

An introduction to coding sequences of graphs

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Abstract. In his pioneering paper on matroids in 1935, Whitney obtained a characterization for binary matroids and left a comment at end of the paper that the problem of characterizing graphic matroids is the same as that of characterizing matroids which correspond to matrices (mod 2) with exactly two ones in each column. Later on Tutte obtained a characterization of graphic matroids in terms of forbidden minors in 1959. But it is clear that Whitney indicated about incidence matrices of simple undirected graphs.

In this paper, we introduce the concept of a segment binary matroid which corresponds to matrices over \mathbb{Z}_2 which has the consecutive 1's property (i.e., 1's are consecutive) for columns and obtained a characterization of graphic matroids in terms of this.

In fact, we introduce a new representation of simple undirected graphs in terms of some vectors of finite dimensional vector spaces over \mathbb{Z}_2 which satisfy consecutive 1's property. The set of such vectors is called a coding sequence of a graph G . Among all such coding sequences we identify the one which is unique for a class of isomorphic graphs. We call it the code of the graph. We characterize several classes of graphs in terms of coding sequences. It is shown that a graph G with n vertices is a tree if and only if any coding sequence of G is a basis of the vector space \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 .

Moreover considering coding sequences as binary matroids, we obtain a characterization for simple graphic matroids and found a necessary and sufficient condition for graph isomorphism in terms of a special matroid isomorphism between their corresponding coding sequences. For this, we introduce the concept of strong isomorphisms of segment binary matroids and show that two simple (undirected) graphs are isomorphic if and only if their canonical sequences are strongly isomorphic segment binary matroids.

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1 Introduction

There are various representations of simple undirected graphs in terms of adjacency matrices, adjacency lists, incidence matrix, unordered pairs etc. In this paper, we introduce another representation of a simple undirected graph with n vertices in terms of certain vectors in the vector space \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 . We call the set of vectors representing a graph G as a coding sequence of G and denote it by $\beta(G, n)$. Among all such coding sequences we identify the one which is unique for a class of isomorphic graphs. We call it the code of the graph. We find characterizations of graphs which are connected, acyclic, bipartite, Eulerian or Hamiltonian in terms of $\beta(G, n)$. We prove that a graph G with n vertices is a tree if and only if any coding sequence of G is a basis of the vector space \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 .

In his pioneering paper [7] on matroids in 1935, Whitney left the problem of characterizing graphic matroid open by making the following comment: “The problem of characterizing linear graphs from this point of view is the same as that of characterizing matroids which correspond to matrices (mod 2) with exactly two ones in each column.” In 1959, Tutte obtained a characterization of graphic matroids in terms of forbidden minors [5]. But it is clear that Whitney indicated about incidence matrices of simple undirected graphs. In this paper, use a variation of incidence matrix for the same characterization.

In section 3, we introduce the concept of a segment binary matroid which corresponds to matrices over \mathbb{Z}_2 that has the consecutive 1’s property (i.e., 1’s are consecutive) for columns and a characterization of graphic matroids is obtained by considering $\beta(G, n)$ as a segment binary matroid. We introduce the concept of a strong isomorphism for segment binary matroids and show that two simple graphs G and H (with n vertices each) are isomorphic if and only if $\beta(G, n)$ and $\beta(H, n)$ are strongly isomorphic segment binary matroids.

For graph theoretic concepts, see [6] and for matroid related terminologies, one may consult [4].

2 Coding sequences

Definition 1. *Let $G = (V, E)$ be a simple undirected graph with n vertices and m edges. Let $V = \{v_0, v_1, \dots, v_{n-1}\}$. Define a map $f : V \rightarrow \mathbb{N}$ by $f(v_i) = 10^i$ and another map $f^* : E \rightarrow \mathbb{N}$ by $f^*(v_i v_j) = |f(v_i) - f(v_j)|$, if $E \neq \emptyset$. Let $\sigma(G, n)$ be the sequence $\{f^*(e) \mid e \in E\}$ sorted in the increasing order. If $E = \emptyset$, then $\sigma(G, n) = \emptyset$. It is worth noticing that for $e = v_i v_j \in E$, $f^*(e) = |10^i - 10^j|$ uniquely determines the pair (i, j) as it is a natural number with i digits, starting with $i - j$ number of 9’s and followed by j number of 0’s, when $i > j$. Thus m entries of $\sigma(G, n)$ are all distinct.*

Now for $E \neq \emptyset$, we define a map $f^\# : E \rightarrow \mathbb{Z}_2^{n-1}$ by $f^\#(e) = (x_1, x_2, \dots, x_{n-1})$, where $x_i = 1$, if $(n-i)^{\text{th}}$ digit of $f^*(e)$ from the right is 9, otherwise $x_i = 0$ for $i = 1, 2, \dots, n-1$. For convenience we write the field \mathbb{Z}_2 as $\{0, 1\}$ instead of $\{\bar{0}, \bar{1}\}$. Let $\beta(G, n)$ be the sequence $\{f^\#(e) \mid e \in E\}$ sorted in the same order as in $\sigma(G, n)$. If $E = \emptyset$, then $\beta(G, n) = \emptyset$. The sequence $\beta(G, n)$ is called a **coding sequence** of the graph G .

Naturally, $\beta(G, n)$ is not unique for a graph G as it depends on the labeling f of vertices. Now there are $n!$ such labellings and consequently we have at most $n!$ different $\sigma(G, n)$ for a graph G . Among which we choose the one, say, $\sigma_c(G, n)$ which is the minimum in the lexicographic ordering of \mathbb{N}^m . The corresponding $\beta(G, n)$ is called the **code** of the graph G and is denoted by $\beta_c(G, n)$. Clearly $\beta_c(G, n)$ is unique for a class of isomorphic graphs with given number of vertices, though it is not always easy to determine generally.

Example 1. Consider the graph G in Figure 1 (left). We have

$$\sigma(G, 4) = (9, 90, 900, 990) \quad \text{and} \quad \beta(G, 4) = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 0)\}$$

according to the labeling of vertices given in Figure 1 (left). One may verify that

$$\sigma_c(G, 4) = (9, 90, 99, 900) \quad \text{and} \quad \beta_c(G, 4) = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0)\}$$

according to the labeling of vertices shown in Figure 1 (right).

