

Title: ASSOCIATED LAMÉ EQUATION,PERIODIC POTENTIALS AND  $sl(2,\mathbb{R})$

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### Abstract

We propose a new approach based on the algebraization of the Associated Lamé equation

$$-\psi''(x) + [m(m+1)k^2 sn^2 x + \ell(\ell+1)k^2 (cn^2 x/dn^2 x)]\psi(x) = E\psi(x)$$

within  $sl(2,\mathbb{R})$  to derive the corresponding periodic potentials. The band edge eigenfunctions and energy spectra are explicitly obtained for integers  $m, \ell$ . We also obtain the explicit expressions of the solutions for half-integer  $m$  and integer or half-integer  $\ell$ .

Periodic potentials have attracted considerable attention of late[1-8]. In the study of quantized Neumann problem,generalized Lamé equation was obtained[1] based on the approach of variables separation. A model on periodic potentials was constructed[2a] in the framework of supersymmetric quantum mechanics. A relation with Lamé equation was established in Ref 2b and it was shown that some families of Lamé potentials turn out to be self-isospectral. A geometric approach on elliptic finite-gap potentials was also studied in Ref 3. Moreover,Andrianov et al [ 4 ] also obtained several elliptic potentials for some integrable two-dimentional systems in connection with Lax method.

Several periodic potentials such as Razavy potential[5],Lamé potential or the associated Lamé potential[6] are known to be quasi-exactly solvable(QES). A study of Lie-algebraic approach has been made extensively to find the hidden symmetry algebra of the QES potentials. It is now known that for a one-dimentional QES Hamiltonian,hidden symmetry is  $sl(2,R)$ . Attempts have been made to find a Lie-algebraic representation of several QES periodic potentials[7,8]. The class of QES periodic Hamiltonians representable in terms of  $sl(2,R)$  generators is known as algebraic QES. The QES property of an algebraic potential immediately follows from the fact that the matrix representing the action of Hamiltonian has finite block structure so that by diagonalizing the matrix, a part of the spectrum can be computed in a pure algebraic way. Unfortunately, the list of such algebraic QES periodic potentials is very short. In this letter we propose an algebraization of the associated Lamé equation based on the underlying group  $SL(2,R)$ . To the best of our knowledge such a study has not been undertaken so far in the literature. Interestingly our solutions include as a subset those discussed in Ref 8.

To begin with, let us consider the following differential realization of the  $sl(2,R)$  generators  $T^\pm, T^0$

$$T^+ = \xi^2 \partial_\xi - n\xi, \quad T^0 = \xi \partial_\xi - \frac{1}{2}n, \quad T^- = \partial_\xi, \quad (1)$$

where  $n$  is a non-negative integer. These generators act on polynomials in real variable  $\xi$  of  $\text{deg} \leq n$ . The  $sl(2,R)$  algebra is given by the commutation relations

$$[T^+, T^-] = -2T^0, \quad [T^0, T^\pm] = \pm T^\pm \quad (2)$$

Now the general quadratic combination of the generators induce the gauged Hamiltonian

$$H_G = - \sum_{a,b=0,\pm} C_{ab} T^a T^b - \sum_{a=0,\pm} C_a T^a - d, \quad (3)$$

Using (1),  $H_G$  can be represented as

$$H_G(\xi) = - \sum_{i=2}^4 B_i(\xi) \partial_\xi^{i-2}, \quad (4)$$

where  $B_i(\xi)$  are the  $i$ th degree polynomial in variable  $\xi$  given by

$$\begin{aligned} B_4(\xi) &= C_{++}\xi^4 + 2C_{+0}\xi^3 + C_{00}\xi^2 + 2C_{0-}\xi + C_{--}, \\ B_3(\xi) &= 2(1-n)C_{++}\xi^3 + \{3(1-n)C_{+0} + C_+\}\xi^2 + \{(1-n)C_{00} + C_0\}\xi \\ &\quad + (1-n)C_{0-} + C_-, \\ B_2(\xi) &= n(n-1)C_{++}\xi^2 + n\{(n-1)C_{+0} - C_+\}\xi + \frac{n^2}{4}C_{00} - \frac{1}{2}nC_0 + d. \end{aligned} \quad (5)$$

Note that the numerical parameters  $\{C_{i,j}\}$  are symmetric with  $C_{+-} = C_{-+} = 0$  and  $d$  is a suitably chosen constant. A coordinate transformation of the type

$$x(\xi) = \int^\xi \frac{d\tau}{\sqrt{B_4(\tau)}} \quad (6)$$

converts  $H_G$  to the form

$$H_G(x) = -\partial_x^2 + \left[ \frac{B'_4 - 2B_3}{2\sqrt{B_4}} \right]_{\xi=\xi(x)} \partial_x - [B_2]_{\xi=\xi(x)}, \quad (7)$$

where the prime denotes derivative with respect to  $\xi$ .

