end{aligned}$$

It is worthwhile to point out, however, that, in absence of additional constraints, neither \mathcal{F}^{-1} nor \mathcal{F} is necessarily positive definite so that, for instance, the equation $\mathcal{F}|\phi\rangle = 0$ might have a non-trivial solution different from $|\phi\rangle = 0$. At this level, $\langle\phi|\phi\rangle_{\mathcal{F}^{-1}}$ does not define a true norm but merely a pseudo-norm [17, 20].

It is evident that solving Eq. (2) amounts to analyzing the compatibility between \mathcal{C} and H ; in other words, \mathcal{C} and H are to be found contextually. Once Eq. (2) is formally solved, one can investigate its supersymmetrization [21]. By this we mean the construction of super-charges

$$Q = \begin{pmatrix} 0 & \mathcal{F} \\ 0 & 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 & 0 \\ \mathcal{F}^* & 0 \end{pmatrix} \quad (6)$$

with anti-commutator

$$K \equiv \{Q, \tilde{Q}\} = \begin{pmatrix} \mathcal{F}\mathcal{F}^* & 0 \\ 0 & \mathcal{F}^*\mathcal{F} \end{pmatrix}, \quad (7)$$

and a polynomial formulae

$$\mathcal{F}\mathcal{F}^* = \sum_{k=0}^n a_k H^k, \quad \mathcal{F}^*\mathcal{F} = \sum_{k=0}^n a_k^* (H^*)^k \quad (8)$$

with the final goal to elucidate the conditions leading to such a type of the closure of the algebra [22].

1.2 Plan of the paper

In Section 2 we elaborate a particular solution to our problem inspired by the specific second-order supersymmetry (SSUSY) results of ref. [11]. Our solution of Eq. (8) will have the form

$$\mathcal{F}\mathcal{F}^* = h_1^2 - \frac{c^2}{4}, \quad \mathcal{F}^*\mathcal{F} = h_2^2 - \frac{c^2}{4}, \quad (9)$$

where h_1 is naturally related to h_2 by Hermitian conjugation,

$$h_1 = h_2^\dagger, \quad (10)$$

if c^2 is real. Our explicit solution to the problem is rendered possible by a SSUSY inspired gluing constraint [21, 22]. We show that

$$\mathcal{F}h_2 = h_1\mathcal{F}, \quad (11)$$

which, because of Eq. (10), is now equivalent to Eq. (2). This amounts to

$$\mathcal{CPT}h_1 = h_1\mathcal{CPT}. \quad (12)$$

Explicit analytic examples of \mathcal{PT} -asymmetric models are expressed in terms of circular or hyperbolic functions.

In Section 3 we perform a detailed investigation of eq. (2) for a charge operator which is of the second order in derivatives,

$$\mathcal{C} = \frac{d^2}{dx^2} + G(x) \frac{d}{dx} + D(x), \quad (13)$$

and where $G(x)$ and $D(x)$ are complex functions of the real coordinate x :

$$\begin{aligned} G(x) &= G_R(x) + iG_I(x), \\ D(x) &= D_R(x) + iD_I(x). \end{aligned}$$

We further derive the polynomial algebra of Eq. (8). In order to show explicitly that our formalism allows to generate \mathcal{PT} -asymmetric models with real spectrum, we discuss in Section 4 a particular polynomial oscillator model.

In Section 5 we generalize the postulate (13) and derive the general form of the charge operator \mathcal{C} of any finite order in the derivative d/dx such that $\mathcal{F} \equiv \mathcal{C}\mathcal{P}$ is Hermitian. At the very end, in section 6 we give some perspectives on the impact of our results on a variety of fields where the use of similar \mathcal{F} might play significant role.

2 SUSY gluing constraint

Starting with a second-order \mathcal{C} of the form (13) we have to guarantee, first of all, the Hermiticity of $\mathcal{F} = \mathcal{C}\mathcal{P}$ and $\mathcal{F}^{-1} = \mathcal{P}\mathcal{C}^{-1}$. It is easy to show (see also section 5 below for an exhaustive discussion of these important conditions for polynomial charges) that the latter Hermiticity condition forces us to impose the necessary and sufficient requirements

$$D_R(x) = D_R(-x) + \frac{d}{dx}G_R(x), \quad D_I(x) = -D_I(-x) + \frac{d}{dx}G_I(x)$$

where $G_R(x) = G_R(-x)$ is even while $G_I(x) = -G_I(-x)$ must be odd.

2.1 Factorization

In the subsequent step of our considerations we factorize our second-order charge operator \mathcal{C} as follows,

$$\mathcal{C} = q_1 q_2, \quad q_1 = \frac{d}{dx} + U(x), \quad q_2 = \frac{d}{dx} + W(x), \quad (14)$$

where

$$U(x) + W(x) = G(x), \quad \frac{d}{dx}W(x) + U(x)W(x) = D(x). \quad (15)$$

In order to simplify the problem at the start we impose the following “gluing” constraint on q_1 and q_2 ,

$$q_2(q_2^\dagger)^* = (q_1^\dagger)^* q_1 + c, \quad (16)$$

where c is a complex number. By inserting Eqs. (14) into Eq. (16), we obtain

$$\left(\frac{d}{dx} + W\right) \left(-\frac{d}{dx} + W\right) = \left(-\frac{d}{dx} + U\right) \left(\frac{d}{dx} + U\right) + c,$$

whence

$$\frac{d}{dx}W(x) + W^2(x) = -\frac{d}{dx}U(x) + U^2(x) + c. \quad (17)$$

We find the following representation for $\mathcal{F}\mathcal{F}^*$ and $\mathcal{F}^*\mathcal{F}$ (Eq. (3))

$$\mathcal{F}\mathcal{F}^* = \mathcal{F}(\mathcal{F}^\dagger)^* = (q_1 q_2 \mathcal{P}) \cdot (\mathcal{P} q_2^\dagger q_1^\dagger)^* = q_1 q_2 (\mathcal{P})^2 (q_2^\dagger)^* (q_1^\dagger)^*,$$

which, taking Eq. (16) into account, becomes

$$\begin{aligned} \mathcal{F}\mathcal{F}^* &= q_1 \left[(q_1^\dagger)^* q_1 + \frac{c}{2} + \frac{c}{2} \right] (q_1^\dagger)^* \\ &= \left[q_1 (q_1^\dagger)^* + \frac{c}{2} + \frac{c}{2} \right] \cdot \left[q_1 (q_1^\dagger)^* + \frac{c}{2} - \frac{c}{2} \right]. \end{aligned}$$