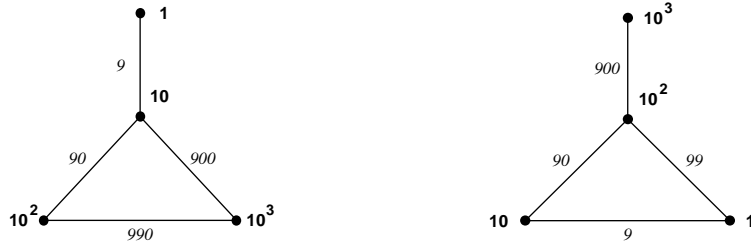


Fig. 1. The graph G in Example 1 with different labellings.

Remark 1. An incidence matrix of a (simple undirected) graph $G = (V, E)$ is obtained by placing its vertices in rows and edges in columns and an entry in a row corresponding to a vertex v and in a column corresponding to an edge e of the matrix is 1 if and only if v is an end point of e , otherwise it is 0. It is important to note that a coding sequence of a graph has a similarity with the incidence matrix of the graph. In fact, given a coding sequence of a graph G ,

one can easily obtain the incidence matrix of G and vice-versa. Also cut-set and circuit subspaces of a vector space of dimension $|E|$ over \mathbb{Z}_2 constructed from edges of G are well known [1–3]. Further, as we mentioned in the introduction, Whitney expected the characterization of graphic matroids would be obtained from the incidence matrix. Here we consider a variation of it with a consecutive 1's representation as it helps us to build a very natural interplay between graph theory, matroids and linear algebra which is evident from Theorems 1 and 4.

Throughout this section by a graph we mean a simple undirected graph.

Definition 2. A non-null vector $e = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{Z}_2^{n-1}$ is said to satisfy the **consecutive 1's property** if 1's appear consecutively in the sequence of coordinates of e . Let

$$C(n-1) = \{v \in \mathbb{Z}_2^{n-1} \mid v \text{ satisfies the consecutive 1's property}\}.$$

Clearly, $|C(n-1)| = \binom{n}{2} = \frac{n(n-1)}{2}$ and G is a complete graph with n vertices if and only if $\beta(G, n) = C(n-1)$. In fact, for every $S \subseteq C(n-1)$, there is a unique graph $G(S)$ of n vertices such that $\beta(G, n) = S$. If $S = \emptyset$, then G is the null graph with n vertices and no edges. If $S \neq \emptyset$, each member e of S represents an edge of $G(S) = (V, E)$ with end points 10^{n-i} and 10^{n-j-1} , where the consecutive stretch of 1's in e starts from the i^{th} entry and ends at the j^{th} entry from the left and $V = \{1, 10, 10^2, \dots, 10^{n-1}\}$. Also it is clear that $C(n-1) \setminus \beta(G, n)$ is a coding sequence of the complement \tilde{G} of a graph G with n vertices.

Let $\emptyset \neq S \subseteq C(n-1)$. Let $\tilde{G}(S)$ be the subgraph of $G(S)$ obtained by removing isolated vertices (if any) from $G(S)$. Then $\tilde{G}(S)$ is the subgraph of the complete graph of n vertices induced by the edges represented by the vectors in S .

We denote the null vector in the vector space \mathbb{Z}_2^{n-1} by $\mathbf{0}$ for any $n \in \mathbb{N}$ and write \mathbb{Z}_2^0 for the zero-dimensional space $\{\mathbf{0}\}$. Let $S = \{e_1, e_2, \dots, e_k\} \subseteq \mathbb{Z}_2^{n-1} \setminus \{\mathbf{0}\}$, ($k \in \mathbb{N}$, $k \leq 2^{n-1}$). As e_i 's are distinct, we have $e_i + e_j \neq \mathbf{0}$ for all $i \neq j$, $i, j \in \{1, 2, \dots, k\}$. Thus S is a set of linearly dependent vectors in \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 if and only if there exists $A \subseteq S$, $|A| \geq 3$ such that $\sum_{e \in A} e = \mathbf{0}$. In other

words, $S \subseteq \mathbb{Z}_2^{n-1}$ is linearly independent over \mathbb{Z}_2 if and only if $S = \emptyset$ or $S = \{e_1, e_2, \dots, e_k\}$ for some $k \in \mathbb{N}$, $k \leq 2^{n-1}$ and $\sum_{e \in A} e \neq \mathbf{0}$ for all $\emptyset \neq A \subseteq S$. We

denote the linear span (over \mathbb{Z}_2) of a subset S of \mathbb{Z}_2^{n-1} by $\text{Sp}(S)$, i.e., $\text{Sp}(S)$ is the smallest subspace of \mathbb{Z}_2^{n-1} containing S .

Proposition 1. Let $S = \{e_1, e_2, e_3\} \subseteq C(n-1)$ for some $n \in \mathbb{N}$, $n \geq 3$. Then $\tilde{G}(S)$ is a 3-cycle if and only if $e_1 + e_2 + e_3 = \mathbf{0}$.

Proof. First suppose that $\tilde{G}(S)$ is the 3-cycle shown in Figure 2, where $\alpha, \beta, \gamma \in \{0, 1, \dots, n-1\}$. Without loss of generality we assume $\alpha > \beta > \gamma$. Then

$$\begin{aligned} e_1 &= (\underbrace{0, 0, \dots, 0}_{n-\alpha-1}, \underbrace{1, 1, \dots, 1}_{\alpha-\beta}, \underbrace{0, 0, \dots, 0}_{\beta}, 0, 0, \dots, 0) \\ e_2 &= (\underbrace{0, 0, \dots, 0}_{n-\alpha-1}, \underbrace{1, 1, \dots, 1}_{\alpha-\gamma}, \underbrace{1, 1, \dots, 1}_{\gamma}, 0, 0, \dots, 0) \\ e_3 &= (\underbrace{0, 0, \dots, 0}_{n-\beta-1}, \underbrace{0, 0, \dots, 0}_{\beta-\gamma}, \underbrace{1, 1, \dots, 1}_{\gamma}, 0, 0, \dots, 0) \end{aligned}$$

Clearly $e_1 + e_2 + e_3 = \mathbf{0}$.