On the other hand, it is well known that the Schrödinger equation

$$H(x)\psi(x) \equiv [-\partial_x^2 + V(x)]\psi(x) = E\psi(x) \quad (8)$$

can be gauge-transformed using an imaginary phase transformation

$$\psi(x) = e^{-\int \mathcal{A}(x) dx} \chi(x) \quad (9)$$

to the form

$$H_G(x)\chi(x) \equiv [-\partial_x^2 + 2\mathcal{A}\partial_x + \frac{d\mathcal{A}}{dx} - \mathcal{A}^2 + V]\chi(x), \quad (10)$$

where  $\mathcal{A}(x)$  is some gauge function.

Comparison between Eqs. (7) and (10) reveal

$$\mathcal{A}(x) = \left[ \frac{B'_4 - 2B_3}{4\sqrt{B_4}} \right]_{\xi=\xi(x)}, \quad V(x) = \left[ \mathcal{A}^2 - \frac{d\mathcal{A}}{dx} - B_2 \right]_{\xi=\xi(x)}. \quad (11)$$

The Schrödinger potential  $V(x)$  can be written in terms of  $B_i(\xi)$  as follows :

$$V(x) = \left[ \frac{(B'_4 - 2B_3)(3B'_4 - 2B_3)}{16B_4} - \frac{1}{4}(B''_4 - 2B'_3 + 4B_2) \right]_{\xi=\xi(x)}. \quad (12)$$

We wish to remark that the family of potentials (12) is reducible [9] to a class of elliptic potentials for a general choice of  $B_4(\xi)$  and the construction of Hamiltonian (3) ensures that these potentials form a quasi-exactly solvable system. Indeed, for the choice

$$B_4(\xi) = (1 + \xi^2)(1 + k'^2 \xi^2) \quad (13)$$

Eq. (6) can be inverted as

$$\xi = sn x / cn x, \quad (14)$$

where  $sn x \equiv sn(x, k)$ ,  $cn x \equiv cn(x, k)$ ,  $dn x \equiv dn(x, k)$  are three Jacobian elliptic functions of real modulus  $k(0 < k^2 < 1)$  and  $k'^2 = 1 - k^2$  is the complementary modulus.

Given  $B_4(\xi)$  as above, the form of (12) becomes

$$V(x) = P sn^2 x + Q sn x cn x + R \frac{sn x cn x}{dn^2 x} + S \frac{cn^2 x}{dn^2 x}, \quad (15)$$

where

$$P = \frac{k^2}{4}n(n+2) - \frac{C_0}{2}(n+1) + \frac{1}{4k^2}[C_0^2 - (C_+ - C_-)^2],$$

$$\begin{aligned}
Q &= \frac{1}{2k^2}(C_+ - C_-)[k^2(n+1) - C_0], \\
R &= \frac{1}{2k^2}(C_+ - k'^2 C_-)[k^2(n+1) + C_0], \\
S &= \frac{k^2}{4}n(n+2) + \frac{C_0}{2}(n+1) + \frac{1}{4k^2}[C_0^2 - \frac{1}{k'^2}(C_+ - k'^2 C_-)^2],
\end{aligned} \tag{16}$$

and  $d$  is chosen as

$$d = \frac{1}{4k^2}[C_-^2 - (C_0^2 + 2C_+C_-) + (\frac{C_+}{k'})^2] - \frac{n(n+2)}{2}. \tag{17}$$