Correspondingly,

$$\begin{aligned} \mathcal{F}^*\mathcal{F} &= (\mathcal{F}^\dagger)^* \mathcal{F} \\ &= \left[\mathcal{P} (q_2^\dagger)^* q_2 \mathcal{P} - \frac{c}{2} - \frac{c}{2} \right] \cdot \left[\mathcal{P} (q_2^\dagger)^* q_2 \mathcal{P} - \frac{c}{2} + \frac{c}{2} \right]. \end{aligned}$$

Defining the Hamiltonian operators

$$\begin{aligned} h_1 &= q_1 (q_1^\dagger)^* + \frac{c}{2} \\ &= \left(\frac{d}{dx} + U\right) \left(-\frac{d}{dx} + U\right) + \frac{c}{2} \\ &= -\frac{d^2}{dx^2} + \frac{d}{dx}U(x) + U^2(x) + \frac{c}{2}, \end{aligned}$$

and

$$\begin{aligned}
h_2 &= \mathcal{P}(q_2^\dagger)^* q_2 \mathcal{P} - \frac{c}{2} \\
&= \mathcal{P} \left(-\frac{d}{dx} + W(x) \right) \left(\frac{d}{dx} + W(x) \right) \mathcal{P} - \frac{c}{2} \\
&= -\frac{d^2}{dx^2} - \frac{d}{dx} W(-x) + W^2(-x) - \frac{c}{2},
\end{aligned}$$

equation (7) provides the following representation for K

$$K = \mathcal{H}^2 - \frac{c^2}{4}, \quad \mathcal{H} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}.$$

Comparing with Section 3.3 below, where $\mathcal{F}\mathcal{F}^* = H^2 + \alpha H + \gamma$, and setting $h_1 = H$, we get in the present case $\alpha = 0$ and, correspondingly, $V_0 = 0$, according to Eq. (53) below, as well as $\gamma = -c^2/4$. This shows explicitly how the present model can be derived from the general results of section 3.

2.2 Hamiltonians

Remembering the first of Eqs. (15), Eq. (17) becomes

$$\frac{d}{dx}G(x) + W^2(x) - (G(x) - W(x))^2 = c,$$

i.e.,

$$\frac{d}{dx}G(x) - G^2(x) + 2G(x)W(x) = c,$$

or

$$G(x)W(x) = \frac{1}{2} \left(G^2(x) - \frac{d}{dx}G(x) + c \right). \quad (18)$$

We immediately deduce that

$$\begin{aligned}
W(x) &= \frac{G^2(x) - \frac{d}{dx}G(x) + c}{2G(x)}, \\
U(x) &= G(x) - W(x) = \frac{G^2(x) + \frac{d}{dx}G(x) - c}{2G(x)}.
\end{aligned} \quad (19)$$

Thus

$$h_1 = -\frac{d^2}{dx^2} + V(x),$$

with

$$V(x) = G'(x) - \frac{(G'(x))^2}{4G^2(x)} + \frac{G''(x)}{2G(x)} + \frac{G^2(x)}{4} + \frac{c^2}{4G^2(x)}. \quad (20)$$

From Eq. (18) we also get that at the zeros \bar{x} of G , we must have

$$\left. \frac{dG}{dx} \right|_{x=\bar{x}} = c,$$

which is a constraint on G , too. In fact, the method would fail if G had several zeros with non-identical values of the first derivative at each of them.

An important comment must be made here since even if a function does not vanish on the real axis, one can investigate its zeros in the complex x plane. For instance, if

$$G(x) = G_0(x) \equiv z(x) = \frac{1 + i \sinh(\alpha x)}{2}, \quad \alpha \in \mathbf{R}, \quad (21)$$

it is immediate to check that $z(x_n) = 0$ at $x_n = -i(2n + 3/2)\pi/\alpha$, $n = 0, \pm 1, \dots$. This would mean that $dG_0(x_n)/dx = (i\alpha/2) \cosh(\alpha x_n) = 0$, thus implying that we must put $c = 0$ in this case.

In the similar spirit, we may consider the whole class of functions which depend on x only via $z(x)$ of Eq. (21) in an arbitrary nonlinear manner, $G_m(x) \equiv G(z(x))$, since, as a function of x , z is \mathcal{PT} -symmetric, and any real function of z is \mathcal{PT} -symmetric, too, and is an acceptable candidate for G .

It becomes convenient to change variables and express the Hamiltonian, $H = -d^2/dx^2 + V(x)$, with $V(x)$ given by formula (20), as a function of z , by observing that

$$\begin{aligned} \frac{d}{dx} &= \frac{dz}{dx} \frac{d}{dz} = i\alpha \sqrt{z(1-z)} \frac{d}{dz}, \\ \frac{d^2}{dx^2} &= \left(\frac{dz}{dx} \frac{d}{dz} \right)^2 = -\alpha^2 \left(\frac{1}{2} - z \right) \frac{d}{dz} - \alpha^2 z(1-z) \frac{d^2}{dz^2}, \end{aligned}$$

and

$$\begin{aligned} V(z) &= i\alpha \sqrt{z(1-z)} \frac{d}{dz} G + \alpha^2 \frac{z(1-z)}{4G^2} \left(\frac{d}{dz} G \right)^2 - \alpha^2 \frac{1-2z}{4G} \frac{d}{dz} G \\ &\quad - \alpha^2 \frac{z(1-z)}{2G} \frac{d^2}{dz^2} G + \frac{G^2}{4} + \frac{c^2}{4G^2}. \end{aligned} \quad (22)$$