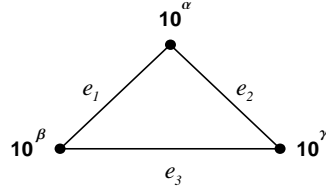


Fig. 2. A 3-cycle

Conversely, let $e_1 + e_2 + e_3 = \mathbf{0}$. Consider the matrix $M = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$, where we represent each e_i as a row matrix consisting of $n-1$ columns for $i = 1, 2, 3$. Since $e_1 + e_2 + e_3 = \mathbf{0}$, in each column where 1's appear, they appear exactly in two rows. Let i be the least column number of M that contains 1. Without loss of generality we assume that 1's appear in the first two rows in the i^{th} column (otherwise we rearrange rows of M). Also suppose that the number of zeros after the stretch of 1's in the first row, say, β is more than that of the second, say, γ (otherwise again we rearrange rows of M). Let $\alpha = n - i$. Then the end points of the edge of $\tilde{G}(S)$ corresponding to e_1 are 10^α and 10^β and those of the edge corresponding to e_2 are 10^α and 10^γ . Since $\beta > \gamma$ and $e_1 + e_2 + e_3 = \mathbf{0}$, we have

$$e_3 = e_1 + e_2 = (\underbrace{0, 0, \dots, 0}_{n-\beta-1}, \underbrace{1, 1, \dots, 1}_{\beta-\gamma}, \underbrace{0, 0, \dots, 0}_{\gamma}).$$

Thus the end points of the edge of $\tilde{G}(S)$ corresponding to e_3 are 10^β and 10^γ . So the vertices labeled by $10^\alpha, 10^\beta$ and 10^γ form a 3-cycle with edges corresponding to e_1, e_2, e_3 , as required.

Definition 3. A set $S \neq \emptyset$ of non-null vectors in \mathbb{Z}_2^{n-1} is called **reduced** if $\sum_{e \in A} e \neq \mathbf{0}$ for all $\emptyset \neq A \subsetneq S$.

Lemma 1. Let $S = \{e_1, e_2, \dots, e_k\} \subseteq C(n-1)$ for some $k, n \in \mathbb{N}$, $3 \leq k \leq n$. Then $\tilde{G}(S)$ is a k -cycle if and only if S is reduced and $e_1 + e_2 + \dots + e_k = \mathbf{0}$.

Proof. We prove by induction on k . By Proposition 1, the result is true for $k = 3$. Suppose the result is true for $k = r - 1 \geq 3$. Let $S = \{e_1, e_2, \dots, e_r\} \subseteq C(n-1)$ for some $r, n \in \mathbb{N}$, $3 < r \leq n$ form the r -cycle shown in Figure 3 (we renumber e_i 's, if necessary). Consider the chord e so that e_1, e_2 and e form a triangle. Then $e_1 + e_2 + e = \mathbf{0}$ by Proposition 1. So $e = e_1 + e_2$. Also $\{e, e_3, e_4, \dots, e_r\}$ form an $(r-1)$ -cycle. So by induction hypothesis $e + e_3 + e_4 + \dots + e_r = \mathbf{0}$ which implies $e_1 + e_2 + e_3 + e_4 + \dots + e_r = \mathbf{0}$. Moreover since S is a cycle, no proper subset of S form a cycle. Thus S is reduced by induction hypothesis.

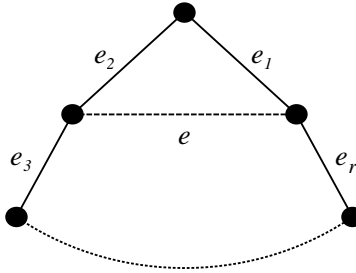


Fig. 3. An r -cycle

Conversely, let $S = \{e_1, e_2, \dots, e_r\} \subseteq C(n-1)$ for some $r, n \in \mathbb{N}$, $3 < r \leq n$ be

reduced and $e_1 + e_2 + \dots + e_r = \mathbf{0}$. Consider the matrix $M = \begin{pmatrix} e_1 \\ e_2 \\ \dots \\ e_r \end{pmatrix}$, where we

represent each e_i as a row matrix consisting of $n-1$ columns for $i = 1, 2, \dots, r$. Let i be the least column number of M that contains 1. Since the row sum of M is zero (the null vector), the i^{th} column contains even number of 1's. So there are at least two rows with 1 in the i^{th} column. Rearrange rows of M such that e_1 and e_2 be two such rows. Since both of these rows begin with 1 in the i^{th} column, the edges corresponding to them have a common end point with label 10^{n-i} . Join the other end points of e_1 and e_2 by an edge, say, e to form a triangle with edges e_1, e_2, e . Then $e_1 + e_2 + e = \mathbf{0}$ which implies $e = e_1 + e_2$. So $e + e_3 + e_4 + \dots + e_r = \mathbf{0}$.

We claim that $S_1 = \{e, e_3, e_4, \dots, e_r\}$ is reduced. Suppose $A \subsetneq S_1$, $|A| \geq 3$ be such that $a = \sum_{x \in A} x = \mathbf{0}$. If $e \notin A$, then $a \neq \mathbf{0}$ as S is reduced. So $e \in A$. Then replacing e by $e_1 + e_2$ in a would again contradict the fact that S is reduced.

So S_1 is reduced. Hence by induction hypothesis, S_1 form an $(r - 1)$ -cycle. Now replacing the edge e by the path consisting of edges e_1 and e_2 gives us an r -cycle formed by S .

The following two corollaries follow immediately from the above lemma.