We now turn to the associated Lamé equation

$$-\psi''(x) + [m(m+1)k^2 sn^2 x + \ell(\ell+1)k^2 \frac{cn^2 x}{dn^2 x}] \psi(x) = E\psi(x), \tag{18}$$

where  $\ell, m \in \mathbb{R}$  and  $m \geq \ell$  without loss of generality. Equation(18) reduces to the ordinary Lamé equation when either of  $\ell$  and  $m$  takes a value 0 or -1. Comparing the Schrödinger equation (8) having the potential(15) with (18) shows the following correspondence:

$$P = k^2 m(m+1), \quad Q = R = 0, \quad S = k^2 \ell(\ell+1). \tag{19}$$

Four nontrivial solutions emerge which are summarized below

$$n = m + \ell : \quad C_+ = C_- = 0, \quad C_0 = k^2(\ell - m), \tag{20}$$

$$n = m - \ell - 1 : \quad C_+ = C_- = 0, \quad C_0 = -k^2(\ell + m + 1), \tag{21}$$

$$n = m - \frac{1}{2} : \quad C_+ = C_- = ik'(2\ell + 1), \quad C_0 = -k^2(m + \frac{1}{2}), \tag{22}$$

$$n = m - \frac{1}{2} : \quad C_+ = C_- = -ik'(2\ell + 1), \quad C_0 = -k^2(m + \frac{1}{2}). \tag{23}$$

The Schrödinger Hamiltonian  $H_G(x)$  can be expressed from(8) and (9) as

$$H(x) = \left[ \mu(\xi) H_G \frac{1}{\mu(\xi)} \right]_{\xi=\xi(x)}, \tag{24}$$

where the gauge factor  $\mu(\xi(x)) = \exp[-\int \mathcal{A}(x) dx]$  and the gauge Hamiltonian  $H_G$  can be easily computed from Eqs (3) and (11) corresponding to each of the four algebraizations(20)-(23).

The band edge wave functions and energy spectra of the associated Lamé potential may now be explicitly formulated using the techniques of Bender and Dunne[10]. We do not give the details of our calculations which will be communicated elsewhere[11]. Here we briefly outline our results. In the following two particular cases are considered.

### Case 1. $m$ and $\ell$ are both non-negative integer

Here  $m$  takes values  $0, 1, 2, \dots$  and for each value of  $m, \ell$  is restricted to take  $(m+1)$  values  $0, 1, \dots, m$ . The lowest value of  $\ell$  gives the ordinary Lamé potential. This case corresponds to the algebraizations(20) and(21). First algebraization gives  $m + \ell + 1$  eigenstates while the second algebraization gives  $m - \ell$  eigenstates. Hence when  $m$  and  $\ell$  are both non-negative integers ( $m \geq \ell$ ), there are  $m$  bound bands followed by a continuum band and associated Lamé Hamiltonian possesses  $(2m+1)$  band edge eigenstates

at the lower and upper edges of each band. The latter generates two distinct families of orthogonal polynomials for the energy variable  $E$ . In the following examples  $\phi_r(x)$  and  $e_r$  denote ordered levels of eigenstates and energy spectra.

**Examples.**

(a)  $\mathbf{m} = 1$

$$(i) \ell = 0 : \text{Lamé potential } V(x) = 2k^2 sn^2 x \quad (25)$$

$$\phi_0(x) = dn x, \quad \phi_1(x) = cn x, \quad \phi_2(x) = sn x$$

$$e_0 = k^2, \quad e_1 = 1, \quad e_2 = 1 + k^2.$$

$$(ii) \ell = 1 : \text{Associated Lamé potential } V(x) = 2k^2 sn^2 x + 2k^2 (cn^2 x / dn^2 x) \quad (26)$$

$$\phi_{0,1}(x) = dn x \pm (k' / dn x), \quad e_{0,1} = 2 + k^2 \mp 2k',$$

$$\phi_2(x) = (sncn x) / dn x, \quad e_2 = 4$$

(b)  $\mathbf{m} = 2$

$$(i) \ell = 0 : \text{Lamé potential } V(x) = 6k^2 sn^2 x \quad (27)$$

$$\phi_{0,4}(x) = 3dn^2 x + k^2 - 2 \pm \sqrt{k^4 - k^2 + 1}, \quad e_{0,4} = 2(1 + k^2 \mp \sqrt{k^4 - k^2 + 1}),$$

$$\phi_1(x) = cn x dn x, \quad \phi_2(x) = sn x dn x, \quad \phi_3(x) = sn x cn x,$$