2.3 Consistency

We prove now an important constraint on the complex number $c = c_R + ic_I$. From the second of Eqs. (15), we have

$$\begin{aligned} D(x) &= \frac{d}{dx} W(x) + U(x)W(x) \\ &= \frac{1}{2G^2(x)} \left[\left(2G \frac{d}{dx} G - \frac{d^2}{dx^2} G \right) G - \frac{d}{dx} G \left(G^2 - \frac{d}{dx} G + c \right) \right] \\ &\quad + \frac{1}{4G^2} \left[G^4 - \left(\frac{d}{dx} G \right)^2 - c^2 + 2c \frac{d}{dx} G \right], \end{aligned}$$

or

$$D(x) = \frac{1}{4G^2} \left[2G^2 \frac{d}{dx} G - 2G \frac{d^2}{dx^2} G + \left(\frac{d}{dx} G \right)^2 + G^4 - c^2 \right], \quad (23)$$

$$D^*(-x) = \frac{1}{4G^2} \left[-2G^2 \frac{d}{dx} G - 2G \frac{d^2}{dx^2} G + \left(\frac{d}{dx} G \right)^2 + G^4 - (c^*)^2 \right] \quad (24)$$

where the functions on the right-hand-sides of Eqs. (23) and (24) are all computed at x .

In deriving Eq. (24), use has been made of the fact that G and d^2G/dx^2 are \mathcal{PT} -symmetric, while dG/dx is \mathcal{PT} -antisymmetric, *i.e.*, $(dG/dx(-x))^* = -dG/dx(x)$. Eq. (23) is obviously consistent with the general form of D as a function of G given by Eqs. (48), (50), with $c^2/4 = -I_0 - D(x_0)G^2(x_0)$. Subtracting Eq. (24) from Eq. (23) side by side, we obtain

$$D(x) - D^*(-x) = \frac{d}{dx} G + \frac{(c^2)^* - c^2}{4G^2}. \quad (25)$$

Combining Eq. (25) with Eq. (33), we obtain the important result

$$(c^2)^* - c^2 = 0 \quad \rightarrow \quad \Im(c^2) = 0 \quad \rightarrow \quad c_R c_I = 0.$$

From Eq. (19) we easily obtain

$$U(x) = W^*(-x) - \frac{c_R}{G(x)},$$

whence

$$\left(\frac{d}{dx} U(x) \right)^* = -\frac{d}{dx} W(-x) + \frac{c_R}{(G^*(x))^2} \frac{d}{dx} G^*, \quad (26)$$

and

$$(U^*(x))^2 = W^2(-x) + \frac{c_R^2}{(G^*(x))^2} - 2c_R \frac{W(-x)}{G^*(x)}. \quad (27)$$

Thus

$$\begin{aligned} h_1^\dagger - h_2 &= \left(\frac{d}{dx} U(x) + U^2(x) \right)^* + \frac{d}{dx} W(-x) - W^2(-x) + c_R \\ &= c_R \left[\frac{1}{(G^2(x))^*} \left(\frac{d}{dx} G^*(x) + c_R \right) - 2 \frac{W(-x)}{G^*(x)} + 1 \right]. \end{aligned}$$

Using Eq. (19) to replace $W(-x)$, we obtain

$$\begin{aligned} h_1^\dagger - h_2 &= c_R \left[\frac{1}{(G^2)^*} \left(\frac{d}{dx} G^* + c_R \right) - \frac{1}{(G^2)^*} \left((G^2)^* + \frac{d}{dx} G^* + c \right) + 1 \right] \\ &= c_R \frac{c_R - c}{(G^2)^*} = -i \frac{c_R c_I}{(G^2)^*} = -i c_R c_I \frac{G^2}{|G|^4}. \end{aligned} \quad (28)$$

Therefore

$$h_1^\dagger = h_2 \quad \Leftrightarrow \quad c_R c_I = 0.$$

2.4 Periodic potential

Let us now give an example which generalizes \mathcal{PT} -symmetric periodic potentials [8, 23]:

$$G(x) = e^{i\alpha x} + r, \quad \alpha \in \mathbf{R}, \quad r \in \mathbf{R}, \quad r \neq \pm 1.$$

In this case we have, for all $x \in \mathbf{R}$,

$$U(x) = \frac{1}{2} (e^{i\alpha x} + r) + \frac{1}{2} (i\alpha e^{i\alpha x} - c) / (e^{i\alpha x} + r)$$

$$W(x) = \frac{1}{2} (e^{i\alpha x} + r) - \frac{1}{2} (i\alpha e^{i\alpha x} - c) / (e^{i\alpha x} + r).$$

Since G never vanishes, we do not have any constraint on the value of c , in addition to the one which requires that c be either real, or imaginary. The spectral analysis of the corresponding Schrödinger operators h_1 and h_2 with periodic potentials can be performed as a generalization to the non- \mathcal{PT} -symmetric case of the investigation done by in Ref. [24].

We now examine the invertibility of \mathcal{C} and the boundedness of \mathcal{C}^{-1} . First notice that \mathcal{C} can be written in the following form

$$\mathcal{C} = \mathcal{C}_1 \mathcal{C}_2, \quad \mathcal{C}_1 = \mathcal{C}_U + \frac{r}{2}, \quad \mathcal{C}_2 = \mathcal{C}_U + \frac{r}{2} \quad (29)$$

where

$$\begin{aligned} \mathcal{C}_U &= \frac{d}{dx} + U_1, & U_1 &= U - \frac{r}{2} \\ \mathcal{C}_W &= \frac{d}{dx} + W_1, & W_1 &= W - \frac{r}{2}. \end{aligned}$$

We will discuss the invertibility of each factor in (29) separately. As for \mathcal{C}_1 we first observe that the numerical range $\{z = \langle \mathcal{C}_U \psi, \psi \rangle : \psi \in H^1(\mathbf{R})\}$ of \mathcal{C}_U is contained in the strip $\{z : |\operatorname{Re} z| \leq a\}$ where $a = \max_{x \in \mathbf{R}} |U_1(x)|$. Hence, if $|r| > 2a$ then $-r/2$ is in the resolvent set of \mathcal{C}_U [25] and, therefore, \mathcal{C}_1 is invertible with bounded inverse on $L^2(\mathbf{R})$. A similar argument holds for \mathcal{C}_W . Thus, for sufficiently large values of $|r|$, operator \mathcal{C} is invertible and \mathcal{C}^{-1} is bounded on $L^2(\mathbf{R})$.