Corollary 1. *A graph G with $n \geq 3$ vertices ($n \in \mathbb{N}$) is Hamiltonian if and only if for any coding sequence $\beta(G, n)$ of G , there exists $S = \{e_1, e_2, \dots, e_n\} \subseteq \beta(G, n)$ such that S is reduced and $e_1 + e_2 + \dots + e_n = \mathbf{0}$.*

Corollary 2. *A graph G with n vertices ($n \in \mathbb{N}$) is acyclic if and only if any coding sequence $\beta(G, n)$ of G is linearly independent over \mathbb{Z}_2 .*

Corollary 3. *A graph G with at most one non-trivial component and with n vertices ($n \in \mathbb{N}$) is Eulerian if and only if $\sum_{e \in \beta(G, n)} e = \mathbf{0}$ for any coding sequence $\beta(G, n)$ of G .*

Proof. Follows from Lemma 1 and the fact that a circuit in a graph can be decomposed into edge-disjoint cycles.

Corollary 4. *A graph G with $n \geq 2$ vertices ($n \in \mathbb{N}$) is bipartite if and only if for any coding sequence $\beta(G, n)$ of G , $\sum_{e \in S} e \neq \mathbf{0}$ for every $S \subseteq \beta(G, n)$ where $|S|$ is odd.*

Proof. Follows from Lemma 1 and the fact that a graph is bipartite if and only if it does not contain any odd cycle.

Theorem 1. *A graph G with n vertices ($n \in \mathbb{N}$) is a tree if and only if any coding sequence $\beta(G, n)$ of G is a basis of the vector space \mathbb{Z}_2^{n-1} over the field \mathbb{Z}_2 .*

Proof. Suppose G is a tree. Then G is acyclic which implies $\beta(G, n)$ is linearly independent over \mathbb{Z}_2 by Corollary 2. Again since G is a tree, the number of entries in $\beta(G, n)$ is $n - 1$, we have $n - 1$ linearly independent vectors in \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 . Thus $\beta(G, n)$ is a basis of \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 .

Conversely, suppose $\beta(G, n)$ is a basis of \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 . Then $\beta(G, n)$ is linearly independent over \mathbb{Z}_2 and so G is acyclic by Corollary 2. Also since $\beta(G, n)$ is a basis of \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 , the number of entries in $\beta(G, n)$ is $n - 1$ which implies G has $n - 1$ edges. Thus G is a tree.

Corollary 5. *A graph G with n vertices ($n \in \mathbb{N}$) is connected if and only if for any coding sequence $\beta(G, n)$ of G , $\text{Sp}(\beta(G, n)) = \mathbb{Z}_2^{n-1}$.*

Proof. We first note that a graph G is connected if and only if G has a spanning tree. Suppose $G = (V, E)$ has a spanning tree $H = (V, E_1)$. Then $\beta(H, n) \subseteq \beta(G, n)$ with the same vertex labeling. But $\beta(H, n)$ is a basis of \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 by Theorem 1. Thus $\text{Sp}(\beta(G, n)) \supseteq \text{Sp}(\beta(H, n)) = \mathbb{Z}_2^{n-1}$. So $\text{Sp}(\beta(G, n)) = \mathbb{Z}_2^{n-1}$.

Conversely, if $\text{Sp}(\beta(G, n)) = \mathbb{Z}_2^{n-1}$, then $\beta(G, n)$ contains a basis, say B of \mathbb{Z}_2^{n-1} over \mathbb{Z}_2 . Then $G(B)$ is a spanning tree of G by Theorem 1 as $B = \beta(G(B), n)$. Thus G is connected.

Corollary 6. *Let G be a connected graph with n vertices and $S \subseteq \beta(G, n)$. Then $G(S)$ is a spanning tree of G if and only if S is a basis of the vector space \mathbb{Z}_2^{n-1} over the field \mathbb{Z}_2 .*

3 Matroid representation

Whitney introduced the concept of a matroid in [7]. There are several ways of defining matroids. We take the one that will serve our purpose. A *matroid* M is an ordered pair (E, \mathcal{B}) consisting of a finite set E of *elements* and a nonempty collection \mathcal{B} of subsets of E , called *bases* which satisfies the properties: (i) no proper subset of a base is a base and (ii) if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, then there exists $f \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{f\} \in \mathcal{B}$. *Independent sets* of M are subsets of bases and minimal dependent sets are *circuits*. The *cycle matroid* $M[G]$ of graph G is the matroid whose elements are edges of G and circuits are cycles of G . Independent sets and bases of $M[G]$ are forests and maximal forests of G respectively. A matroid is *graphic* (*simple graphic*) if it is a cycle matroid of a graph (respectively, simple graph).

Let E be the set of column labels of an $n \times m$ matrix A over a field F , and \mathcal{B} be the set of maximal subsets X of E for which the multiset of columns labeled by X is linearly independent in the vector space F^m over F . Then the pair (E, \mathcal{B}) is the *column (vector) matroid* of A and is denoted by $M[A]$. In particular, if $F = \mathbb{Z}_2$, then $M[A]$ is a *binary matroid*. A binary matroid $M[A]$ is *simple* if A does not contain zero columns and no two columns of A are identical (i.e., columns of A are non-zero and distinct).

Definition 4. *A binary matroid $M[A]$ is called a **segment binary matroid** if A satisfies the consecutive 1's property for columns. Moreover, if it is simple, then we call it a **simple segment binary matroid**. For any $\emptyset \neq S \subseteq \mathbb{Z}_2^{n-1}$, $M[S]$ denotes the column (vector) matroid of the matrix whose columns are precisely the elements of S . Clearly, $M[S]$ is a binary matroid.*

Remark 2. In particular, when $\emptyset \neq S \subseteq C(n-1)$, $M[S]$ becomes a simple segment binary matroid. So for any simple graph G with n vertices, $M[\beta(G, n)]$ is a simple segment binary matroid. Conversely, every simple segment binary matroid $M[A]$ with $n-1$ rows is same as $M[S]$, where S is the set of column vectors of A over \mathbb{Z}_2 . Also in this case $S \subseteq C(n-1)$.

Two matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ are *isomorphic* if there is a bijection ψ from E_1 onto E_2 such that for all $X \subseteq E_1$, X is independent in M_1 if and only if $\psi(X)$ is independent in M_2 (or, equivalently, X is a circuit in M_1 if and only if $\psi(X)$ is a circuit in M_2). In this case, we denote by $M_1 \cong M_2$. Also

abusing notations we sometimes identify elements of $M[A]$ with its corresponding column vector representation. Thus a simple binary matroid $M[A]$ may be considered as the set of column vectors of A . The following theorem characterizes isomorphisms of simple binary matroids in terms of linear transformations.

