

Cosmological solutions of the Einstein equation with heat flow.

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Abstract

Cosmological solutions of Einstein's equation for fluids with heat flow in a generalized Robertson-Walker metric are obtained, generalizing the results of Bergmann.

1 Introduction:

We consider Einstein's equation of general theory of relativity for a fluid with heat flow having the following energy-momentum tensor

$$T^{\alpha\beta} = (\rho + p)v^\alpha v^\beta - pg^{\alpha\beta} + q^\alpha v^\beta + q^\beta v^\alpha, \quad (1)$$

where, p and ρ are the isotropic pressure and matter density of the fluid respectively, q_α is the heat flux in the radial direction, and v_α is the velocity vector. In the co-moving coordinate system, $v^\alpha = \delta_0^\alpha$, $v_\alpha v^\alpha = -1$ and $q_\alpha v^\alpha = 0$, along with the generalized Robertson-Walker line element

$$ds^2 = A^2 dt^2 - B^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (2)$$

where A and B are functions of r and t . Components of Einstein's equation $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi GT_{\alpha\beta}$, had been reduced by Bergmann [1] employing a technique formulated by Glass [2], to the following single equation,

$$A'' + 2\frac{F'}{F}A' - \frac{F''}{F}A = 0. \quad (3)$$

In the above, prime denotes differentiation with respect to $x = r^2$, and $F = B^{-1}$. Clearly, one physically relevant assumption is required in order to solve the above differential equation containing a pair of variables A and $F = B^{-1}$. However, a physically meaningful assumption on the metric coefficients A and/ or B is obscure. Bergmann [1] therefore obtained a simple solution under the choice $A = 1$. In this paper, we opt for more general solutions. It is important to mention that once the forms of A and B are known, it is quite trivial to compute the radial component of heat flow, which is given by,

$$q = \left(\frac{4r}{GB^2} \right) \left(\frac{B}{AB} \right)', \quad (4)$$

where, G is the Newtonian gravitational constant.

2 Generating solutions:

Case:1. $A'' = 0$.

Under this choice, one obtains

$$A' = Q(t); \quad \text{and} \quad A(x, t) = Q(t)x + P(t). \quad (5)$$

Thus equation (3) reads as,

$$2QF' - QxF'' - PF'' = 0. \quad (6)$$

Integrating the above equation and thereafter dividing throughout by $(Qx + P)^4$, one obtains

$$\left(\frac{F}{Qx + P} \right)' + \frac{h(t)}{(Qx + P)^4} = 0. \quad (7)$$

Further integration yields,

$$F = \frac{h}{3Q} + (Qx + P)^3 L, \quad (8)$$

and thus,

$$B(x, t) = F^{-1} = \left[\frac{h}{3Q} + (Qx + P)^3 L \right]^{-1}, \quad (9)$$

where, h , Q , P , L are all functions of time. Equations (5) and (9) may be used to find explicit form of the radial component of heat flow q , in view of the expression (4).

Case:2. $A'' \neq 0$.

Under this choice, $F' \neq 0$, as may be seen from equation (3) and thus one can express A as,

$$A = A(F, t); \quad A' = A_F F'; \quad A'' = A_{FF} F'^2 + A_F F'', \quad (10)$$

where, suffix stands for derivative. So, equation (3) in this case reduces to

$$\frac{A_{FF} + 2\frac{A_F}{F}}{A_F - \frac{A}{F}} dF + \frac{dF'}{F'} = 0. \quad (11)$$

Integrating the above equation one obtains,

$$\int \left[\frac{A_{FF} + 2\frac{A_F}{F}}{A_F - \frac{A}{F}} \right] dF + \ln F' = \ln \alpha(t), \quad (12)$$

or,

$$\exp \int \left[\frac{A_{FF} + 2\frac{A_F}{F}}{A_F - \frac{A}{F}} dF \right] = \alpha(t) \frac{dx}{dF}. \quad (13)$$

Integrating yet again one obtains,

$$\int \left[\exp \int \left(\frac{A_{FF} + 2\frac{A_F}{F}}{A_F - \frac{A}{F}} dF \right) \right] dF = \alpha(t)x + \beta(t). \quad (14)$$

Therefore, if A is given as a function of F and t , then the above integral can be evaluated and hence the solutions may be obtained. Nevertheless, for a particular case, simple solutions may be obtained as follows.