3 Second-order charge operator \mathcal{C}

3.1 Re-construction of the potential

We already noticed that in the second-order charge operator (13), the notation of Section 5 below implies that we have the correspondences $\gamma_2(x) = 1$, $\gamma_1(x) = G(x)$ and $\gamma_0(x) = D(x)$, so that the Hermiticity constraints on

the real and imaginary parts of $\gamma_\ell(x)$, Eqs. (70) and (71), with $\omega = 2$ and $\ell = 0, 1$, immediately give

$$G_R(x) - G_R(-x) = 0; \quad G_I(x) + G_I(-x) = 0; \quad (30)$$

$$D_R(x) - D_R(-x) = \frac{d}{dx}G_R(x); \quad D_I(x) + D_I(-x) = \frac{d}{dx}G_I(x). \quad (31)$$

As a consequence of Eq. (30), G is \mathcal{PT} -symmetric

$$G(x) = G^*(-x), \quad (32)$$

while Eq. (31) yields

$$D(x) - D^*(-x) = \frac{d}{dx}G(x). \quad (33)$$

We assume that \mathcal{F} and H satisfy the intertwining condition (2) and that H depends on a local complex potential, $V(x)$:

$$H = -\frac{d^2}{dx^2} + V(x), \quad (34)$$

with $V(x) = V_R(x) + iV_I(x)$. In turn, $V_R(x)$ and $V_I(x)$ are conveniently decomposed into their even and odd parts:

$$\begin{aligned} V_R(x) &= V_R^E(x) + V_R^O(x), \\ V_I(x) &= V_I^E(x) + V_I^O(x), \end{aligned}$$

with $V_K^E(x) = V_K^E(-x)$ and $V_K^O(x) = -V_K^O(-x)$, ($K = R, I$). We write now condition (2) explicitly and obtain three non-trivial equations by imposing that the coefficients of $(d/dx)^2$, d/dx and $(d/dx)^0$ vanish,

$$-2(V_R^O + iV_I^E) + 2\frac{d}{dx}(G_R + iG_I) = 0, \quad (35)$$

$$\begin{aligned} 2\frac{d}{dx}(V_R^E + iV_I^O) - 2\frac{d}{dx}(V_R^O + iV_I^E) + 2\frac{d}{dx}(D_R + iD_I) \\ + \frac{d^2}{dx^2}(G_R + iG_I) - 2(V_R^O + iV_I^E)(G_R + iG_I) = 0, \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{d^2}{dx^2}(V_R^E + iV_I^O) - \frac{d^2}{dx^2}(V_R^O + iV_I^E) + \frac{d^2}{dx^2}(D_R + iD_I) \\ + (G_R + iG_I)\frac{d}{dx}(V_R^E(x) + iV_I^O(x)) - (G_R + iG_I)\frac{d}{dx}(V_R^O + iV_I^E) \\ - 2(D_R + iD_I)(V_R^O + iV_I^E) = 0 \end{aligned} \quad (37)$$

while the coefficients of $(d/dx)^4$ and $(d/dx)^3$ are identically zero.

3.2 Integrability

The first of the above equations (35) yields

$$V_R^O = \frac{d}{dx}G_R; \quad V_I^E = \frac{d}{dx}G_I. \quad (38)$$

The second equation (36) yields

$$\begin{aligned} \frac{d}{dx}V_R^E - \frac{d}{dx}V_R^O - V_R^O G_R + V_I^E G_I + \frac{d}{dx}D_R + \frac{1}{2} \frac{d^2}{dx^2}G_R &= 0; \\ \frac{d}{dx}V_I^O - \frac{d}{dx}V_I^E - V_I^E G_R - V_R^O G_I + \frac{d}{dx}D_I + \frac{1}{2} \frac{d^2}{dx^2}G_I &= 0, \end{aligned}$$

and is easily integrated for the other two components of the potential, $V_R^E(x)$ and $V_I^O(x)$, as functions of $G_R(x)$, $G_I(x)$, $D_R(x)$ and $D_I(x)$, by replacing $V_R^O(x)$ and $V_I^E(x)$ with their expressions (38):

$$V_R^E(x) = \frac{1}{2} \frac{d}{dx}G_R(x) + \frac{1}{2} (G_R(x))^2 - \frac{1}{2} (G_I(x))^2 - D_R(x) + V_0 \quad (39)$$

$$V_I^O(x) = \frac{1}{2} \frac{d}{dx}G_I(x) + G_R(x)G_I(x) - D_I(x). \quad (40)$$

Here, V_0 is a real integration constant. The corresponding integration constant in the equation for $V_I^O(x)$ must be zero, because the function is odd. Both equations can be recombined as

$$V(x) = \frac{3}{2} \frac{d}{dx}G(x) + \frac{1}{2}G^2(x) - D(x) + V_0. \quad (41)$$

Finally, the third equation (37) allows us to express the $G_J(x)$'s, ($J = R, I$), as functions of the $D_K(x)$'s, ($K = R, I$), or, more conveniently, viceversa.

$$\begin{aligned} & -\frac{1}{2} \frac{d^3}{dx^3}G_R + \frac{G_R}{2} \frac{d^2}{dx^2}G_R + \left(\frac{d}{dx}G_R \right)^2 + (G_R^2 - G_I^2 - 2D_R) \frac{d}{dx}G_R \\ & -\frac{G_I}{2} \frac{d^2}{dx^2}G_I - \left(\frac{d}{dx}G_I \right)^2 + 2(D_I - G_I G_R) \frac{d}{dx}G_I - G_R \frac{d}{dx}D_R + G_I \frac{d}{dx}D_I \\ & = 0 \quad (42) \\ & -\frac{1}{2} \frac{d^3}{dx^3}G_I + \frac{1}{2} G_R \frac{d^2}{dx^2}G_I + (-G_I^2 + G_R^2 - 2D_R) \frac{d}{dx}G_I + \frac{1}{2} G_I \frac{d^2}{dx^2}G_R \\ & + 2 \frac{d}{dx}G_R \frac{d}{dx}G_I + 2(G_R G_I - D_I) \frac{d}{dx}G_R - G_I \frac{d}{dx}D_R - G_R \frac{d}{dx}D_I \\ & = 0. \end{aligned}$$