Theorem 2. *Let $M[A]$ and $M[A_1]$ be two simple binary matroids such that both A and A_1 are of same order $n \times m$, ($m, n \in \mathbb{N}$). Then $M[A] \cong M[A_1]$ if and only if there exists a bijective linear operator T on \mathbb{Z}_2^n such that T restricted on $M[A]$ is a bijective map from $M[A]$ onto $M[A_1]$.*

Proof. Let ψ be an isomorphism from $M[A]$ onto $M[A_1]$. Let B be a base in $M[A]$. Then B is linearly independent over \mathbb{Z}_2 . We extend B to a basis B_1 (say) of \mathbb{Z}_2^n over \mathbb{Z}_2 . Now since ψ is an isomorphism, $\psi(B)$ is also a base in $M[A_1]$ and $|\psi(B)| = |B|$. We also extend $\psi(B)$ to B_2 , a basis of \mathbb{Z}_2^n over \mathbb{Z}_2 . Then $|B_1 \setminus B| = |B_2 \setminus \psi(B)| = n - |B|$. Let f be a bijection from $B_1 \setminus B$ onto $B_2 \setminus \psi(B)$. Now define a map $T_1 : B_1 \rightarrow B_2$ by

$$T_1(e) = \begin{cases} \psi(e), & e \in B \\ f(e), & e \in B_1 \setminus B \end{cases}$$

We next verify that ψ is ‘linear’ on $M[A]$, i.e., if e_1, e_2 are columns of A such that $e_1 + e_2$ is also a column of A , then $\psi(e_1 + e_2) = \psi(e_1) + \psi(e_2)$. Let $e = e_1 + e_2$. Then $e + e_1 + e_2 = \mathbf{0}$ which implies that $\{e, e_1, e_2\}$ is a circuit of $M[A]$. Again since ψ is an isomorphism, $\{\psi(e), \psi(e_1), \psi(e_2)\}$ is also a circuit in $M[A_1]$. Hence $\psi(e) + \psi(e_1) + \psi(e_2) = \mathbf{0}$, i.e., $\psi(e_1 + e_2) = \psi(e) = \psi(e_1) + \psi(e_2)$. This completes the verification. We extend T_1 linearly to obtain a linear operator T on \mathbb{Z}_2^n over \mathbb{Z}_2 . Then T is bijective as T_1 maps a basis bijectively to another basis of \mathbb{Z}_2^n over \mathbb{Z}_2 and the restriction of T on $M[A]$ is ψ due to the above verification.

Conversely, let T be a bijective linear operator on \mathbb{Z}_2^n such that the map ψ , the restriction of T on $M[A]$ is a bijective map from $M[A]$ onto $M[A_1]$. Let X be a circuit in $M[A]$. Then $\sum_{e \in X} e = \mathbf{0}$ and $\sum_{e \in A} e \neq \mathbf{0}$ for all $\emptyset \neq A \subsetneq X$. Now since T is bijective and linear, we have $\sum_{e \in A} e = \mathbf{0}$ if and only if $\sum_{e \in A} T(e) = \mathbf{0}$ for all $\emptyset \neq A \subseteq X$. Thus X is a circuit in $M[A]$ if and only if $\psi(X)$ is a circuit in $M[A_1]$. Hence ψ is an isomorphism from $M[A]$ onto $M[A_1]$.

Corollary 7. *Let $M[A]$ and $M[A_1]$ be two simple binary matroids such that both A and A_1 are of same order $n \times m$, ($m, n \in \mathbb{N}$). Then $M[A] \cong M[A_1]$ if and only if there exist a non-singular matrix P of order $n \times n$ and a permutation matrix Q of order $m \times m$ such that $PAQ = A_1$.*

Proof. If $M[A] \cong M[A_1]$, then following the proof of the direct part of the above theorem, consider two bases B_1 and B_2 of \mathbb{Z}_2^n over \mathbb{Z}_2 and the bijective linear operator T that maps B_1 onto B_2 . Let P be the matrix representation of T with respect to these bases. Then P is a non-singular matrix and $PA = A_2$ where A_2 is obtained from A_1 by rearranging columns such that i^{th} column of A_2 is the

image of the i^{th} column of A under T . Thus $PAQ = A_1$ for some permutation matrix Q .

Conversely, let $A_1 = PAQ$ for some non-singular matrix P and some permutation matrix Q . Let $A_2 = A_1Q^{-1}$. Then $PA = A_2$. Since P is non-singular, it corresponds to a bijective linear operator T on \mathbb{Z}_2^n (over \mathbb{Z}_2) defined by $T(e) = Pe$ (considering elements of \mathbb{Z}_2^n as column matrices) such that the restriction of T on $M[A]$ is a bijective map from $M[A]$ onto $M[A_2]$. Then $M[A] \cong M[A_2]$ by the above theorem. Since $M[A_1] = M[A_2]$, we have $M[A] \cong M[A_1]$.

Now we proceed to characterize simple graphic matroids.

Lemma 2. *Let G be a simple graph with n vertices. Then $M[G] \cong M[\beta(G, n)]$ for any coding sequence $\beta(G, n)$ of G .*

Proof. It follows from Lemma 1 that cycles of G are precisely the circuits of the matroid $M[\beta(G, n)]$. So $M[G] \cong M[\beta(G, n)]$ as matroids.

Theorem 3. *A matroid is simple graphic if and only if it is isomorphic to a simple segment binary matroid.*

Proof. Let M be a simple graphic matroid. Then $M \cong M[G]$ for a simple graph G . By Lemma 2, we have $M[G] \cong M[\beta(G, n)]$ where n is the number of vertices of G . Thus M is isomorphic to a simple segment binary matroid by Remark 2.