Let us consider $F'' = mF$, where m is a function of time alone. So equation (3) may be written as,

$$\frac{U''}{U} = 2\frac{F''}{F} = \pm k^2, \quad i.e., \quad U'' = \pm k^2 U \quad (15)$$

where, $U = AF$, and k is a function of time. Solutions of the above equation (16) may now be easily found as given below,

$$\begin{aligned} U &= C_1 e^{kx} + D_1 e^{-kx}, \quad \text{where, } m \text{ is positive } m = k^2, \\ U &= C_1 \cos(kx) + D_1 \sin(kx), \quad \text{where, } m \text{ is negative } m = -k^2, \\ U &= qx + r, \quad \text{where, } m = 0. \end{aligned} \quad (16)$$

Subcase-I: $m = k^2$:

When $m > 0$, equation (15) may be solved to obtain

$$F = C_2 e^{\frac{kx}{\sqrt{2}}} + D_2 e^{-\frac{kx}{\sqrt{2}}}. \quad (17)$$

Now since, $AF = U$ and $B = F^{-1}$, so

$$A = \frac{C_1 e^{kx} + D_1 e^{-kx}}{C_2 e^{\frac{kx}{\sqrt{2}}} + D_2 e^{-\frac{kx}{\sqrt{2}}}}; \quad B = \frac{1}{C_2 e^{\frac{kx}{\sqrt{2}}} + D_2 e^{-\frac{kx}{\sqrt{2}}}}, \quad (18)$$

where, C_1, C_2, D_1, D_2 and k are all functions of time. Solution (18) may be used to evaluate q from expression (4).

Subcase-II: $m = -k^2$:

When $m < 0$, equation (15) may be solved to obtain

$$F = C_3 \cos\left(\frac{kx}{\sqrt{2}}\right) + D_3 \sin\left(\frac{kx}{\sqrt{2}}\right), \quad (19)$$

where, C_3 and D_3 are functions of time. As before, one can find A and B as,

$$A = \frac{C_1 \cos(kx) + D_1 \sin(kx)}{C_3 \cos\left(\frac{kx}{\sqrt{2}}\right) + D_3 \sin\left(\frac{kx}{\sqrt{2}}\right)}; \quad B = \frac{1}{C_3 \cos\left(\frac{kx}{\sqrt{2}}\right) + D_3 \sin\left(\frac{kx}{\sqrt{2}}\right)}, \quad (20)$$

and hence q may be evaluated as well, from the expression (4).

Subcase-III: $m = 0$:

In this case $k^2 = 0$, and so equation (15) may be solved to obtain,

$$F = k(t)x + C(t), \quad (21)$$

which when substituted in equation (3), one obtains

$$kx \frac{d}{dx} \left(\frac{dA}{dx} \right) + C \frac{d}{dx} \left(\frac{dA}{dx} \right) + 2k \left(\frac{dA}{dx} \right) = 0. \quad (22)$$

Integration yields,

$$A = \frac{f(t)x + g(t)}{k(t)x + C(t)}; \quad B = F^{-1} = \frac{1}{k(t)x + C(t)}. \quad (23)$$

Equation (23) may be used to find the expression for q from equation (3).

3 Conclusion:

Summarily, the present paper gives the complete set of cosmological solutions of Einstein's equation with heat flow which was reduced by Bergman to equation (3), either explicitly or implicitly. For $A'' = 0$, solutions have been obtained explicitly and are presented in (5) and (9). For $A'' \neq 0$, on the contrary, solutions are given implicitly by (14). However, some explicit solutions can be obtained for $F'' = +\frac{1}{2}k^2F$ as presented in equation (18), $F = -\frac{1}{2}k^2F$ as in (20) and $F'' = 0$, as revealed in equation (23).

It has already been stated that the solution of equation (3) gives the solution of Einstein's equation for the metric (2) and the energy-momentum tensor (1), where $B = F^{-1}$ and q is the heat flow given by equation (4). Having obtained these solutions, it remains to be shown that these are physically acceptable. Certain energy conditions have to be satisfied, particularly that the energy density is positive everywhere.

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References

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