Eqs. (42) can be recombined in the following first-order linear equation expressing the unknown function $D(x)$ in terms of the known function $G(x)$

and its derivatives up to third order

$$\frac{1}{2} \frac{d^3}{dx^3} G - \frac{1}{2} G \frac{d^2}{dx^2} G - \left(\frac{d}{dx} G \right)^2 - G^2 \frac{d}{dx} G + 2 \left(\frac{d}{dx} G \right) D + G \frac{d}{dx} D = 0. \quad (43)$$

Eq. (43) is easily solved by direct integration. Let us define the auxiliary functions

$$g(x) \equiv 2 \frac{d}{dx} G, \quad (44)$$

$$f(x) \equiv -\frac{1}{2} \frac{d^3}{dx^3} G + \frac{1}{2} G \frac{d^2}{dx^2} G + \left(\frac{d}{dx} G \right)^2 + G^2 \frac{d}{dx} G, \quad (45)$$

$$\frac{1}{p(x)} \frac{d}{dx} p(x) \equiv \frac{g(x)}{G(x)}. \quad (46)$$

Eq. (46) is promptly integrated by use of definition (44) to

$$p(x) = \exp \left(2 \int_{x_0}^x d \ln G(x') \right) = \frac{G^2(x)}{G^2(x_0)}, \quad (47)$$

where x_0 is an initial point where G is different from zero. It is now easy to check that the general solution to Eq. (43) can be written in the form

$$p(x)D(x) = \int_{x_0}^x dx' \frac{p(x')f(x')}{G(x')} + p(x_0)D(x_0),$$

or

$$D(x) = \frac{1}{G^2(x)} \int_{x_0}^x dx' G(x')f(x') + \frac{D(x_0)G^2(x_0)}{G^2(x)}. \quad (48)$$

The integral on the right-hand side of Eq. (48) is computed by elementary methods in the form

$$\int_{x_0}^x dx' G(x')f(x') = \frac{G^4(x)}{4} + \frac{G^2(x)G'(x)}{2} - \frac{G(x)G''(x)}{2} + \frac{(G'(x))^2}{4} + I_0, \quad (49)$$

with

$$I_0 \equiv -\frac{G^4(x_0)}{4} - \frac{G^2(x_0)G'(x_0)}{2} + \frac{G(x_0)G''(x_0)}{2} - \frac{(G'(x_0))^2}{4}, \quad (50)$$

where $G' \equiv dG/dx$, and so on, thus providing the most general expression of D as a function of G and of its derivatives.

3.3 SSUSY algebra

Assuming a charge operator, $\mathcal{C}(x)$, of the form (13), we now verify that the operator

$$\mathcal{F}(x)\mathcal{F}^*(x) = \mathcal{C}(x)\mathcal{P}\mathcal{C}^*(x)\mathcal{P} = \mathcal{C}(x)\mathcal{C}^*(-x)$$

can be written as a particular case of formula (8)

$$\mathcal{F}(x)\mathcal{F}^*(x) = H^2 + \alpha H + \gamma,$$

where α and γ are constants to be determined and H is Hamiltonian (34) with V given in (41). In fact, we have

$$\begin{aligned} \mathcal{C}(x)\mathcal{C}^*(-x) &= \left(\frac{d^2}{dx^2} + G(x)\frac{d}{dx} + D(x) \right) \cdot \left(\frac{d^2}{dx^2} - G^*(-x)\frac{d}{dx} + D^*(-x) \right) \\ &= \left(\frac{d^2}{dx^2} + G(x)\frac{d}{dx} + D(x) \right) \cdot \left(\frac{d^2}{dx^2} - G(x)\frac{d}{dx} + D(x) - G'(x) \right) \end{aligned}$$

where use has been made of relations (30), (31) stemming from Hermiticity of $\mathcal{C}(x)$. After some algebra, the right-hand side of the above expression is brought to the form

$$\begin{aligned} \mathcal{C}(x)\mathcal{C}^*(-x) &= \frac{d^4}{dx^4} + \left(2D(x) - G^2(x) - 3G'(x) \right) \frac{d^2}{dx^2} \quad (51) \\ &\quad + \left(2D'(x) - 3G''(x) - 2G(x)G'(x) \right) \frac{d}{dx} \\ &\quad + D''(x) - G'''(x) + G(x)D'(x) \\ &\quad - G(x)G''(x) + D^2(x) - D(x)G'(x), \end{aligned}$$

and is to be compared with

$$\begin{aligned} H^2 + \alpha H + \gamma &= \left(-\frac{d^2}{dx^2} + V(x) \right)^2 + \alpha \left(-\frac{d^2}{dx^2} + V(x) \right) + \gamma \quad (52) \\ &= \frac{d^4}{dx^4} - (2V(x) + \alpha) \frac{d^2}{dx^2} - 2V'(x) \frac{d}{dx} + V^2(x) \\ &\quad - V''(x) + \alpha V(x) + \gamma, \end{aligned}$$

where $V(x)$ may be expressed as a function of $D(x)$ and $G(x)$ according to Eq. (41). Direct comparison of the right-hand sides of the above formulae allows us to determine the α constant as

$$\alpha = -2V_0. \quad (53)$$

The value of γ expresses the compatibility between \mathcal{C} and the polynomial algebra through the equation

$$V^2(x) - V''(x) + \alpha V(x) + \gamma =$$

$$= D''(x) - G'''(x) + G(x)D'(x) - G(x)G''(x) + D^2(x) - D(x)G'(x).$$

Here, we insert the expressions of $V(x)$ and $V''(x)$ in terms of $G(x)$, $D(x)$ and of their derivatives obtained from formula (41), and making use of Eq. (43), as well as of its general solution (48), (49), we obtain the final result

$$\gamma = V_0^2 + I_0 + D(x_0)G^2(x_0), \quad (54)$$

where I_0 is defined in Eq. (50). This makes it possible to interpret γ as a kind of integration constant. Thus, \mathcal{CPT} invariance leads to the SSUSY polynomial algebra, Eqs. (7), (8).