Conversely, let $M[A]$ be a simple segment binary matroid. Then we may consider $M[A]$ as $M[S]$ where S is the set of columns of A . By Remark 2, we have $S \subseteq C(n-1)$, where $A \in M_{n-1, m}(\mathbb{Z}_2)$. Then by Definition 2, there is a unique simple graph G such that $S = \beta(G, n)$. Therefore, by Lemma 2, $M[S] = M[\beta(G, n)] \cong M[G]$. Thus, $M[A]$ is a simple graphic matroid.

Corollary 8. *A simple binary matroid $M[A]$ (where A is of order $(n-1) \times m$) is simple graphic if and only if $m \leq \binom{n}{2}$ and there exists a non-singular matrix P such that PA satisfies the consecutive 1's property for columns.*

Proof. Follows from Theorem 3 and Corollary 7.

Remark 3. Since any non-singular matrix is obtained from identity matrix by finite number of elementary row operations, a simple binary matroid $M[A]$ is simple graphic if and only if the consecutive 1's property for columns can be obtained from A by finite number of elementary row operations.

It is well known [4] that an ordinary matroid isomorphism does not guarantee the corresponding graph isomorphism for graphic matroids. We now introduce the concept of a strong isomorphism of simple segment binary matroids.

Definition 5. *Two simple segment binary matroids $M[A_1]$ and $M[A_2]$ are called strongly isomorphic if*

- (1) $A_1, A_2 \in M_{n-1, m}(\mathbb{Z}_2)$ for some $m, n \in \mathbb{N}$, $n \geq 2$.

- (2) There exists a bijective linear operator T on \mathbb{Z}_2^{n-1} such that:
- (i) T restricted on $C(n-1)$ is a bijection onto itself.
 - (ii) T restricted on $M[A_1]$ is a bijective map from $M[A_1]$ onto $M[A_2]$.

We write $M[A_1] \cong_s M[A_2]$ to denote that $M[A_1]$ is strongly isomorphic to $M[A_2]$.

Note that, if $M[A_1] \cong_s M[A_2]$, then the restriction of T on $M[A_1]$ is a matroid isomorphism onto $M[A_2]$ and the restriction of T on $C(n-1)$ is a matroid automorphism. These follow from the fact that T is linear and injective, as then for any subset X of the set of columns of A_1 , $\sum_{e \in X} e = \mathbf{0}$ if and only if $\sum_{e \in X} T(e) = \mathbf{0}$. In the sequel, we show that strong isomorphism of simple segment binary matroids would guarantee the corresponding graph isomorphism.

Let $G = (V, E)$ be a (simple undirected) graph with $|V| = n$. Then for any $e \in E$, we use the symbol $p \sim_n |10^x - 10^y|$ if $f^*(e) = |10^x - 10^y|$ and $p = f^\#(e) \in \beta(G, n)$.

Lemma 3. *Let p_1, p_2 be distinct elements in $\beta(G, n)$ for any coding sequence $\beta(G, n)$ of a graph G with n vertices. If $p_1 \sim_n |10^x - 10^y|$ and $p_2 \sim_n |10^x - 10^z|$, then $p_1 + p_2 \sim_n |10^y - 10^z|$.*

Proof. Let $p_3 = p_1 + p_2$. So $p_1 + p_2 + p_3 = \mathbf{0}$. From the converse part of the proof of Proposition 1, we have that p_3 corresponds to the end points 10^y and 10^z . Thus, $p_1 + p_2 \sim_n |10^y - 10^z|$.

Lemma 4. *Let $S = \{e_1, e_2, \dots, e_k\} \subseteq \beta(G, n)$ for any coding sequence $\beta(G, n)$ of a graph G with n vertices. Then $\tilde{G}(S)$ induces a path in G if and only if $\sum_{j=1}^k e_j \in C(n-1)$ and S is reduced.*

Proof. If e_1, e_2, \dots, e_k induce a path (in that order) then it is easy to see that we have $e_i \sim_n |10^{x_{i+1}} - 10^{x_i}|$ for some distinct $x_1, x_2, \dots, x_{k+1} \in \mathbb{N} \cup \{0\}$.

By applying Lemma 3 repetitively, we have $\sum_{j=1}^k e_j \sim_n |10^{x_{k+1}} - 10^{x_1}|$. Thus

$\sum_{j=1}^k e_j \in C(n-1)$. Let $e = \sum_{j=1}^k e_j$. Then elements of $S \cup \{e\}$ form a cycle. Then by Lemma 1, $S \cup \{e\}$ is reduced and so S is reduced.

Conversely, let $\sum_{j=1}^k e_j \in C(n-1)$ and S is reduced. Let $\sum_{j=1}^k e_j = e$. So $e + \sum_{j=1}^k e_j = \mathbf{0}$ and since S is reduced, $S \cup \{e\}$ is also reduced. Now consider the graph G' such that $\beta(G', n) = \beta(G, n) \cup \{e\}$. Clearly, $S \cup \{e\}$ induces a cycle in $\beta(G', n)$ by Lemma 1. One edge of that cycle corresponds to e , all the other edges correspond precisely to the members of S . Hence, $\tilde{G}(S)$ induces a path in G .

Corollary 9. *Suppose $p \sim_n |10^i - 10^j|, q \sim_n |10^r - 10^s|$ where $p \neq q$ and $i, j, r, s \in \{0, 1, 2, \dots, n-1\}$. Then we have $p + q \in C(n-1)$ if and only if $|\{i, j\} \cap \{r, s\}| = 1$. Moreover, $p + q \sim_n |10^x - 10^y|$, where $x \in \{i, j\}, y \in \{r, s\}$ and $x, y \notin \{i, j\} \cap \{r, s\}$.*

Proof. Consider a graph G with n vertices such that $p, q \in \beta(G, n)$ for some coding sequence $\beta(G, n)$ of G . Clearly, p corresponds to end-points 10^i and 10^j , and q corresponds to end-points 10^r and 10^s . First, let $p + q \in C(n-1)$. So by Lemma 4, it follows that $\tilde{G}(\{p, q\})$ induces a path in G . This implies that the two edges corresponding to p and q have a common vertex. Since $p \neq q$, this gives that $|\{i, j\} \cap \{r, s\}| = 1$.