$$e_1 = 1 + k^2, \quad e_2 = 1 + 4k^2, \quad e_3 = 4 + k^2.$$

$$(ii) \ell = 1 : \text{Associated Lamé potential } V(x) = 6k^2 sn^2 x + 2k^2 (cn^2 x / dn^2 x) \quad (28)$$

$$\phi_0(x) = dn^2 x, \quad e_0 = 4k^2,$$

$$\phi_{1,3}(x) = cn x [3dn^2 x - 1 \pm \sqrt{4 - 3k^2}] / dn x, \quad e_{1,3} = 5 + k^2 \mp 2\sqrt{4 - 3k^2},$$

$$\phi_{2,4}(x) = sn x [3dn^2 x - k'^2 \pm \sqrt{k^4 - 5k^2 + 4}] / dn x, \quad e_{2,4} = 5 + 2k^2 \mp 2\sqrt{k^4 - 5k^2 + 4}.$$

$$(iii) \ell = 2 : \text{Associated Lamé potential } V(x) = 6k^2 sn^2 x + 6k^2 (cn^2 x / dn^2 x) \quad (29)$$

$$\phi_{0,4}(x) = [1 - \eta_{\mp}(k) sn^2 x + \{\eta_{\mp}(k) - k^2\} sn^4 x] / dn^2 x, \quad e_{0,4} = 2\eta_{\mp}(k) + 4k^2,$$

$$\phi_1(x) = [1 - 2sn^2 x + k^2 sn^4 x] / dn^2 x, \quad e_1 = 4(1 + k^2),$$

$$\phi_{2,3}(x) = sn x cn x [1 + (\pm k' - 1) sn^2 x] / dn^2 x, \quad e_{2,3} = 10 + k^2 \mp 6k',$$

$$\text{where} \quad \eta_{\pm}(k) = 4 - k^2 \pm \sqrt{k^4 - 16k^2 + 16}.$$

Similarly, higher integer values of  $m$  can be dealt with. Let us point out that the Lamé potential corresponding to  $\ell = 0$  in the above examples was obtained algebraically in Ref 8. However, to the best of our knowledge, the algebraic approach for the remaining solutions are new.

**Case 2.  $m$  is half an odd positive integer and  $\ell$  is either a non-negative integer or half an odd positive integer**

Here  $m$  takes values  $(1/2), (3/2), \dots$  and for each values of  $m, \ell$  is restricted to take  $(2m + 1)$  values  $0, (1/2), 1, (3/2), \dots, m$ . The lowest value of  $\ell$  gives the ordinary Lamé potential. Each of the two algebraizations(22) and (23) gives  $[m + (1/2)]$  eigenstates, conjugate to each other, generating the same family of orthogonal polynomials for the energy variable  $E$ . This implies that energies are doubly degenerate and  $2m + 1$  eigenstates are given by real and imaginary parts. Thus there are  $[m + (1/2)]$  characteristic values of  $E$  for each of which we obtain two linearly independent solutions. In the following examples the parenthesized superscript in the eigenstates indicates the degeneracy of the eigenvalue.

**Example.**

(a)  $m = \frac{1}{2}$

$$(i) \ell = 0 : \text{Lamé potential } V(x) = \frac{3}{4}k^2 sn^2 x \quad (30)$$

$$\phi_0^{(1)}(x) = \sqrt{dn x + cn x}, \quad \phi_0^{(2)} = sgn(sn x) \sqrt{dn x - cn x}, \quad e_0 = (1 + k^2)/4.$$

$$(ii) \ell = \frac{1}{2} : \text{Associated Lamé potential } V(x) = \frac{3}{4}k^2 sn^2 x + \frac{3}{4}k^2 (cn^2 x / dn^2 x) \quad (31)$$

$$\phi_0^{(1)}(x) = cn x / \sqrt{dn x}, \quad \phi_0^{(2)}(x) = sn x / \sqrt{dn x}, \quad e_0 = 1 + (k^2/4).$$