4 Polynomial oscillators

The simplest factorization of \mathcal{C} reads

$$\mathcal{C}(x) = \left(\frac{d}{dx} + \frac{G(x)}{2} \right) \cdot \left(\frac{d}{dx} + \frac{G(x)}{2} \right), \quad (55)$$

so that, correspondingly,

$$D(x) = \frac{G'(x)}{2} + \frac{G^2(x)}{4}. \quad (56)$$

In this case, Eq. (43) yields $G'''(x) = 0$, *i.e.*,

$$G(x) = ax^2 + ibx + c \quad (57)$$

where a, b and c are real numbers, owing to the fact that $G(x)$ is \mathcal{PT} -symmetric. From Eq. (41) we obtain:

$$\begin{aligned} V(x) &= \frac{1}{4}G^2(x) + G'(x) + V_0 \\ &= \frac{1}{4}a^2x^4 - \frac{1}{4}(b^2 - 2ac)x^2 + \frac{1}{2}iabx^3 + \frac{1}{2}x(ibc + 4a) + ib + \frac{c^2}{4} + V_0. \end{aligned} \quad (58)$$

If we make the additional assumption $c = 0$, for the sake of simplicity, the polynomial algebra provides the constraint

$$\gamma = V_0^2 + \frac{b^2}{4}$$

on γ [Eq. (54)].

4.1 The problem of invertibility

We will now make the spectral analysis for H and study the invertibility of \mathcal{F} in the case $c = 0$. Then

$$V(x) = \frac{1}{4}a^2x^4 - \frac{1}{4}b^2x^2 + \frac{1}{2}iabx^3 + 2ax + ib + V_0. \quad (59)$$

Setting $\mu^2 = \frac{a^2}{4}$ and $\nu^2 = \frac{b^2}{4}$, we obtain an expression for the Schrödinger operator H of the same type as that presented in Eqs. (22), (23) of Ref. [19], namely

$$H = -\frac{d^2}{dx^2} + \mu^2x^4 - \nu^2x^2 + 2i\mu\nu x^3 + 4\mu x + 2i\nu + V_0 \quad (60)$$

and $D(H) = H^2(\mathbf{R}) \cap D(x^4)$, $\forall \mu, \nu \in \mathbf{R}$, $\mu \neq 0$. As in Ref. [19], H has discrete spectrum, *i.e.*, the spectrum consists of a sequence of isolated eigenvalues with finite multiplicity.

In order to prove the reality of the spectrum of H , we first notice that H can be rewritten as

$$H = -\frac{d^2}{dx^2} + x^2(\mu x + i\nu)^2 + 4\mu x + 2i\nu + V_0. \quad (61)$$

Let us now perform the complex translation $x \rightarrow x - \frac{i\nu}{2\mu}$. Then $H = S^{-1}H_1S$ where $S\psi(x) = \psi(x - \frac{i\nu}{2\mu})$ on a dense set of functions $\psi \in L^2(\mathbf{R})$ and

$$\begin{aligned} H_1 &= -\frac{d^2}{dx^2} + \left(x - \frac{i\nu}{2\mu}\right)^2 \left(\mu x - \frac{i\nu}{2} + i\nu\right)^2 + 4\mu x - 2i\nu + 2i\nu + V_0 \\ &= -\frac{d^2}{dx^2} + \mu^2 \left(x - \frac{i\nu}{2\mu}\right)^2 \left(x + \frac{i\nu}{2\mu}\right)^2 + 4\mu x + V_0 \\ &= -\frac{d^2}{dx^2} + \mu^2 \left(x^2 + \frac{\nu^2}{4\mu^2}\right)^2 + 4\mu x + V_0 \end{aligned} \quad (62)$$

Hence H has the same spectrum of H_1 . In turn H_1 is selfadjoint on $D(H_1) = D(H) = H^2(\mathbf{R}) \cap D(x^4)$, thus it has real spectrum for all $\mu, \nu, V_0 \in \mathbf{R}$, $\mu \neq 0$.

We may stress that Hamiltonian (60) is not \mathcal{PT} -invariant but has still a real spectrum because it is related by explicit similarity to the standard self-adjoint anharmonic oscillator. In our opinion this is an exceptional example since in general the proof of the reality of the spectra of non-Hermitian Hamiltonians cannot proceed in such a straightforward manner and, generically, the necessary maps are non-local [26]. Moreover, by our construction, the reality of the spectrum is robust insofar as its \mathcal{CPT} -symmetry cannot be spontaneously broken. In this sense, our example (60) may be perceived as a \mathcal{PT} -*asymmetric* parallel to the \mathcal{PT} -symmetric quartic oscillator of Buslaev and Grecchi [27].

4.2 The problem of boundedness

Let us now turn to the operator $\mathcal{F} = \mathcal{C}\mathcal{P}$. In order to prove the invertibility of \mathcal{F} and the boundedness of \mathcal{F}^{-1} on $L^2(\mathbf{R})$ it is enough to demonstrate the same facts for C . Factorization (55) implies that it will suffice to prove that $C_1 = (\frac{d}{dx} + \frac{G}{2})$ is invertible and that C_1^{-1} is bounded on $L^2(\mathbf{R})$ if G is given by (57). Indeed, we have

$$C_1 = \frac{d}{dx} + \frac{1}{2}ax^2 + \frac{i}{2}bx \quad (63)$$

and we now proceed as in Ref. [19]. More precisely

$$C_1 = \frac{d}{dx} + \frac{a}{2} \left(x + \frac{ib}{2a} \right)^2 + \frac{b^2}{8a} \quad (64)$$

is similar to

$$C_2 = \frac{d}{dx} + \frac{a}{2}x^2 + \frac{b^2}{8a} \quad (65)$$

via the complex translation $x \rightarrow x - \frac{ib}{2a}$. Hence C_1 has the same spectrum as C_2 . In turn C_2 is unitarily equivalent, via the Fourier transformation, to

$$C_3 = -\frac{a}{2} \frac{d^2}{dx^2} + ix + \frac{b^2}{8a}. \quad (66)$$

Therefore C_1 has the same spectrum as C_3 . Finally, we perform the unitary dilation $(U\psi)(x) = (a/2)^{1/6}\psi[(a/2)^{1/3}x]$ and obtain that C_1 has the same spectrum as

$$C_4 = UC_3U^{-1} = \left(\frac{a}{2}\right)^{1/3} \left[-\frac{d^2}{dx^2} + ix + \left(\frac{a}{2}\right)^{-1/3} \frac{b^2}{8a} \right]. \quad (67)$$

Now, since the Schrödinger operator $-\frac{d^2}{dx^2} + ix$ has an empty spectrum (see Ref. [28]), so does C_1 . In particular $z = 0$ belongs to resolvent set of C_1 , so that C_1 is invertible and its inverse is bounded and defined on the whole of $L^2(\mathbf{R})$.