Conversely, let $|\{i, j\} \cap \{r, s\}| = 1$. Suppose $j = r$, without loss of generality. Then by Lemma 3, $p + q \sim_n |10^i - 10^s|$ (which proves the next part also). Thus, $p + q \in C(n-1)$.

Lemma 5. *Suppose $p_1 \sim_n |10^i - 10^j|, p_2 \sim_n |10^k - 10^l|, p_3 \sim_n |10^r - 10^s|$, where $i, j, k, l, r, s \in \{0, 1, 2, \dots, n-1\}$ and p_1, p_2, p_3 are distinct. If $p_1 + p_2, p_2 + p_3, p_1 + p_3 \in C(n-1)$, then either $p_1 + p_2 + p_3 = \mathbf{0}$ or $\{i, j\} \cap \{k, l\} = \{i, j\} \cap \{r, s\} = \{k, l\} \cap \{r, s\}$.*

Proof. Consider a graph G with n vertices such that $p_1, p_2, p_3 \in \beta(G, n)$ for some coding sequence $\beta(G, n)$ of G . Clearly, p_1 corresponds to end-points 10^i and 10^j , p_2 corresponds to end-points 10^k and 10^l and p_3 corresponds to end-points 10^r and 10^s . Now since $p_1 + p_2 \in C(n-1)$, by Corollary 9 we have $|\{i, j\} \cap \{k, l\}| = 1$. Without loss of generality, let $j = k$. Then 10^j is the common end-point between edges corresponding to p_1 and p_2 . Since we also have that $p_1 + p_3 \in C(n-1)$ and $p_2 + p_3 \in C(n-1)$, it follows that the edge corresponding to p_3 has a common end-point with the edge corresponding to p_1 and a common end-point with the edge corresponding to p_2 . If the common end-point in both the cases is $10^j (= 10^k)$ then we have $\{i, j\} \cap \{k, l\} = \{i, j\} \cap \{r, s\} = \{k, l\} \cap \{r, s\}$. Otherwise, the common end-point between p_1 and p_3 must be 10^i and the common end-point between p_2 and p_3 must be 10^l . Thus, edges corresponding to p_1, p_2, p_3 form a cycle involving the vertices $10^i, 10^j, 10^l$. So by Proposition 1, we have $p_1 + p_2 + p_3 = \mathbf{0}$.

Lemma 6. *Let G, H be two simple graphs with n vertices each and suppose $M[\beta(G, n)] \cong_s M[\beta(H, n)]$. Let T be any bijective linear operator on \mathbb{Z}_2^{n-1} giving a strong isomorphism between $M[\beta(G, n)]$ and $M[\beta(H, n)]$. Then for $e_1, e_2 \in C(n-1)$, we have $e_1 + e_2 \in C(n-1)$ if and only if $T(e_1) + T(e_2) \in C(n-1)$.*

Proof. Let $e_1, e_2 \in \beta(G, n)$. First, let $e_1 + e_2 \in C(n-1)$. Now $T(e_1) + T(e_2) = T(e_1 + e_2) \in T(C(n-1)) = C(n-1)$ as restriction of T maps $C(n-1)$ onto itself. Conversely, let $T(e_1) + T(e_2) \in C(n-1)$. So $T(e_1 + e_2) \in C(n-1)$. Again since restriction of T maps $C(n-1)$ onto itself, there exists some e in $C(n-1)$ such that $T(e) = T(e_1 + e_2)$. Finally, since T is injective, we have $e = e_1 + e_2$. So $e_1 + e_2 \in C(n-1)$.

Now we prove the theorem which gives a necessary and sufficient condition for two simple graphs to be isomorphic.

Theorem 4. *Let G and H be two simple graphs of n vertices each. Then $G \cong H$ if and only if $M[\beta(G, n)] \cong_s M[\beta(H, n)]$ for any coding sequences $\beta(G, n)$ and $\beta(H, n)$ of G and H , respectively.*

Proof. We consider vertices of both G and H are labeled by $1, 10, 10^2, \dots, 10^{n-1}$. First, let $G \cong H$. So there exists a permutation g on the set $\{0, 1, 2, \dots, n-1\}$ such that for any $r, s \in \{0, 1, 2, \dots, n-1\}$, we have 10^r and 10^s are adjacent in G if and only if $10^{g(r)}$ and $10^{g(s)}$ are adjacent in H . We consider the graphs $K^{(1)}, K^{(2)}$, where $K^{(1)} = G \cup \bar{G}$ and $K^{(2)} = H \cup \bar{H}$, where \bar{G} and \bar{H} are complements of graphs G and H respectively. Now for each $e \in C(n-1)$, if $e \sim_n |10^i - 10^j|$, we define $T(e) \sim_n |10^{g(i)} - 10^{g(j)}|$. Clearly, T is a well-defined mapping from $C(n-1)$ into itself, since $|10^p - 10^q|$ uniquely determines the pair $\{p, q\}$ for any p, q .

Now g , being a permutation, is a bijection. Suppose e_1, e_2 are distinct elements of $C(n-1)$. Let $e_1 \sim_n |10^a - 10^b|$ and $e_2 \sim_n |10^c - 10^d|$. Clearly, $\{a, b\} \neq \{c, d\}$. Bijectiveness of g implies that $\{g(a), g(b)\} \neq \{g(c), g(d)\}$. This shows that $T(e_1) \neq T(e_2)$. So we have that T is one-to-one. Since T is a mapping from a finite set into itself, injectiveness of T implies that T is a bijective mapping from $C(n-1)$ onto itself. Again, since 10^r and 10^s are adjacent in G if and only if $10^{g(r)}$ and $10^{g(s)}$ are adjacent in H for distinct $r, s \in \{0, 1, 2, \dots, n-1\}$, we have that T restricted on $\beta(G, n)$ is a mapping from $\beta(G, n)$ into $\beta(H, n)$. Injectiveness of T ensures the injectiveness of T restricted to $\beta(G, n)$. Since $\beta(G, n)$ and $\beta(H, n)$ are finite sets with equal cardinality, we have that T restricted to $\beta(G, n)$ is a bijection from $\beta(G, n)$ onto $\beta(H, n)$.