(b)  $m = \frac{3}{2}$

$$(i) \ell = 0 : \text{Lamé Potential } V(x) = \frac{15}{4}k^2 sn^2 x \quad (32)$$

$$e_{0,1} = 5(1 + k^2)/4 \mp \sqrt{k^4 - k^2 + 1},$$

$$\phi_{0,1}^{(1)}(x) = \sqrt{(dn x + cn x)[k^2 cn x + \alpha_{\pm}(k) dn x]},$$

$$\phi_{0,1}^{(2)}(x) = sgn(sn x) \sqrt{(dn x - cn x)[k^2 cn x - \alpha_{\pm}(k) dn x]},$$

$$\text{where } \alpha_{\pm}(k) = k'^2 \pm \sqrt{k^4 - k^2 + 1}.$$

The above  $\ell = 0$  cases namely, those for the potentials(30) and (32), have already been reported in Ref 8.

$$(ii) \ell = 1/2 : \text{Associated Lamé potential } V(x) = \frac{15}{4}k^2 sn^2 x + \frac{3}{4}k^2 (cn^2 x / dn^2 x) \quad (33)$$

$$\phi_0(x) = dn^{3/2} x, \quad e_0 = 9k^2/4,$$

$$\phi_1^{(1)}(x) = [2sn^2x - 1]/\sqrt{dnx}, \quad \phi_1^{(2)}(x) = [snxcn x]/\sqrt{dnx}, \quad e_1 = 4 + (k^2/4).$$

$$(iii)\ell = 1 : \text{Associated Lamé potential } V(x) = \frac{15}{4}k^2sn^2x + 2k^2(cn^2x/dn^2x) \quad (34)$$

$$e_{0,1} = \frac{3}{2}k^2 + \frac{13}{4} \mp \sqrt{k^4 + 9k'^2},$$

$$\phi_{0,1}^{(1)}(x) = \sqrt{(dnx + cnx)[\beta_{\mp}(k)dn^2x + \gamma_{\mp}(k)cnxdnx + \delta_{\mp}(k)]/dnx},$$

$$\phi_{0,1}^{(2)}(x) = \sqrt{(dnx - cnx)[\beta_{\mp}(k)dn^2x - \gamma_{\mp}(k)cnxdnx + \delta_{\mp}(k)]/dnx},$$

$$\text{where} \quad \beta_{\pm}(k) = -7k^4 + 72K^2 - 96 \pm 8(k^2 - 4)\sqrt{k^4 + 9k'^2},$$

$$\gamma_{\pm}(k) = 8k^2[-3k^2 + 6 \pm 2\sqrt{k^4 + 9k'^2}],$$

$$\delta_{\pm}(k) = 48k^4 - 144k^2 + 96 \pm 32k'^2\sqrt{k^4 + 9k'^2}.$$

Thus from our algebraic approach we can find a new QES periodic potential(34) whose solutions can be written analytically.

$$(iv)\ell = 3/2 : \text{Associated Lamé potential } V(x) = \frac{15}{4}k^2sn^2x + \frac{15}{4}k^2(cn^2x/dn^2x) \quad (35)$$

$$e_{0,1} = 5 + \frac{5}{4}k^2 \mp \sqrt{k^4 - 16k^2 + 16},$$

$$\phi_{0,1}^{(1)}(x) = cnx[\phi_{\pm}(k)sn^2x + 2k^2 + \epsilon_{\pm}(k)]/(dnx)^{3/2},$$

$$\phi_{0,1}^{(2)}(x) = snx[\rho_{\pm}(k)sn^2x + 3\epsilon_{\pm}(k)]/(dnx)^{3/2},$$

$$\text{where} \quad \phi_{\pm}(k) = 3k^4 - 20k^2 + 16 \pm (3k^2 - 4)\sqrt{k^4 - 16k^2 + 16},$$

$$\rho_{\pm}(k) = 2k^4 - 12k^2 + 16 \pm (k^2 - 4)\sqrt{k^4 - 16k^2 + 16},$$

$$\epsilon_{\pm}(k) = 3k^2 - 4 \pm \sqrt{k^4 - 16k^2 + 16}.$$

Proceed in this way we can determine the potentials corresponding to  $m = (5/2), (7/2), \dots$ . We may mention that the results of the cases  $\ell = 1/2, 1, 3/2, \dots, m$  given above using our algebraic approach are new.

To conclude, we have presented in this work new algebraizations of the associated Lamé equation. Within  $sl(2, \mathbb{R})$  these have enabled us to determine a general class of periodic potentials which include the ones derived recently [8].

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