5 Towards operators \mathcal{C} of any finite order

We shall postulate that the charge-operator component \mathcal{C} of the pseudo-metric $\mathcal{C}\mathcal{P}$, where \mathcal{P} denotes parity, is a polynomial of any finite degree $\omega = 0, 1, \dots$ in the momentum operator p ,

$$\mathcal{C} = \sum_{k=0}^{\omega} \gamma_k(x) \frac{d^k}{dx^k}, \quad \gamma_k(x) = \gamma_k^R(x) + i \gamma_k^I(x). \quad (68)$$

The functions $\gamma_k^R(x)$ and $\gamma_k^I(x)$ are both assumed real, and our main task here is just to guarantee, at any integer ω , that the operator candidate for the metric $\mathcal{C}\mathcal{P}$ is Hermitian.

5.1 The metric \mathcal{CP} in differential form

From

$$\begin{aligned} \mathcal{C}^\dagger &= \sum_{k=0}^{\omega} (-1)^k \sum_{\ell=0}^k \binom{k}{\ell} \left[\frac{d^{(k-\ell)}}{dx^{(k-\ell)}} \gamma_k^*(x) \right] \frac{d^\ell}{dx^\ell} = \\ &= \sum_{\ell=0}^{\omega} (-1)^\ell \left\{ \sum_{m=0}^{\omega-\ell} (-1)^m \binom{\ell+m}{\ell} [\gamma_{\ell+m}^{R(m)}(x) - i \gamma_{\ell+m}^{I(m)}(x)] \right\} \frac{d^\ell}{dx^\ell}, \end{aligned} \quad (69)$$

where the superscripts (m) at the functions γ^R and γ^I indicate their m -tuple differentiation, one obtains that the Hermiticity condition $\mathcal{CP} = \mathcal{PC}^\dagger$ is equivalent to the $(\omega + 1)$ -plet of relations

$$\mathcal{P} \gamma_\ell \mathcal{P} = \gamma_\ell^R(-x) + i \gamma_\ell^I(-x) = \sum_{m=0}^{\omega-\ell} (-1)^m \binom{\ell+m}{\ell} [\gamma_{\ell+m}^{R(m)}(x) - i \gamma_{\ell+m}^{I(m)}(x)]$$

with a trivial decoupling into its real and imaginary parts

$$\gamma_\ell^R(-x) - \gamma_\ell^R(+x) = \sum_{m=1}^{\omega-\ell} (-1)^m \binom{\ell+m}{\ell} \gamma_{\ell+m}^{R(m)}(x) \quad (70)$$

and

$$\gamma_\ell^I(-x) + \gamma_\ell^I(+x) = - \sum_{m=1}^{\omega-\ell} (-1)^m \binom{\ell+m}{\ell} \gamma_{\ell+m}^{I(m)}(x), \quad (71)$$

respectively, with $\ell = \omega - k = 0, 1, \dots, \omega$.

5.2 Functional freedom in complex coefficients $\gamma_k(x)$

At the first few $k = 0, 1, \dots$ the above Hermiticity constraints degenerate to the comparatively elementary relations,

$$\gamma_\omega^R(x) - \gamma_\omega^R(-x) = 0, \quad k = 0,$$

$$\gamma_{\omega-1}^R(x) - \gamma_{\omega-1}^R(-x) = \binom{\omega}{1} \gamma_\omega^{R(1)}(x), \quad k = 1,$$

$$\gamma_{\omega-2}^R(x) - \gamma_{\omega-2}^R(-x) = \binom{\omega-1}{1} \gamma_{\omega-1}^{R(1)}(x) - \binom{\omega}{2} \gamma_\omega^{R(2)}(x), \quad k = 2,$$

etc, or, in parallel,

$$\gamma_\omega^I(x) + \gamma_\omega^I(-x) = 0, \quad k = 0,$$

$$\gamma_{\omega-1}^I(x) + \gamma_{\omega-1}^I(-x) = \binom{\omega}{1} \gamma_\omega^{I(1)}(x), \quad k = 1,$$

$$\gamma_{\omega-2}^I(x) + \gamma_{\omega-2}^I(-x) = \binom{\omega-1}{1} \gamma_{\omega-1}^{I(1)}(x) - \binom{\omega}{2} \gamma_\omega^{I(2)}(x), \quad k = 2,$$

etc. This means that the symmetric parts $H_\ell(x) = H_\ell(-x)$ of all $\gamma_\ell^R(x)$ are arbitrary functions while, in parallel, the antisymmetric parts $h_\ell(x) = -h_\ell(-x)$ of all $\gamma_\ell^I(x)$ are also arbitrary. We may conjecture that the remaining components $R_\ell(x) = \gamma_\ell^R(x) - H_\ell(x) = -R_\ell(-x)$ and $r_\ell(x) = \gamma_\ell^I(x) - h_\ell(x) = r_\ell(-x)$ obey the rules

$$R_\omega = 0, \quad R_{\omega-1}(x) = \frac{\omega}{2}H_\omega^{(1)}(x), \quad R_{\omega-2}(x) = \frac{\omega-1}{2}H_{\omega-1}^{(1)}(x), \quad \dots \quad (72)$$

while

$$r_\omega = 0, \quad r_{\omega-1}(x) = \frac{\omega}{2}h_\omega^{(1)}(x), \quad r_{\omega-2}(x) = \frac{\omega-1}{2}h_{\omega-1}^{(1)}(x), \quad \dots \quad (73)$$

and are fully determined by the respective recurrent relations (70) and (71).