Next we observe that T is ‘linear’ on $C(n-1)$, i.e., if $p, q \in C(n-1)$ such that $p+q \in C(n-1)$, then $T(p)+T(q) = T(p+q)$. Let $p \sim_n |10^i - 10^j|$, $q \sim_n |10^r - 10^s|$ such that $p+q \in C(n-1)$. Then by Corollary 9, we have $|\{i, j\} \cap \{r, s\}| = 1$. Without loss of generality, we assume $j = r$. So $g(j) = g(r)$ and $p+q \sim_n |10^i - 10^s|$. Now $T(p) \sim_n |10^{g(i)} - 10^{g(j)}|$ and $T(q) \sim_n |10^{g(r)} - 10^{g(s)}|$. Since $g(j) = g(r)$, we have $T(p)+T(q) \in C(n-1)$ and $T(p)+T(q) \sim_n |10^{g(i)} - 10^{g(s)}|$. Since $T(p+q) \sim_n |10^{g(i)} - 10^{g(s)}|$, we have $T(p+q) = T(p)+T(q)$.

Now let $e_i \sim_n |10^i - 10^{i-1}|$ for all $i = 1, 2, \dots, n$. We know that $B = \{e_i \mid i = 1, 2, \dots, n\}$ is a basis of \mathbb{Z}_2^{n-1} . Since T is defined on each e_i as the latter is in $C(n-1)$ (in fact, $T(e_i) \sim_n |10^{g(i)} - 10^{g(i-1)}|$), we can extend T linearly to a linear operator T_1 on \mathbb{Z}_2^{n-1} . Bijectiveness of T_1 follows from injectiveness of T on $C(n-1)$ (which ensures distinct images under T for distinct elements of B , thus ensuring injectiveness of T_1) and finiteness of \mathbb{Z}_2^{n-1} . As $M[\beta(G, n)]$ and $M[\beta(H, n)]$ are also of the same order, we have $M[\beta(G, n)] \cong_s M[\beta(H, n)]$.

Conversely, let $M[\beta(G, n)] \cong_s M[\beta(H, n)]$. So there exists a bijective linear operator T satisfying the properties mentioned in the Definition 5. We find a

permutation g on the set $\{0, 1, 2, \dots, n-1\}$ such that for any distinct $r, s \in \{0, 1, 2, \dots, n-1\}$, 10^r and 10^s are adjacent in G if and only if $10^{g(r)}$ and $10^{g(s)}$ are adjacent in H . Clearly, such a g acts as an isomorphism between G and H . Now for $i = 1, 2, \dots, n$, define e_i as the element in $C(n-1)$ which has 1 in its i^{th} co-ordinate from the right ($(n-i)^{\text{th}}$ co-ordinate from the left) and 0 in remaining coordinates. Then $e_i \sim_n |10^i - 10^{i-1}|$. Now for $i = 1, 2, \dots, n-1$, we have $e_i + e_{i+1} \in C(n-1)$. So from Lemma 6, we have $T(e_i) + T(e_{i+1}) \in C(n-1)$. From Corollary 9, the edges corresponding to $T(e_i)$ and $T(e_{i+1})$ have a (unique) common point, say 10^x . We define $g(i-1) = x$. This defines g as a mapping from $\{0, 1, 2, \dots, n-2\}$ into $\{0, 1, 2, \dots, n-1\}$. Now we show that g is one-to-one.

If possible, let $g(i-1) = g(j-1)$ for some $i \neq j$, $i, j \in \{1, 2, \dots, n-1\}$. By the above definition of g , $g(i-1)$ and $g(j-1)$ are one of the common end points of $T(e_i), T(e_{i+1})$ and $T(e_j), T(e_{j+1})$ respectively. So the edges corresponding to $T(e_i)$ and $T(e_j)$ have a common end-point $g(i-1)$ ($= g(j-1)$). So from Corollary 9, we have $T(e_i) + T(e_j) \in C(n-1)$. Linearity of T implies that $T(e_i + e_j) \in C(n-1)$. Since T restricted to $C(n-1)$ is a bijection from T onto itself, we have $e_i + e_j \in C(n-1)$. Again, $e_j + e_{i+1} \in C(n-1)$ for the same reason as $g(i-1)$ is a common end-point between edges corresponding to $T(e_j)$ and $T(e_{i+1})$. We also have $e_i + e_{i+1} \in C(n-1)$. So $e_i + e_j, e_{i+1} + e_i, e_j + e_{i+1} \in C(n-1)$. If possible, let $e_i + e_j + e_{i+1} = \mathbf{0}$. However, then $e_j = e_i + e_{i+1}$, which is impossible by definition of e_i 's. Thus, from Lemma 5, 10^i is either 10^j or 10^{j-1} , i.e., $i = j$ or $j-1$. Similar argument on e_i, e_j, e_{j+1} gives us $j = i$ or $i-1$. But $i = j-1$ and $j = i-1$ both cannot be true simultaneously. So we have $i = j$ which is a contradiction. Thus g is injective. So we define $g(n-1) = x$, where $x \in \{0, 1, 2, \dots, n-1\} \setminus \{g(0), g(1), \dots, g(n-2)\}$. Then g is defined on $\{0, 1, 2, \dots, n-1\}$ into itself. Moreover since g is injective, it is a permutation on the set $\{0, 1, 2, \dots, n-1\}$.

References

1. W. K. Chen, *On vector spaces associated with a graph*, SIAM J. Appl. Math. **20** (1971), 526–529.
2. Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice Hall of India Pvt. Ltd., 1997.
3. R. Gould, *Graphs and vector spaces*, J. Math. Phys. **37** (1958), 193–214.
4. J. G. Oxley, *Matroid Theory*, Oxford University Press, 1992.
5. W. T. Tutte, *Matroids and graphs*, Trans. Amer. Math. Soc. **90** (1959), 527–552.
6. D. B. West, *Introduction to Graph Theory*, Prentice-Hall of India Pvt.Ltd., 2003.
7. H. Whitney, *On the Abstract Properties of Linear Dependence*, American Journal of Mathematics **57** (1935), 509–533.