5.3 Proof

We see that both the sequences $R_{\omega-k}(x)$ and $r_{\omega-k}(x)$ have precisely the same structure so that just the sequence of $R_{\omega-k}(x)$ may be considered without any loss of generality. Its elements should be evaluated in the recurrent manner with respect to the growing k . The appropriate *Ansätze* may be written in the finite-series form where, formally, $H_{\omega+1}(x) = H_{\omega+2}(x) = \dots = 0$ and $h_{\omega+1}(x) = h_{\omega+2}(x) = \dots = 0$,

$$\gamma_{\omega-k}^R(x) = H_{\omega-k}(x) + \sum_{m=1}^k c_m \frac{(\omega-k+m)!}{(\omega-k)!} H_{\omega-k+m}^{(m)}(x), \quad (74)$$

$$\gamma_{\omega-k}^I(x) = h_{\omega-k}(x) + \sum_{m=1}^k c_m \frac{(\omega-k+m)!}{(\omega-k)!} h_{\omega-k+m}^{(m)}(x). \quad (75)$$

With an auxiliary $c_0 = 1$ these *Ansätze* describe all the ω -dependence of our functions $\gamma = \gamma^R + i\gamma^I$ in closed form.

As already stated above, the first term and the subsequent sum are of an opposite parity in both these formulae since $c_{2n} = 0$ at all $n = 1, 2, \dots$. This observation is easily proved since after the insertion of the latter two *Ansätze*, the complicated recurrences (70) are replaced by their simplified version

$$2c_1 = \frac{c_0}{1!}, \quad 2c_2 = \frac{c_1}{1!} - \frac{c_0}{2!}, \quad \dots,$$

i. e.,

$$2c_k = \sum_{m=0}^{k-1} (-1)^{k-m-1} \frac{c_m}{(k-m)!}. \quad (76)$$

It is worthwhile to point out that the c_k coefficients with odd k can be written in terms of Bernoulli numbers (see, *e.g.*, Ref. [29])

$$c_{2n-1} = \frac{2(2^{2n} - 1)}{(2n)!} B_{2n} \quad (n > 0). \quad (77)$$

The key idea of an explicit solution of these recurrences is that the generating function $f(x) = \sum c_k x^k$ of the coefficients c_m must satisfy the functional equation $f(x) - 2 = -f(x)/e^x$ which is, in its turn, easily solvable. In this way we arrive at the solution of recurrences (76) in the following compact form,

$$\begin{aligned} f(x) = c_0 + c_1 x + c_2 x^2 + \dots &= \frac{2}{1 + \exp(-x)} = 1 + \tanh \frac{x}{2} = \\ &= 1 + \sum_{n=1}^{\infty} \frac{2(2^{2n} - 1)}{(2n)!} B_{2n} x^{2n-1}. \end{aligned} \quad (78)$$

Obviously, all the possible parity-violating terms in the right-hand side of our Hermiticity conditions (70) vanish. This makes the form of our polynomial charge \mathcal{C} extremely flexible and confirms the consistency of its present construction.

We may conclude that the requirement of Hermiticity of the metric \mathcal{CP} defines all the antisymmetric components $R_\ell(x)$ and their spatially symmetric partners $r_\ell(x)$. It does not impose any additional constraint either upon the real and spatially symmetric coefficient functions $H_\ell(x)$ or upon their purely imaginary spatially antisymmetric partners $h_\ell(x)$.

6 Outlook

We now sketch some possible applications of our methods to a variety of problems where quasi-Hermitian or pseudo-Hermitian operators are involved.

In the context of the Klein-Gordon description of the free motion of a spinless particle in the “usual” Hilbert space \mathcal{H} the relativistic evolution is generated by the Feshbach-Villars [30] “Hamiltonian” $H_{(FV)}$ which proves non-Hermitian,

$$|\psi(t)\rangle = e^{-iH_{(FV)}(t-t_0)} |\psi(t_0)\rangle, \quad H_{(FV)} = -\frac{1}{2} \begin{pmatrix} 1 - \Delta & -\Delta \\ \Delta & \Delta - 1 \end{pmatrix} \quad (79)$$

(cf., e.g., p. 341 in ref. [31]). One should notice that this model works with the differential pseudo-Hermitian operator with structure which strongly resembles the usual Schrödinger operators in the simplest non-trivial, two-dimensional coupled-channel case. Thus, we may expect that the methods described in our previous study might find an immediate extension to the similar problems.

The idea may also find applications in a broader context, say, of the boson mappings in nuclear physics which were comprehensively discussed in the paper [13]. It is shown there that a consistent quantum mechanical framework,

and in particular a viable variational calculation for non-hermitian Hamiltonians, can indeed be constructed after the introduction of a non-trivial metric. In the context of Holstein-Primakoff type mappings this freedom defines the link with so-called Dyson-Maleev type mappings (see [32]). In practical computations, a puzzling non-Hermiticity of observables proved more than compensated by the advantages, as has been amply demonstrated in applications of generalized Dyson-Maleev mappings (see [33] and references cited therein).

All the technical conditions imposed upon the “true physical metric” Θ in review [13] are important, especially if one tries to work within a truly infinite-dimensional Hilbert space. This has been emphasized by Kretschmer and Szymanowski [14] who showed that the use of the toy metric operators might require a careful scrutiny because these operators remain unbounded. In this context, ref. [19] as well as our present paper demonstrated persuasively that a switch to the use of the differential operators \mathcal{C} might be understood as an important new idea.

All the similar observations must be perceived as individual steps of a systematic improvement of the mathematically correct understanding of the use of the differential operators in connections with many applications of the quasi-Hermitian observables which seems to range, at present, from the elementary descriptions of the localization transitions in solid state physics [34] up to many ambitious \mathcal{PT} -symmetric models in quantum field theory [35].

The experience gained during our study of the simple Schrödinger equations might equally well find applications on the very boundary of quantum mechanics (like, say, in cosmology [36]) or even in the domain of the classical model-building (e.g., in the magneto-dynamics of fluids [37]) and in the various physical models of different origin characterized by the simple matrix structure of their description (see a number of their most elementary samples mentioned in the short and nice review [38]) where the eigenvalues coalesce or almost coalesce in the manner which contradicts the standard and robust finite-dimensional Hermitian-matrix mathematics. Of course, all these mathematical problems and not entirely standard physical situations may impose new and challenging tasks and motivate a deeper future analysis of the questions outlined in our present paper.

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